ASYMPTOTICAL LIMITS OF YANG-MILLS FLOW IN HOLOMORPHIC VECTOR BUNDLES ON COMPACT KÄHLER MANIFOLDS

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1. Introduction

Let $X$ be a Kähler manifold of complex dimension $m$ and let $E$ be a holomorphic vector bundle over $X$. Let $\Omega^0(E) = \Gamma(E)$ denote the set of all smooth sections of $E$ and let $\mathcal{A}$ denote the set of all connections on $E$ which are unitary with respect to its structure.

For each connection $D_A$, the Yang-Mills functional is defined by

\begin{equation}
YM(A; X) = \int_X |F_A|^2 \, dV_g,
\end{equation}

where $F_A$ is the curvature of $D_A$. In local trivialization, we can express $D_A$ as $d + A$, where $A \in \Gamma(\text{End}E \otimes T^*X)$ is the connection matrix.

We say that a connection $D_A$ in a (holomorphic) vector bundle $E$ over $X$ is a Hermitian Yang-Mills connection if $A$ satisfies the Yang-Mills equation

\begin{equation}
D_A^* F_A = 0.
\end{equation}
It is of great interests to establish relations between the stability of bundles and the existence of Hermitian-Einstein metrics in a holomorphic vector bundle $E$ over a Kähler manifold $X$.

- $m = 1$. Narasimhan-Seshadri (1965) first proved existence theorem of Hermitian-Einstein metrics for stable bundles over a compact Riemann surface.

- $m = 2$. Donaldson (1985) proved that an irreducible holomorphic vector bundle $E$ over a compact Kähler surface $X$ admits a unique Hermitian-Einstein connection if and only if it is stable.

- $m \geq 2$. Uhlenbeck-Yau (1986) established that an irreducible holomorphic vector bundle $E$ over a compact Kähler manifold $X$ admits a unique Hermitian-Einstein connection if and only if it is stable. Donaldson (1987) also proved this result in the case of algebraic manifolds.
Heat flow for the Yang-Mills equations, suggested by Atiyah and Bott, has played an important role in the Yang-Mills theory.

The Yang-Mills flow is

$$\frac{\partial A}{\partial t} = -D^*_A F_A$$

with initial value $A(0) = A_0$, where $A_0$ is a connection corresponding to a Hermitian metric $H_0$ in $E$.

The first contribution of Yang-Mills flow was made by Donaldson [D1] in the case of a holomorphic vector bundle by using it to establish an important relationship between the Hermitian Yang-Mills connection and stable holomorphic vector bundles.

When the bundle over $X$ is stable, it is can be proved that as $t \to \infty$, the solution $A(t)$ of (1.3) converges to a Hermitian-Einstein connection.
In 1994, Bando and Siu proved that a reflexive sheaf on an $m$-dimensional compact manifold has an admissible Hermitian-Einstein metric if and only if the sheaf is polystable.

When the bundle $E$ over $X$ is unstable, for a given initial connection $A_0$, there is a global smooth solution $A(t)$ of Yang-Mills flow (1.3) over $X$ by the Donaldson result, but we can not expect that $A(t)$ converges to the unique Hermitian-Einstein connection since there is no Hermitian-Einstein connection in $E$ over $X$ by the Donaldson-Uhlenbeck-Yau theorem (also Bando-Siu).

Bando and Siu conjectured about the relation between the breaks up into a direct sum of Hermitian-Einstein via Yang-Mills flow and the Harder-Narashimhan filtration on Kähler manifolds.
By analyzing asymptotical limits of Yang-Mills flow, jointed with Tian, we proved the existence of a singular Hermitian Yang-Mills connection in a holomorphic vector bundle $E$ over a Kähler manifold $X$, where the Hermitian Yang-Mills connection is smooth away from a singular set $\Sigma$ of complex codimension 2. More precisely, for any sequence $t \to \infty$, there exists a subsequence $t_k$ such that as $t_k \to \infty$, $A(t_k)$ converges to a singular Yang-Mills connection $A_\infty$ with the blow-up locus $(\Sigma, \Theta)$.

In order to settle the conjecture of Bando and Siu, there is an open question concerning the uniqueness of a limiting connection $A_\infty$ of the Yang-Mills heat flow and whether the blow-up locus $(\Sigma, \Theta)$ is unique.

In this talk, we will outline the approach of the above results.
Geometric setting

For a connection $D_A$, we mark up $D_A = \partial_A + \bar{\partial}_A$ with

$$\partial_A : \Omega^0(E) \to \Omega^{1,0}(E),$$

$$\bar{\partial}_A : \Omega^0(E) \to \Omega^{0,1}(E).$$

For a Hermitian metric $H$ in a holomorphic vector bundle $E$, there is a unique Hermitian connection $D_A : \Omega^0(E) \to \Omega^1(E)$ characterized by

$$d \langle \sigma, \tau \rangle_H = \langle D_A \sigma, \tau \rangle_H + \langle \sigma, D_A \tau \rangle_H, \quad \forall \sigma, \tau \in \Omega^0(\text{End} E)$$

and

$$\bar{\partial}_A s = \bar{\partial}_E s, \quad \forall s \in \Omega^0(\text{End} E)$$

where $\bar{\partial}_A$ is the $(0, 1)$ component of the connection $D_A$. 
For the Kähler metric on $X$, we define the Kähler form $\omega$ as

$$\omega = i \sum g_{k\bar{j}} dz_i \wedge d\bar{z}_j; \quad i = \sqrt{-1}.$$ 

Using Kähler form $\omega$ we define the contraction $\Lambda$ on $(1,1)$-forms by

$$\Lambda \eta = \langle \eta, \omega \rangle$$

where $\langle , \rangle$ denotes the pointwise inner product on $(1,1)$-forms by the Kähler metric on $X$.

If $F_A$ is of the form $(1,1)$, then the Yang-Mills equation $D_A^* F_A = 0$ with $D_A F_A = 0$ yields

$$\nabla_A (\Lambda F_A) = 0$$

This implies $\Lambda F_A$ has constant eigenvalues.
Relation between stable bundles and Hermitian-Einstein metric

Donaldson-Uhlenbeck-Yau Theorem.

An irreducible holomorphic vector bundle admits a unique Hermitian-Einstein connection if and only if it is stable.

A metric $H$ on $E$ is called “Hermitian-Einstein” if the corresponding connection satisfies

$$\Lambda F_H = \lambda I,$$

where

$$i\lambda = \frac{2\pi}{\text{vol } X} \frac{\deg E}{\text{rank } E}.$$

The degree of $E$ is

$$\deg E = \int_X C_1(E) \wedge *\omega = \int_X C_1(E) \wedge \frac{\omega^{m-1}}{(m-1)!}.$$

where $C_1(E) = \frac{\sqrt{-1}}{2\pi} \text{tr}(F_A)$ is the first Chern form.
A holomorphic vector bundle $E$ over $X$ is stable if for every proper subsheaf $V \subset E$,

$$\frac{\deg(V)}{\text{rk}(V)} < \frac{\deg(E)}{\text{rk}(E)}.$$ 

Some generalizations of the Donaldson-Uhlenbeck-Yau Theorem:

- In 1988 C. Simpson generalized the Donaldson-Uhlenbeck-Yau theorem in holomorphic bundles $E$ over some non-compact Kähler manifolds $X$.

- In 1991, Bradlow generalized the result for Yang-Mills-Higgs field. (In 2001, I gave a different proof by the Yang-Mill-Higgs flow.)

- In 1996 P. De Bartolomeis and G. Tian generalized the result in a complex vector bundle over a compact almost Hermitian regularized manifolds

When the bundle $E$ over $X$ is unstable, there is no Hermitian-Einstein connection by using the Donaldson-Uhlenbeck-Yau theorem.
Singular admissible connections


Given a sequence of smooth Yang-Mills connections \( \{A_i\} \) with uniformly \( L^2 \)-bounded curvatures, by taking a subsequence if necessary, \( A_i \) converges, modulo gauge transformations, to a Yang-Mills connection \( A \) outside a closed subset \( \Sigma \) of Hausdorff real codimension 4, moreover, the set \( \Sigma \) (blow-up locus) is rectifiable.

Tian also studied admissible (singular) connections \( A \) which are smooth over \( M \setminus S(A) \) for a closed subset \( S(A) \) of Hausdorff dimension \( n - 4 \) and the compactification of the moduli space \( M_{\Omega,E} \) of \( \Omega \)-anti-self-dual instantons of \( E \).

In this talk, we discuss the asymptotical limits of Yang-Mills flow, which yields the existence of singular Hermitian Yang-Mills connection in a holomorphic bundles over a Kähler manifold.
Limiting connections of Yang-Mills flow

Hong-Tian.

Let $A(x,t)$ be a global smooth solution of the Yang-Mills flow in a holomorphic vector bundle $E$. For any sequence $t_k \to \infty$, $A(t_k)$ converges (up-to a subsequence )to a singular Hermitian Yang-Mills connection $A_\infty$ having curvature of type $(1,1)$ with the blow-up locus $(\Sigma, \Theta)$, such that $\Sigma = \bigcup_\alpha \Sigma_\alpha$ and $\Theta|_{\Sigma_\alpha} = 8\pi^2 m_\alpha$, where $\Sigma_\alpha$ are holomorphic subvarieties of complex codimension two in $X$ and $m_\alpha$ are positive integers.

The above limiting connection $A_\infty$ can be uniquely extended in a Hermitian Yang-Mills connection in a reflexive sheaf $\mathcal{E}$ over $X$ such that $A$ is smooth outside a closed set $\tilde{\Sigma}$ of complex codimension three in $X$, where $\mathcal{E}$ is local free on $X\setminus \tilde{\Sigma}$.

2008. For each subsequence $t \to \infty$, the limiting connection $A_\infty$ is unique up to a bundle isomorphism.
Outline proof

- The global existence of the solution of Yang-Mills heat flow for all $t > 0$.
- The asymptotical behavior of the solution of flow (1.3) as $t \to \infty$, by establishing a local monotonicity for Yang-Mills flow.
- Following Tian’s result on blow-up loci of a sequence of smooth Yang-Mills connections, we get a similar result on blow-up loci of a sequence connections from Yang-Mills flow, involving a result of King (or R. Harvey and B. Shiffmann) on holomorphic subvarieties $\Sigma_\alpha$.
Global existence of Yang-Mills heat flow

We consider the Yang-Mills flow

(2.1) \[ \frac{\partial A}{\partial t} = -D^*_A F_A \]

with initial value \( A(0) = A_0 \), where \( A_0 \) is a connection corresponding to a Hermitian metric \( K \) in \( E \).

For the initial metric \( K \) in \( E \), set \( H(t) = K h(t) \). If \( K = \langle \cdot, \cdot \rangle_K \) and \( H = \langle \cdot, \cdot \rangle_H \) are two Hermitian metrics with \( \langle s, t \rangle_H = \langle hs, t \rangle_K \) for a \( h = K^{-1}H \in \text{End}E \). Then the corresponding connections satisfy

(2.2) \[ \bar{\partial}_H = \bar{\partial}_K, \quad \partial_H = h^{-1} \circ \partial_K \circ h = \partial_K + h^{-1} \partial_K h \]

and the associated curvatures \( F_H \) and \( F_K \in \Omega^{1,1}(\text{End}E) \) by

(2.3) \[ F_H = F_K + \bar{\partial}_K (h^{-1} \partial_K h). \]
For the initial metric $K$, set $H(t) = K \ h(t)$ the metric corresponding the connection $A$. We consider a flow of complex gauge transformations:

\[
(2.6) \quad \frac{\partial h}{\partial t} = -2i h [\Lambda F_H - \lambda I]
\]

with initial value

\[ h(0) = I. \]

**Proposition 1.** Let $E$ be a holomorphic bundle over a compact Kähler manifold $X$. Then there exists a global solution $H$ to (2.6).

Through an unitary gauge transformation, we prove the global existence of the YM flow (2.1).
Let $A_0$ be the unitary connection on $E$ with curvature $F_{A_0} = F_K$ of type $(1, 1)$ corresponding to the Hermitian metric $K$. The complex gauge group $\mathcal{G}^\mathbb{C}$ of general linear automorphism acts on the space $\mathcal{A}^{1,1}$ of connections with curvature of type $(1, 1)$ by

\begin{align}
\bar{\partial}_g^*(A_0) &= g \circ \bar{\partial}_{A_0} \circ g^{-1}, \\
\partial_g^*(A_0) &= \bar{g}^{t-1} \circ \partial_{A_0} \circ \bar{g}^t,
\end{align}

that is,

\begin{equation}
g^{-1} \circ D_g^*(A_0) \circ g = \bar{\partial}_{A_0} + h^{-1} \partial_{A_0} h,
\end{equation}

and the curvature transforms by

\begin{equation}
g^{-1} F_g^*(A_0) g = F_{A_0} + \bar{\partial}_{A_0} (h^{-1} \partial_{A_0} h)
\end{equation}

where $h = \bar{g}^t g$. 
Theorem 2. Let $A_0$ be a given smooth unitary connection on $E$ with curvature $F_{A_0}$ of type $(1,1)$. Then for any $T > 0$ there exists a smooth solution $A(t)$ of the Yang-Mills flow (1.3) in $X \times [0,T)$ with initial values $A(0) = A_0$ on $X$.

Proof. We take any $g \in G^C$ with $\bar{g}^t g = h$ (for example, $g = h^{1/2}$.) Since $h$ solves (2.6), we have

$$\frac{\partial g}{\partial t} = -i \left[ \Lambda F_{g^*(A_0)} - \lambda I \right] g$$

with initial value $g(0) = I$. Let $g(t) = h^{1/2}(t)$ be a solution of (2.8). Then the connection $A(t) = g^*(A_0)$ a solution to

$$\frac{\partial A}{\partial t} = \partial_A (g^{t^{-1}} \partial_t \bar{g}^t) - \tilde{\partial}_A (\partial_t gg^{-1})$$

$$= -D^*_A F_A + D_A(\alpha(t)),$$

where $\alpha(t) = \frac{1}{2} (\bar{g}^{t^{-1}} \partial_t \bar{g}^t - \partial_t gg^{-1}) \in \Omega^0(\text{Ad}E)$ since $\tilde{\alpha}^t = -\alpha$. (2.9) is equivalent to the Yang-Mills flow (1.3).
3. Asymptotical behavior of Yang-Mills flow

**Lemma 3.** (Energy inequality) Let $A$ be a smooth solution to the Yang-Mills flow (1.3) in $X \times [0, \infty)$ with initial value $A_0$. Then, for any $t < \infty$, we have

$$
\int_X |F_{A(t)}|^2 \, dV_g + 2 \int_0^t \int_X \left| \frac{\partial A}{\partial s} \right|^2 \, dV_g \, ds \leq \int_X |F_{A_0}|^2 \, dV_g.
$$

**Theorem 4.** Let $A$ be a global smooth solution of the Yang-Mills flow equation (1.3) in $X \times [0, \infty)$ with initial value $A_0$. For any a sequence $\{t_k\}$, there is a subsequence, still denoted by $\{t_k\}$, such that as $t_k \to \infty$, $A(x, t_k)$ converges in $C^\infty(X \setminus \Sigma)$ to a solution $A_\infty$ of the Yang-Mills equation (1.2), where $\Sigma$ is a closed set in $X$. Moreover $\mathcal{H}^{2m-4}(\Sigma)$ is finite with

(2.9)

$$
\Sigma = \bigcap_{\varepsilon_0 > r > 0} \left\{ x \in X : \liminf_{k \to \infty} r^{4-n} \int_{B_r(x)} |F_A|^2(\cdot, t_k)^2 \, dV_g \geq \varepsilon_1 \right\},
$$

where $\varepsilon_1$ is a positive constant.
Let $z = (x, t)$ be points in $X \times \mathbb{R}$. For a fixed point $z_0 = (x_0, t_0) \in X \times \mathbb{R}_+$, we write

$$T_R(t_0) = \{ z = (x, t) : t_0 - 4R^2 < t < t_0 - R^2, x \in X \} ,$$

and

$$P_R(z_0) = B_R(x_0) \times [t_0 - R^2, t_0 + R^2].$$

The fundamental solution is

$$G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp \left( -\frac{(x - x_0)^2}{4(t_0 - t)} \right), \quad t < t_0$$

for the (backward) heat equation with singularity at $z_0$. 
Theorem 5. (Local monotonicity) Let $\phi \in C^\infty_0(B_R(x_0))$ be a cut-off function such that $\phi \equiv 1$ on $B_{R/2}$, $|\phi| \leq 1$ and $|\nabla \phi| \leq \frac{C}{R}$ in $B_R \setminus B_{R/2}$. Then for $z_0 = (x_0, t_0) \in M \times (0, \infty)$ and for two real numbers $R_1, R_2$ with $0 < R_1 \leq R_2 \leq R$, we have

$$\int_{T_{R_1}(z_0)} R_1^2 |F_A|^2(x, t) \phi^2 G_{z_0} \ dV_g \ dt \\ \leq C \exp(C(R_2 - R_1)) \int_{T_{R_2}(z_0)} R_2^2 |F_A|^2(x, t) \phi^2 G_{z_0} \ dV_g \ dt \\ + C(R_2 - R_1) YM(A_0) + CR^{2-n} \int_{P_{R}(z_0)} |F_A|^2(x, t) \ dV_g \ dt .$$

The well-known Weitzenböck formula:

$$\nabla_A^\ast \nabla_A \phi = \triangle_A \phi + R_X \# \phi + F_A \# \phi ,$$

where $\phi \in \Omega^p(E)$, $R_X$ is the curvature of $X$ and $\#$ denotes a multi-linear map with smooth coefficients.

We derive a Bochner type inequality in the following:
Proposition 6. Let $A$ be a regular solution of the heat flow equation (2.1) with initial values $A_0$. Then

$$\left(\frac{\partial}{\partial t} - \triangle_X\right)(|F_A|^2) \leq C(|F_A| + |R_X|)|F_A|^2,$$

where $C$ is a constant independent of $u$ and $A$, $R_X$ is the Riemannian curvature of $X$.

Theorem 7. Suppose that $A$ is a solution of the Yang-Mills heat flow (2.1) in $X \times [0, \infty)$ with initial value $A_0$. There exists a positive constant $\varepsilon_0 < i(M)$ such that if, for some $R$ with $0 < R < \min\{\varepsilon_0, \frac{t_1}{2}\}$, the inequality

$$R^{2-2m} \int_{P_R(x_0,t_0)} |F_A|^2 \, dx \, dt \leq \varepsilon_0$$

holds, then we have

$$\sup_{P_{\delta R}(x_0,t_0)} |F_A|^2 \leq C(\delta R)^{-4},$$

where $C$ is a constant depending on $M$, $\delta > 0$, $YM(A_0)$ and $R$. 
4. Limiting connections of Yang-Mills flow

**Proposition 8.** Let $A$ be a smooth solution of Yang-Mills flow. Then there exists a nonnegative constant $a$ such that for $0 < \sigma \leq \rho < r_p$ and for each $t \in (0, \infty)$,

$$\pm \rho^{4-m}e^{\pm a \rho^2} \int_{B_\rho} \phi |F_A|^2(\cdot, t) \, dV_g \pm \sigma^{4-m}e^{\pm a \sigma^2} \int_{B_\sigma} \phi |F_A|^2(\cdot, t) \, dV_g$$

$$\mp 4 \int_{B_\rho \backslash B_\sigma} r^{4-m}e^{\pm ar^2} \phi |F_A|_{\frac{\partial}{\partial r}}^2(\cdot, t) \, dV_g$$

$$\geq -2 \int_{\sigma}^{\rho} r^{4-m}e^{\pm ar^2} dr \int_{B_{r(p)}} 2 |F_A|_{\frac{\partial}{\partial r}} \left| F_A \nabla \phi \right|(\cdot, t) \, dV_g.$$

**Proposition 9.** Let $A$ be a solution of the Yang-Mills heat flow (1.3) in $X \times (0, \infty)$ with $F_A \in \Omega^{1,1}$. Then $\int_X \left| \frac{dA}{dt} \right|^2 \, dV_g$ tends to 0 as $t$ tends to infinity.
Let $A_0$ be a given smooth unitary connection on $E$ over $X$ with curvature $F_{A_0}$ of type $(1, 1)$. Let $A$ solve the Yang-Mills flow (1.3) in $X \times [0, T)$ with initial values $A(0) = A_0$ on $X$. Then, for a sequence $t \to \infty$, there exists a sequence $\{t_k\}$ such that $A(t_k)$ converges, modulo gauge transformation, to $A_\infty$ smoothly except for a closed singular set $\Sigma(\{A_{t_k}\}_{k=1}^{\infty})$ as $t_k \to \infty$.

Let $\mu$ be the limit Radon measure of $\mu_k = |F_{A(t_k)}|^2 dV_g$; i.e., for any continuous function $\phi$ with compact support in $X$,

$$\lim_{k \to \infty} \mu_k(\phi) = \lim_{k \to \infty} \int_X \phi |F_{A(t_k)}|^2 dV_g = \int_X \phi d\mu = \mu(\phi).$$

For $x \in X$, $e^{ar^2} r^{4-2m} \mu(B(x))$ is a nondecreasing in $r$. Then the density

$$\Theta(\mu, x) = \lim_{r \to 0} r^{4-2m} \mu(B_r(x))$$

exists for every $x \in X$ and $x \in \Sigma(\{A_{t_k}\}_{k=1}^{\infty})$ if and only if $\Theta(\mu, x) \geq \varepsilon_1 > 0$. 
By Fatou’s lemma, there is a nonnegative Radon measure $\nu$ on $X$ such that
\[
\mu = |F_{A_\infty}|^2 dV_g + \nu.
\]
Then it follows from [T] that $\nu(x) = \Theta(\mu, x)H^{2m-4} \sum \{A_{t_k}\}_{k=1}^\infty$ for $H^{2m-4}$-a.e. $x \in \Sigma(\{A_{t_k}\})$. Moreover, $\Sigma(\{A_{t_k}\}_{k=1}^\infty) = \Sigma(A) \cup \Sigma$, where

\[
\Sigma = \{x \in \Sigma(\{A_{t_k}\}_{k=1}^\infty) : \Theta(\mu, x) > 0, \lim_{r \to 0} r^{4-2m} \int_{B_r(x)} |F_{A_\infty}|^2 dV_g = 0\}
\]

We call $(\Sigma, \Theta)$ the blow-up locus of the convergent sequence $\{A(t_k)\}_{k=1}^\infty$.

For any $x \in X$ and sufficiently small $\lambda$, we define the scaled measure $\mu_{x,\lambda}$ as follows: for any $E \in T_xX$,
\[
\mu_{x,\lambda}(E) = \lambda^{4-2m} \mu(\exp_x(\lambda E)),
\]
where $\exp_x : T_xX \to X$ is the exponential map of the metric $g$ and
\[
\lambda E = \{y \in T_xX : \lambda^{-1}y \in E\}.
\]
For simplicity, we denote by $A_k$ the connection $A(x, t_k)$. Let $A_{k,x,\lambda}$ be the scaled connection on $T_x X$ defined by

$$A_{k,x,\lambda} = \tau_\lambda^* \exp_x^* A_k,$$

where $\tau_\lambda(v) = \lambda v$ for any $v \in T_x X$.

As $\lambda \to 0$, $\mu_{x,\lambda}$ converges weakly to $\Theta(\mu, x) H^{2m-4} \mu V$ weakly. Similarly, as $k \to \infty$, $|F_{A_k, x, \lambda}|^2 dV_{x, \lambda}$ converges to $\mu_{x, \lambda}$.

Let $\{\lambda_k\}$ be a sequence with $\lim_{k \to \infty} \lambda_k = 0$ and

$$\lim_{k \to \infty} \lambda_k^{4-2m} \int_M |\frac{\partial A}{\partial t}|^2 (\cdot, t_k) \, dV_g = 0.$$

Using Propositions 9 and 10, it follows from an argument similar to that in [T] and [HT] that there is a subsequence sequence $\lambda_k$ such that, as $\lambda_k \to 0$, the Radon measure $|F_{A_k, x, \lambda_k}|^2 dV_{x, \lambda_k}$ converges to $\Theta(\mu, x) H^{2m-4} \mu V$,

$$\Theta(\mu, x) = \lim_{k \to \infty} \int_{B_1(0, g_x, 0)} |F_{A_k, x, \lambda_k}|^2 \, dV_{g_x, \lambda_k}.$$
Moreover, modulo gauge transformations, $A_{k, x, \lambda_k}$ converges to 0 uniformly on any compact subset in $T_x X \setminus V$ and

$$\lim_{k \to \infty} \left( \sum_{\alpha=1}^{2m-4} \int_{B_2(0, g_x, 0)} |F_{A_{k, x, \lambda_k}} \frac{\partial}{\partial z_\alpha}|^2 dV_{g_x, 0} \right) = 0,$$

where $\{z_1, ..., z_{2m-4}\}$ is an orthogonal coordinate system of $V$. We set $z = (z', z'') \in T_x X$ with $z' \in V$, $z'' \in V^\perp$.

For any a sequence $\{t_k\}$, there is a subsequence, still denoted by $\{t_k\}$, that as $t_k \to \infty$, $A(t_k)$ converges, modulo gauge transformation, to $A_\infty$ smoothly except for a closed singular set $\Sigma = \Sigma(\{A_k\}_{k=1}^\infty)$ satisfying

\begin{equation}
D_{A_\infty}^* F_{A_\infty} = 0 \quad \text{in} \quad X \setminus \Sigma; \quad \text{then} \quad \nabla_{A_\infty}(\Lambda F_{A_\infty}) = 0 \quad \text{in} \quad X \setminus \Sigma.
\end{equation}

This means that $\Lambda F_A$ has constant eigenvalues $\lambda_j$, $j = 1, .., l$, such that for a.e. $x \in X \setminus \Sigma$, up to a gauge transformation, we can assume

\begin{equation}
\Lambda F_{A_\infty} = \begin{pmatrix}
\lambda_1 I_1 & 0 & \ldots & 0 \\
0 & \lambda_2 I_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_l I_l
\end{pmatrix}
\end{equation}
where each $I_j$ is a unit matrix for each $j = 1, \ldots, l$. Moreover, for all $x \in \Sigma$,

\[(4.4) \quad \Theta(\mu, x) \geq \varepsilon > 0, \quad \lim_{r \to 0} r^{4-2m} \int_{B_r(x)} |F_{A_{\infty}}|^2 \, dy = 0.\]

Using (4.3), the induced bundle $E^\infty$ from $A^\infty$ over $X \setminus \Sigma$ can be split into the sum of holomorphic bundles $E_j^\infty$ over $X \setminus \Sigma$ for $j = 1, \ldots, l$; i.e.

we have

$$E^\infty|_{X \setminus \Sigma} = E_1^\infty|_{X \setminus \Sigma} \oplus \cdots \oplus E_l^\infty|_{X \setminus \Sigma}.$$  

For each $E_j^\infty$, there exists an induced Hermitian metric $H_j$ and the corresponding Yang-Mills connection $A_{H_j}$ of $E_j^\infty$ satisfying $\Lambda F_{H_j} = \lambda_j I_j$ in $E_j^\infty$ over $X \setminus \Sigma$. Then

**Theorem 10.** The holomorphic bundle $E^\infty|_{X \setminus \Sigma}$ extends to a reflexive sheaf $\tilde{E}^\infty$ over $X$ such that the Hermitian Yang-Mills metric $H$ is smooth and extended to $X \setminus \tilde{\Sigma}$ where the reflexive sheaf $\tilde{E}^\infty$ fails to locally free on the closed set $\tilde{\Sigma}$ of complex codimension three.
For the above limiting Hermitian-Yang-Mills connection $A_\infty$ in $X \setminus \Sigma$, the second Chern class $C_2$ is defined by

$$C_2(A_\infty) = \frac{1}{8\pi^2} [\text{tr}(F_{A_\infty} \wedge F_{A_\infty}) - \text{tr}F_{A_\infty} \wedge \text{tr}F_{A_\infty}].$$

Then, $C_2(A_\infty)$ extends to a closed form on $X$.

**Theorem 11.** Let $A(x,t)$ be a solution of the Yang-Mills flow (1.3) in $X \times [0, \infty)$ with curvature $F_A$ of type $(1,1)$. Then, for any sequence $t \to \infty$, there exists a subsequence $t_k$ such that as $t_k \to \infty$, $A(t_k)$ converges to a singular Yang-Mills connection $A_\infty$ except for a singular set $\Sigma$. Moreover, there exist holomorphic subvarieties $\Sigma_\alpha$ and positive integers $m_\alpha$ such that for any smooth $\phi$,

$$\lim_{k \to \infty} \int_X \phi \wedge C_2(A(t_k)) = \int_X \phi \wedge C_2(A_\infty) + \sum_\alpha m_\alpha \int_{\Sigma_\alpha} \phi.$$
Theorem 12. Let $A$ be a solution of the Yang-Mills flow equation (1.3) in $X \times (0, \infty)$. Let $\{A(t_k)\}_{k=1}^{\infty}$ be a convergent sequence such that as $t_k \to \infty$, $A(t_k)$ convergent to $A_\infty$ with the blowup locus $(\Sigma, \Theta)$. Let $\{A'(t'_k)\}_{k=1}^{\infty}$ be a convergent sequence such that as $t'_k \to \infty$, $A(t'_k)$ convergent to $A'_\infty$ with the blow-up locus $(\Sigma', \Theta')$. Then two limiting connection are unique.