The Equivariant Noncommutative Atiyah-Patodi-Singer Index Theorem

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1. Background

a) Atiyah-Patodi-Singer proved their famous Atiyah-Patodi-Singer index theorem for manifolds with boundary.

M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry*, Math. Proc. Cambridge Philos. Soc. 77 (1975), 43-69; 78 (1975), 405-432; 79 (1976), 71-99.

b) Donnelly extended this theorem to the equivariant case by modifying the Atiyah-Patodi-Singer original method.

H. Donnelly, *Eta invariants for G-space*, Indiana Univ. Math. J. 27 (1978), 889-918.

c) Zhang proved the regularity of equivariant eta invariant by a direct geometric method of Lafferty-Yu-Zhang and he also got this equivariant Atiyah-Patodi-Singer index theorem.

W. P. Zhang, *A note on equivariant eta invariants*, Proc. AMS. 108 (1990), 1121-1129.

d) Wu introduced the total eta invariant (called the higher eta invariant by Wu) which is the generalization of the classical Atiyah-Patodi-Singer eta invariants. Using the variation formula of total eta invariant, Wu proved the Atiyah-Patodi-Singer index theorem in the framework of noncommutative geometry

[Wu] F. Wu, *The Chern-Connes character for the Dirac operators on manifolds with boundary*, K-Theory 7 (1993), 145-174.

e) Using superconnection, Getzler gave another proof of the noncommutative Atiyah-Patodi-Singer index theorem, which was more difficult, but avoided mention of the operators b and B of cyclic cohomology.

E. Getzler, *Cyclic homology and the Atiyah-Patodi-Singer index theorem*, Contemp. Math. 148 (1993), 19-45.

f) It is **a natural question to ask whether Wu' result admits an equivariant generalization.** In this talk, we will give an equivariant extension of the Wu' theorem.

2. The Chern-Connes character theory for θ-summable spectral triple

a) Noncommutative space

Theorem (R. Descartes) Euclidean geometry is the study of three functions x, y, z on R^3.

In other words, one can recover the geometry of the space R^3 by studying functions on it (the coordinates). For example, the unit sphere is associated to the equation $x^2 + y^2 + z^2 = 1$.

· Algebraic geometry: geometric spaces$\Leftrightarrow$ commutative algebras

The purpose of noncommutative geometry : extend this correspondence to the noncommutative case in the framework of real analysis.

Philosophy of NCG

I) classical space $\Leftrightarrow$ commutative $C^\star$ algebras $\Rightarrow$ NC $C^\star$ algebras

Theorem (Gelfand-Naimark) Let X,Y be two compact Hausdoff topological spaces and the set of continuous functions on X denoted by $C(X)$ is a unital $C^\star$ algebra, then
i) $C(X) \cong C(Y) \Leftrightarrow X, Y$ are homeomorphic;
ii) if A is a commutative unital $C^\star$ algebra, then $A \cong C(X)$.

By the Gelfand-Naimark theorem:
compact Hausdoff topological spaces $\Leftrightarrow$ commutative unital $C^\star$ algebras.
Similarly,
locally compact Hausdoff topological spaces $\Leftrightarrow$

commutative nonunital $C^\star$ algebras.
So commutative $C^\star$ algebras are usually called commutative spaces.
Noncommutative $C^\star$ algebras are called non-commutative spaces.
There are some spaces associated to NC $C^\star$ algebras naturally.
Example: 1. The space of leave of a foliation (generalize family index theorem to the foliation case)
2. Duals of discrete groups (Novikov conjecture).
More examples can be found in Connes' book.

II) classical tools (such as measure theory, topology, differential calculus and Riemannian geometry) $\Leftrightarrow$ algebraic reformulation $\Rightarrow$ non-commutative generalization

Algebraic reformulation is never straightforward: i) completely new phenomena arise in the noncommutative case. ii) The constraint of developing the theory in the noncommutative framework leads to a new point of view and new tools even in the commutative case., such as cyclic cohomology and the quantized differential calculus.

b) Index theory in the framework of noncommutative geometry

In the following, we give the NC counterpart of the classical Atiyah-Singer index theorem. In order to be compatible with the odd case, we will give the Connes and Getzler-Szenes index theorem for the Dirac operator by the heat kernel method (Ref. Connes and Moscovici Topology 1990). On the general Getzler-Szenes index theorem, we can find it in the following references.

J. Gracia-Bondía and J. Várilly,H. Figueroa *Elements of noncommutative geometry*, Birkhäuser Boston, 2001.
E. Getzler, and A. Szenes, *On the Chern character of theta-summable Fredholm modules*, J. Func. Anal. 84 (1989), 343-357.

An even spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$**:** $\mathcal{A}$ is an $(C^\star)$ unital algebra represented by bounded operators on the Hilbert space $\mathcal{H}$, D is a selfadjoint operator on $\mathcal{H}$, with compact resolvent, such that $[D, a]$ is bounded for every $a \in \mathcal{A}$. γ is a grading operator on $\mathcal{H}$, commuting with $\mathcal{A}$ and anticommuting with D.
An even spectral triple is called θ-**summable** if tr$(e^{-tD^2}) < +\infty$. for any $t > 0$

Example Dirac triple $(C^\infty(M), L(M, S), D, \gamma)$. Here M be a closed, connected and oriented

Riemannian manifold of even dimension with a fixed spin structure, and S be the bundle of spinors on M. Denote by D the associated Dirac operator on $L(M;S)$, the L^2 completion of the space of smooth sections of the bundle S. γ is a grading operator on S.

Connes' spin manifolds theorem Let M be a closed, connected and oriented Riemannian manifold, then spin structures on M are determined by spectral triples over $C^\infty(M)$ satisfying some conditions.
So by Connes' theorem, then

Spectral triples $\Leftrightarrow$ noncommutative spin geometry.

First we review the classical Atiyah-Singer index theorem. Let M be a closed, oriented even dimensional Riemannian manifold with a fixed spin structure, and $S = S^+ \oplus S^-$ be the bundle of spinors on M. E is a complex Hermitian vector bundle on M with a unitary connection ∇^E. Denote by D (D^E) the associated (twisted) Dirac operator on $\Gamma(M; S)$ ($\Gamma(M; S \otimes E)$) . Denote by $D^{E,+}$ the restriction of D^E on $\Gamma(M; S^+ \otimes E)$. Write:

$$\widehat{A}(M, \nabla^{TM}) = \det\left(\frac{R^{TM}/4\pi i}{\sinh(R^{TM}/4\pi i)}\right)^{\frac{1}{2}};$$

$$\mathrm{ch}(E, \nabla^E) = \mathrm{tr}[\exp(-\frac{R^E}{2\pi i})].$$

Theorem (Atiyah-Singer)

$$\mathrm{Ind} D^{E,+} = \int_M \widehat{A}(M, \nabla^{TM}) \mathrm{ch}(E, \nabla^E).$$

Serre-Swan Theorem: $E \Leftrightarrow p = p^* = p^2 \in M_r(C^\infty(M))$ and $E = \mathrm{Im} p = p(M \times C^r)$.

Take the connection pd on $E = \text{Im}p$. Then ([GVF,p.340])

$$\text{Ch}(\text{Im}p, pd) = \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!} \text{Tr}_{2k}(p(dp)^{2k}).$$

Define by Getzler and Szenes :

$$\text{Ch}(p) := \text{Tr}(p) + \sum_{k \geq 1} \frac{(-1)^k (2k)!}{k!} \text{Tr}_{2k}((p - \frac{1}{2})(dp)^{2k}).$$

the factor $p - \frac{1}{2}$ replacing p leads to $\text{Ch}(p)$ being $(B+b)$-closed ([GVF,p.447]).

Definition Let (A, H, D, γ) be an even θ-summable spectral triple associated to a Banach algebra A with identity, then its Chern character $\text{ch}_*(A, H, D, \gamma) = \{\text{ch}_k(D) |\ k \geq 0 \text{ and even}\}$ represented by the JLO cocycle in the entire cyclic cohomology is defined by

$$\text{ch}_k(D)(f^0, \cdots, f^k) = \int_{\triangle_k} \text{str}(f^0 e^{-s_0 D^2} [D, f^1]$$

$$e^{-s_1D^2}[D,f^2]\cdots e^{-s_{k-1}D^2}[D,f^k]e^{-s_kD^2})ds,$$

where $f^i\in A$ and $\triangle_k=\{(s_0,\cdots,s_k)|\ s_0+\cdots s_k=1\}$.

By Getzler symbol calculus (CMP 92 (1983), 163-178) or Yanlin Yu' Clifford asymptotics, we get

$$\mathrm{lim}_{t\to 0}\mathrm{ch}_k(\sqrt{t}D)(f^0,\cdots,f^k)$$

$$=\int_M\widehat{A}(M,\nabla^{TM})f^0df^1\wedge\cdots\wedge df^k.$$

By Mckean-Singer formula, When $k=0$, $f^0=1$, then we get the Atiyah-Singer index formula in untwisted case. Direct computations shows that

$$\mathrm{lim}_{t\to 0}\langle\mathrm{ch}(\sqrt{t}D),\mathrm{ch}p\rangle=\int_M\widehat{A}(M,\nabla^{TM})\mathrm{Ch}(\mathrm{Im}p,pd).$$

Define:

$$\langle D,p\rangle=\mathrm{Ind}[p(D^+\otimes I_r)p:$$

$$p(L^2(M,S^+)\otimes C^r)\to p(L^2(M,S^+)\otimes C^r)],$$

then

$$\mathrm{Ind}D^{E,+}=\langle D,p\rangle.$$

· **The homotopy invariance of** $\mathrm{ch}(\sqrt{t}D)$

Given a Banach algebra with unit A, let $C^n(A)$ be the space of continuous $(n+1)$-linear forms on A. Define operators

$b:\ C^n(A)\to C^{n+1}(A)$; $B:\ C^{n+1}(A)\to C^n(A)$,

$$b\phi^n(a_0,\cdots,a_{n+1})$$

$$=\sum_{0\leq j\leq n}(-1)^j\phi^n(a_0,\cdots,a_ja_{j+1},\cdots,a_{n+1})$$

$$+(-1)^{n+1}\phi^n(a_{n+1}a_0,\cdots,a_n);$$

$$B\phi^{n+1}(a_0,\cdots,a_n)$$

$$= \sum_{0\le j\le n} (-1)^{nj}[\phi^{n+1}(1, a_{n-j+1}, \cdots, a_n, a_0, \cdots, a_{n-j})$$

$$+(-1)^n \phi^{n+1}(a_{n-j+1}, \cdots, a_n, a_0, \cdots, a_{n-j}, 1)].$$

Then

$$b^2 = B^2 = bB + Bb = 0$$

Lemma (Getzler-Szenes)
i) Chern-connes character is $b - B$ closed,

$$b\mathrm{ch}_k(D) + B\mathrm{ch}_{k+2}(D) = 0.$$

ii) The transgression formula

$$-\frac{d}{dt}\mathrm{ch}_k(tD) = b\mathrm{ch}_{k-1}(tD, D) + B\mathrm{ch}_{k-1}(tD, D),$$

where

$$\mathrm{ch}_k(tD, D)(a^0, \cdots, a^k) := t^k \sum_{i=0}^{k} (-1)^i$$

$$\cdot\langle a_0, [D, a_1], \cdots, [D, a_i], D, [D, a_{i+1}], \cdots, [D, a_k]\rangle_t,$$

$$\langle A_0, \cdots, A_n\rangle_t = \int_{\triangle_n} \mathrm{Tr}(A_0 e^{-s_0 t^2 D^2} A_1$$

$$\cdot e^{-s_1 t^2 D^2} \cdots A_n e^{-s_n t^2 D^2}) ds.$$

By Atiyah-Singer index theorem and the homotopy invariance of $\mathrm{ch}(\sqrt{t}D)$, then we get

Theorem (Getzler-Szenes)

$$\langle D, p \rangle = \langle \mathrm{ch}(D), \mathrm{ch} p \rangle.$$

Remark 1 This theorem holds for any θ-summable spectral triple. In fact, in the general case, Getzler-Szenes just use the homotopy invariance of $\mathrm{ch}(\sqrt{t}D)$ in order to prove the above theorem. Getzler give another proof of the above theorem for the Dirac case using superconnection (Contemp. Math. 148 (1993), 19-45)).

Remark 2 Since the eta invariant is well defined just for Dirac operators (more generally for elliptic self adjoint operators), but for general selfadjoint traceclass operators, eta invariant is not well defined and we can just define the truncated eta invariant (see Getzler, Topology,1993). The reason is that in order to prove the regularity of eta invariant at zero, we need use heat kernel method (Bismut-Freed). In our NC APS index theorem, a generalization of eta invariants on the boundary are naturally considered, so we need heat kernel method.

even case	odd case
Ind(pD^+p)	$\eta(pDp)$
$\{\mathrm{ch}_{2k}(D)\}$	$\{\eta_{2k}(D)\}$
$\lim_{t\to 0}\mathrm{ch}(\sqrt{t}D)$	regularity of $\eta_{2k}(D)$
entire cyclic cohomology	radius of convergence
$\langle \mathrm{ch}(D), \mathrm{ch}p\rangle$	$\langle \eta_\star(D), \mathrm{ch}p\rangle$

3. The Atiyah-Patodi-Singer index theorem proved by Bismut-Cheeger

Let M be a smooth connected compact manifold with smooth compact boundary N. Assume M has even dimension, is oriented and spin. Let

$$C_1(N) = N \times (0,1]; \quad Z = M \cup_{N\times\{1\}} C_1(N),$$

and $\mathcal{U} \approx N \times [1,2)$ be a collar neighborhood of N in M. For $\varepsilon > 0$, we take a metric g^ε of Z such that on $\mathcal{U} \cup_{N\times\{1\}} C_1(N)$

$$g^\varepsilon = \frac{dr^2}{\varepsilon} + r^2 g^N.$$

Let $S = S^+ \oplus S^-$ be spinors bundle associated to (Z, g^ε) which is an extension of spinors bundle on M. Let E be a Hermitian complex vector bundle on M with unitary connection ∇^E. Assume that the metric and the connection on E are splitting on $\mathcal{U}$. Then we have an extension of E on Z. Let H^∞ be the set of $\xi \in \Gamma(Z, S\otimes E)$ which satisfying ξ and its derivatives are zero near the vertex of cone. Denote by $L^2_c(Z, S\otimes E)$ the L^2-completion of H^∞ (similar define $L^2_c(Z, S^+\otimes E)$ and $L^2_c(Z, S^-\otimes E)$). Let

$$D_{\varepsilon,E}: \; H^\infty \to H^\infty; \quad D_{+,\varepsilon,E}: \; H^\infty_+ \to H^\infty_-,$$

be the Dirac operators associated to (Z, g^ε).

Cheeger-chow $D^+_{\varepsilon,E}$ is a Fredholm operator for the sufficient small ε. and $\mathrm{Ind}(D^+_{\varepsilon,E})$ is independent of ε.

Bismut and Cheeger proved that

Theorem (Atiyah-Patodi-Singer)

$$\mathrm{Ind}(D^+_{\varepsilon,E}) = \int_M \widehat{A}(M, \nabla^{TM})\mathrm{ch}(E, \nabla^E) - \bar{\eta}(D^E_N),$$

where D_N^E is the twisted Dirac operator on boundary N and

$$\bar{\eta}(D_N^E) = \frac{1}{2}[\mathrm{dimker} D_N^E + \eta(D_N^E)];$$

$$\eta(D_N^E) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{t^{\frac{1}{2}}} \mathrm{tr}(D_N^E e^{-tD_N^{E\,2}}) dt.$$

4. The equivariant Atiyah-Patodi-Singer index theorem

Suppose that G is a compact connected Lie group acting on M by orientation-preserving isometries and has a product action near the boundary. E is a G-vector bundle. Assume that G has an extended action on Z and for any $g \in G$, there are lifts of g:

$$g_1: \ L^2(N, S_N \otimes E|_N) \to L^2(N, S_N \otimes E|_N));$$

$$g_2: \ L_c^2(Z, S \otimes E) \to L_c^2(Z, S \otimes E),$$

which commute with D_N and $D_{E,\varepsilon}$ respectively. Define for sufficient ε

$$\mathrm{Ind}_g D^+_{E,\varepsilon} = \mathrm{tr} g|_{\ker D^+_{E,\varepsilon}} - \mathrm{tr} g|_{\ker D^-_{E,\varepsilon}}.$$

Theorem (Donnelly)

$$\mathrm{Ind}_g D^+_{E,\varepsilon} = \sum_{i=1}^{k} \int_{F_i} \widehat{A}(TF_i) \times$$

$$\left\{ \mathrm{Pf}\left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \mathrm{ch}_g(E)$$

$$-\bar{\eta}_g(D_N),$$

where $\{F_1, \cdots, F_k\}$ are components of the fixed point set of g acting on M. Ω is the curvature matrix of the normal bundle $N(F_i)$ and θ is a rotation matrix on $N(F_i)$.

$$\mathrm{ch}_g(E, \nabla^E) = \mathrm{tr}[g j^\star \exp(-\frac{R^E}{2\pi i})]; j : F_i \to M;$$

$$\bar{\eta}_g(D_N^E) = \frac{1}{2}[\mathrm{tr}(g|\mathrm{ker} D_N^E) + \eta_g(D_N^E)];$$

$$\eta_g(D_N^E) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{t^{\frac{1}{2}}} \mathrm{tr}(g D_N^E e^{-t {D_N^E}^2}) dt.$$

5. The equivariant noncommutative Atiyah-Patodi-Singer index theorem

The fundamental setup is the same as the Donnelly index theorem.

The equivariant eta cochains

For $(C^\infty(N), L^2(N, S), D, G, \gamma)$ which is an equivariant spetral triple (for definition, see S. Klimek and A. Lesniewski K-theory 1991 or Chern and Hu Michigan J. Math 1997), we can define the equivariant cochain $\widetilde{\mathrm{ch}}_k^G(tD,D)$ (k is even) by the formula:

$$\widetilde{\mathrm{ch}}_k^G(tD,D)(f^0,\cdots,f^k)(g):=t^k\sum_{i=0}^{k}(-1)^i$$

$$\times\langle f^0,c(df^1),\cdots,c(df^i),D,c(df^{i+1}),\cdots,c(df^k)\rangle_t(g),$$

where $f^0,\cdots,f^k\in C^\infty(N)$, $g\in G$. If A_i $(0\leq i\leq n)$ are operators on H, we define:

$$\langle A_0,\cdots,A_n\rangle_t(g)=\int_{\triangle_n}\mathrm{Tr}(A_0e^{-t^2s_1D^2}A_1$$

$$\cdot e^{-t^2(s_2-s_1)D^2}\cdots A_ne^{-t^2(1-s_n)D^2}g)ds,$$

where $\triangle_n=\{(s_1,\cdots,s_n)|\ 0\leq s_1\leq\cdots\leq s_n\leq 1\}$ is the simplex in $\mathbf{R^n}$.

Formally, the equivariant total η-invariant of the Dirac operator D is defined to be a sequence of even equivariant cochains on $C^\infty(N)$, by the formula:

$$\eta_k^G(D) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \widetilde{\mathrm{ch}_k^G}(tD, D)dt,$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Then $\eta_0^G(D)(1)(g)$ is the half of the equivariant Atiyah-Patodi-Singer eta invariants defined by Donnelly and Zhang. In order to prove that the above definition is well defined, it is necessary to check the integrality near the two ends of the integration, i.e.

Fact I) The regularity of the equivariant total eta invariant

For the idempotent $p \in \mathcal{M}_r(C^\infty(N))$, Let

$$||dp|| = ||[D, p]|| = \sum_{i,j} ||dp_{i,j}||,$$

where $p_{i,j}$ $(1 \leq i, j \leq r)$ is the entry of p.

Fact II) *Suppose that D is invertible with λ the smallest positive eigenvalue of D and $||dp|| < \lambda$, then the pairing $\langle \eta^G(D), Ch(p)\rangle$ is well-defined.*

Let $C_G^\infty(N)$ is the set

$$\{f \in C^\infty(N) | f(g.x) = f(x), \forall g \in G \ x \in N\}.$$

If the above $p \in \mathcal{M}_r(C_G^\infty(N))$ and the lift of g commutes with D for any $g \in G$, then we have

Fact III)

$$\frac{1}{2}\eta^G(p(D \otimes I_r)p) = \langle \eta^G(D), \mathrm{Ch}(p)\rangle,$$

where the left term is the equivariant Atiyah-Patodi-Singer eta invariant.

Let $C_G^\infty(M) = \{f \in C^\infty(M) | f$ is independent of the normal coordinate x_n near the boundary and $f|_N \in C^\infty_G(N)\}$.

Definition The equivariant Chern-Connes character on M, $\tau^G = \{\tau^G_0, \tau^G_2, \cdots, \tau^G_{2q} \cdots\}$ is defined by

$$\tau^G_{2q}(f^0, f^1, \cdot, f^{2q})(g)$$

$$:= -\eta^G_{2q}(D_N)(f^0|_N, f^1|_N, \cdot, f^{2q}|_N)(g)$$

$$+\frac{1}{(2q)!(2\pi\sqrt{-1})^q} \sum_{i=1}^{k} \int_{F_i} \widehat{A}(TF_i)$$

$$\cdot \left\{ \mathrm{Pf} \left[2\mathrm{sinh}(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1}$$

$$\cdot f^0 df^1 \wedge \cdots \wedge df^{2q}.$$

$(\mathbf{H_1})$ Assume that the boundary Dirac operator D_N is invertible and $p = p^* = p^2 \in \mathcal{M}_r(C^\infty_G(M)) \subset \mathcal{M}_r(C^\infty(Z))$ such that $||d(p|_N)|| <$

λ, where λ is the smallest positive eigenvalue of D_N.

Consider $D^+_{p,\varepsilon} := p(D^+_\varepsilon \otimes I_r)p$:

$$p(L^2_c(Z, S^+) \otimes \mathbf{C^r}) \to p(L^2_c(Z, S^-) \otimes \mathbf{C^r}),$$

which is the Dirac operator with the coefficient from G-vector bundle $p(\mathbf{C^r})$ over Z. We also assume that

$(\mathbf{H_2})$ For any $g \in G$, there are lifts of g:

$$g_1 : L^2(N, S_N \otimes \mathrm{Im}(p|_N)) \to L^2(N, S_N \otimes \mathrm{Im}(p|_N));$$

$$g_2 : L^2_c(Z, S \otimes \mathrm{Im}(p)) \to L^2_c(Z, S \otimes \mathrm{Im}(p)),$$

which commute with D_N and $D_{p,\varepsilon}$ respectively.

By $(\mathbf{H_1})$, then $D_{N,p|_N} = p|_N(D_N \otimes I_r)p|_N$:

$$L^2(N, S_N \otimes \mathrm{Im}(p|_N)) \to L^2(N, S_N \otimes \mathrm{Im}(p|_N))$$

is invertible. By Donelly-Zhang index theorem and fact I) II) III), we have

Theorem 1 *Under the assumption* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*, then*

$$\text{Ind}_g D^+_{p,\varepsilon} = \sum_{r=0}^{m} \sum_{i=1}^{k} \frac{(-1)^r}{r!(2\pi\sqrt{-1})^r} \int_{F_i} \widehat{A}(TF_i)$$

$$\cdot \left\{ \text{Pf} \left[2\text{sinh}(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \text{Tr}[p(dp)^{2r}]$$

$$-\langle \eta^G(D_N)(g), \text{Ch}(p) \rangle,$$

when $g = \text{Id}$,we get the Wu' theorem.

Theorem 2 *Suppose that* g *acting on* N *has no fixed points. Under the assumption* $(\mathbf{H_1})$

and $(\mathbf{H_2})$, *then*

$$\mathrm{Ind}_g D^+_{p,\varepsilon} = \langle \tau^G(D), \mathrm{Ch}(p)\rangle(g).$$

6. The regularity of the equivariant total eta invariant

In the following, we will give the idea of the proof of the facts I),II),III). In the non-equivariant case, Wu proved the regularity of the total eta invariant by Getzler symbol calculus. Since g is not a pseudodifferential operator, we use Clifford asymptotics of Yu in order to prove the regularity of the equivariant total eta invariant. This may be seen as the odd analogy of the Chern-Hu theorem or a generalization of the theorem of Zhang.

S. Chern and X. Hu, *Equivariant Chern character for the invariant Dirac operators*, Michigan

Math. J. 44 (1997), 451-473.
H. Feng, *A note on the noncommutative Chern character (in Chinese)*, Acta Math. Sinica 46 (2003), 57-64.
W. P. Zhang, *A note on equivariant eta invariants*, Proc. AMS. 108 (1990), 1121-1129.

The idea of the proof of the fact I)

a) The regularity at infinity. Since g is a bounded operator, this comes from a Connes-Moscovici' lemma in nonequivariant case (CMP 155 (1993), 103-122.)

Lemma

$\widetilde{\mathrm{ch}}_k^G(tD, D)(f^0, \cdots, f^k)(g) = O(t^{-2})$, as $t \to \infty$.

b) The regularity at zero. This comes from the estimate

$$\widetilde{\mathrm{ch}}_k^G(tD, D)(f^0, \cdots, f^k)(g) \sim O(t); \quad i.e.$$

$$\widetilde{\mathrm{ch}}_k^G(\sqrt{t}D, D)(f^0, \cdots, f^k)(g) \sim O(t^{\frac{1}{2}}).$$

The $i-$th term of $\widetilde{\mathrm{ch}}_k^G(\sqrt{t}D, D)(f^0, \cdots, f^k)(g)$ up to sign for $i = 0, \cdots, k$ is:

$$t^{\frac{k}{2}} \langle f^0, c(df^1), \cdots, c(df^i), D, c(df^{i+1}), \cdots, c(df^k) \rangle_{\sqrt{t}}(g)$$

$$= t^{\frac{k}{2}} \int_{\triangle_k} \mathrm{Tr}[f^0 e^{-s_1 tD^2} c(df^1) e^{-(s_2 - s_1)tD^2}$$

$$\cdot c(df^2) \cdots c(df^i)(s_{i+1} - s_i) D e^{-(s_{i+1} - s_i)tD^2}$$

$$\cdot c(df^{i+1}) \cdots c(df^k) e^{-(1-s_k)tD^2} g] ds_1 \cdots ds_k.$$

Let B be an operator and l be a positive integer. Write

$$B^{[l]} = [D^2, B^{[l-1]}], \ B^{[0]} = B.$$

Lemma A.1 (Chern-Hu; Feng) *Let B a finite order differential operator, then for any $s > 0$, we have:*

$$e^{-sD^2}B = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l B^{[l]} e^{-sD^2} + (-1)^N s^N B^{[N]}(s),$$

where $B^{[N]}(s)$ is given by

$$B^{[N]}(s) = \int_{\triangle_N} e^{-u_1 sD^2} B^{[N]} e^{-(1-u_1)sD^2} du_1 du_2 \cdots du_N$$

Lemma A.2 *Let B a finite order differential operator, then for any $s > 0$, we have:*

$$Be^{-sD^2} = \sum_{l=0}^{N-1} \frac{1}{l!} s^l e^{-sD^2} B^{[l]} + s^N B_1^{[N]}(s),$$

where $B_1^{[N]}(s)$ is given by

$$B_1^{[N]}(s) = \int_{\triangle_N} e^{-(1-u_1)sD^2} B^{[N]} e^{-u_1 sD^2} du_1 du_2 \cdots du_N$$

Write

$$D_i^\lambda = [c(df^{i+1})]^{[\lambda_{i+1}]} \cdots [c(df^k)]^{[\lambda_k]} g$$

$$\cdot f^0 [c(df^1)]^{[\lambda_1]} \cdots [c(df^i)]^{[\lambda_i]}.$$

Using the lemmas A.1, A.2 and the property of trace, then we have:

Proposition

$$\widetilde{\mathrm{ch}}_k^G(\sqrt{t}D, D)(f^0, \cdots, f^k)(g)$$

$$= \sum_{i=0}^{k} (-1)^i \sum_{0 \leq \lambda_1, \cdots, \lambda_k \leq N-1} \frac{(-1)^{|\lambda|} C_0 t^{|\lambda| + \frac{k}{2}}}{\lambda!}$$

$$\cdot \mathrm{Tr}\{D_i^\lambda D e^{-tD^2}\} + \sum_{i=0}^{k} (-1)^i \int_{\triangle_k} (A_1^i + A_2^i) ds,$$

where C_0 is a constant.

By functional analysis technique and the Weyl asymptotics on the heat kernel, we can prove when $t \to 0^+$,

$$\int_{\triangle_k} A_2^i ds \sim O(t^{\frac{1}{2}}); \quad \int_{\triangle_k} A_1^i ds \sim O(t^{\frac{1}{2}}).$$

So we have

Proposition 1) If $k \leq \mathrm{dim} N + 1$, then when $t \to 0^+$, we have:

$$\widetilde{\mathrm{ch}}_k^G(\sqrt{t}D, D)(f^0, \cdots, f^k)(g)$$

$$= \sum_{i=0}^{k} (-1)^i \sum_{0 \leq \lambda_1, \cdots, \lambda_k \leq N-1} \frac{(-1)^{|\lambda|} C_0 t^{|\lambda| + \frac{k}{2}}}{\lambda!}$$

$$\cdot \mathrm{Tr}\{D_i^\lambda D e^{-tD^2}\} + O(t^{\frac{1}{2}}).$$

2) If $k > \mathrm{dim} N + 1$, then when $t \to 0^+$, we have:

$$\widetilde{\mathrm{ch}}_k^G(\sqrt{t}D, D)(f^0, \cdots, f^k)(g) \sim O(t^{\frac{1}{2}}).$$

By Clifford asymptotics for heat kernel, we can prove that when $t \to 0$,

$$t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}\{D_i^{\lambda} D e^{-tD^2}\} \sim O(t^{\frac{1}{2}}).$$

So we prove the regularity of the equivariant eta cochains.

The proof of the fact II) comes from the following lemma. Let $C^1(N)$ be Banach algebra of once differentiable function on N with the norm

$$||f||_1 := \mathrm{sup}_{x\in N}|f(x)| + \mathrm{sup}_{x\in N}||df(x)||.$$

Let

$$\phi^G = \{\phi_0^G, \cdots, \phi_{2q}^G, \cdots\}$$

be an equivariant even cochains sequence in the bar complex of $C^1(N)$, then

$$||\phi_{2q}^G|| = \mathrm{sup}_{||f_i||\leq 1;\ 0\leq i\leq 2q}\{||\phi_{2q}^G(f_0, \cdots, f_{2q})||_{C(G)}\}.$$

Definition The radius of convergence of ϕ^G is defined to be that of the power series $\sum q!||\phi^G_{2q}||z^q$. The space of cochains sequence with radius of convergence at least $r > 0$ is denoted by $C_r^{\text{even},G}(C^1(N))$.

In general, the sequence

$$\eta^G(D) = \{\cdots, \eta^G_{2q}(D), \eta^G_{2q+2}(D), \cdots\}$$

is not an entire cochain.

Lemma *Suppose that D is invertible with λ the smallest positive eigenvalue of D. Then the equivariant total eta invariant $\eta^G(D)$ has radius of convergence r satisfying the inequality: $r \geq 4\lambda^2 > 0$ i.e. $\eta^G(D) \in C_{4\lambda^2}^{\text{even},G}(C^1(N))$.*

The proof of this lemma is the same as non equivariant case which given by Wu.

The proof of the fact III) follows the Getzler' superconnection proof in the non equivariant case.

Thanks!