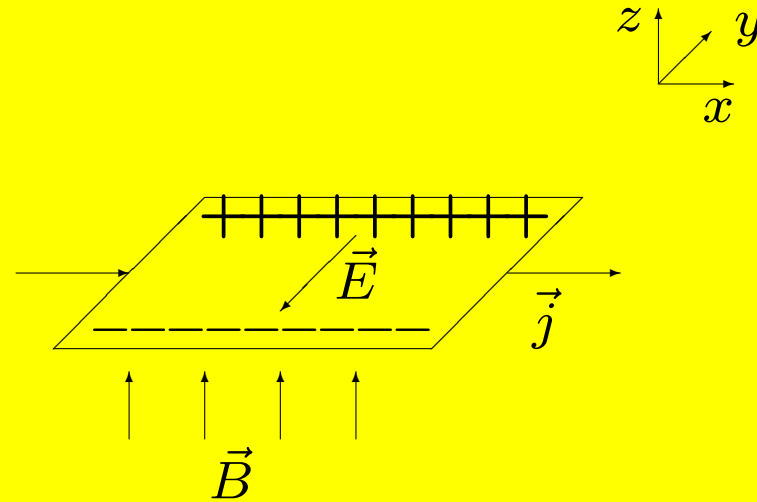


Noncommutative geometry and the fractional quantum Hall effect: higher index theory



Mathai Varghese
School of Mathematical Sciences
University of Adelaide
AUSTRALIA

Type I & Type II Index Theorems

The hyperbolic orbifolds $\Sigma(g; \nu_1, \dots, \nu_n)$ are **good** orbifolds, that is they can be presented as a quotient of a smooth manifold M by a infinitesimally free, proper action by a locally compact Lie group G . However, there are usually many such presentations, and therefore many noncommutative manifolds associated to the same orbifold. However, we saw yesterday that fortunately, they all give rise to Morita equivalent noncommutative manifolds. e.g.

$$\Sigma(g; \nu_1, \dots, \nu_n) = \Gamma \backslash \mathbb{H} = G \backslash \Sigma_{g'} = SO(2) \backslash P(g; \nu_1, \dots, \nu_n),$$

then the C^* algebras

$$C(\Sigma_{g'}) \rtimes G, \quad C^*(\Sigma(g, \nu_1, \dots, \nu_n)), \quad C_0(\mathbb{H}) \rtimes \Gamma$$

are Morita equivalent to each other, and so have isomorphic K-theory.

Proposition. [Farsi] The K -theory of the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$, viewed as a noncommutative manifold, is given by

$$\begin{aligned} K_0(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) &\cong \mathbb{Z}^{2-n+\sum_{j=1}^n \nu_j} \\ K_1(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) &\cong \mathbb{Z}^{2g} \end{aligned}$$

The analytic twisted Baum-Connes map

The Γ -invariant Levi-Civita connection on \mathbb{H} determines a Γ -invariant connection $\nabla^{\mathcal{S}}$ on spinors on \mathbb{H} ,

$$\nabla^{\mathcal{S}} : C^\infty(\mathbb{H}, \mathcal{S}^+) \rightarrow C^\infty(\mathbb{H}, T^*\mathbb{H} \otimes \mathcal{S}^+)$$

One then gets a connection

$$\nabla^{\mathcal{S} \otimes E} : C^\infty(\mathbb{H}, \mathcal{S}^+ \otimes E) \rightarrow C^\infty(\mathbb{H}, T^*\mathbb{H} \otimes \mathcal{S}^+ \otimes E), \quad (1)$$

defined as $\nabla^{\mathcal{S} \otimes E} = \nabla^{\mathcal{S}} \otimes 1 + 1 \otimes \nabla^E$, where ∇^E is a Γ -invariant connection on the Γ -invariant vector bundle E over \mathbb{H} . Now the contraction given by Clifford multiplication defines a map

$$C : C^\infty(\mathbb{H}, T^*\mathbb{H} \otimes \mathcal{S}^+ \otimes E) \rightarrow C^\infty(\mathbb{H}, \mathcal{S}^- \otimes E). \quad (2)$$

The Γ -equivariant Dirac operator with coefficients in E is defined as the composition $\not{D}_E^+ = C \circ \nabla^{\mathcal{S} \otimes E}$,

$$\not{D}_E^+ : L^2(\mathbb{H}, \mathcal{S}^+ \otimes E) \rightarrow L^2(\mathbb{H}, \mathcal{S}^- \otimes E).$$

Note that B is *exact*, $B = d\eta$, since \mathbb{H} is contractible. Define $\nabla = d + i\eta$. Then ∇ is a Hermitian connection on the trivial line bundle \mathcal{L} over \mathbb{H} , and the curvature of ∇ is $(\nabla)^2 = iB$. Then ∇ defines a projective action of Γ on L^2 spinors as follows:

For $u \in L^2(\mathbb{H}, \mathcal{S} \otimes E \otimes \mathcal{L})$, let $S_\gamma u = e^{i\varphi_\gamma} u$ (where φ is the phase for σ as explained earlier at least in the scalar case), $U_\gamma u = \gamma^{-1*} u$, and $T_\gamma = U_\gamma S_\gamma$ be the composition, for all $\gamma \in \Gamma$. Then T defines a projective (Γ, σ) -action on $L^2(\mathbb{H}, \mathcal{S} \otimes E \otimes \mathcal{L})$, meaning that for any $\gamma, \gamma' \in \Gamma$ one has

$$T_\gamma T_{\gamma'} = \sigma(\gamma, \gamma') T_{\gamma\gamma'}.$$

We can further twist the Dirac operator $\not{\partial}_E^+$ by the connection ∇ , which is denoted

$$\not{\partial}_{E \otimes \mathcal{L}}^+ : L^2(\mathbb{H}, \mathcal{S}^+ \otimes E \otimes \mathcal{L}) \rightarrow L^2(\mathbb{H}, \mathcal{S}^- \otimes E \otimes \mathcal{L})$$

Lemma. *The twisted Γ -equivariant Dirac operator on \mathbb{H} ,*

$$\not\partial_{E \otimes \mathcal{L}}^+ : L^2(\mathbb{H}, \mathcal{S}^+ \otimes E \otimes \mathcal{L}) \rightarrow L^2(\mathbb{H}, \mathcal{S}^- \otimes E \otimes \mathcal{L}),$$

commutes with the projective (Γ, σ) -action.

Proof. To simplify notation, set $D_\eta = \not\partial_{E \otimes \mathcal{L}}^+$ and $D_0 = \not\partial_E^+$ where we emphasize the dependence on η . Then $D_\eta = D_0 + ic(\eta)$, where $c(\eta)$ denotes Clifford multiplication by the one-form η . An easy computation establishes that $U_\gamma D_\eta = D_{\gamma^{-1}^* \eta} U_\gamma$ and that $S_\gamma D_{\gamma^{-1}^* \eta} = D_\eta S_\gamma$ for all $\gamma \in \Gamma$. Then $T_\gamma D_\eta = D_\eta T_\gamma$, where $T_\gamma = U_\gamma S_\gamma$ denotes the projective (Γ, σ) -action. $\diamond \quad \square$

Heat kernels and the analytic index The following well-known smoothness properties and Gaussian off-diagonal estimates for the heat kernel, cf. [Greiner]

Lemma. *The Schwartz kernels $k_{\pm}(t, x, y)$ of the heat operators $e^{-tD^{\pm}D^{\mp}}$ are smooth for all $t > 0$. Moreover, for any $t > 0$ there are positive constants C_1, C_2 such that the following off-diagonal estimate holds*

$$|k_{\pm}(t, x, y)| \leq C_1 e^{-C_2 d(x,y)^2}, \quad x \in \mathbb{H}, \quad y \in \mathbb{H},$$

where d denotes the Riemannian distance function on \mathbb{H} . Also a similar estimate holds for all derivatives of the Schwartz kernels $k_{\pm}(t, x, y)$.

For fixed $t > 0$, we will use the Lemma above to show the following.

Corollary *For fixed $t > 0$, the heat operators $e^{-tD^{-}D^{+}}$ and $e^{-tD^{+}D^{-}}$ belong to $\mathcal{R}(\Gamma, \sigma) \otimes \Psi^{-\infty}_{+}$ and $\mathcal{R}(\Gamma, \sigma) \otimes \Psi^{-\infty}_{-}$ respectively, where $\Psi^{-\infty}_{\pm}$ denotes the algebra of smoothing operators on the Hilbert space $\mathcal{H}_{\pm} = L^2(\tilde{\mathcal{F}}, \mathcal{S}^{\pm} \otimes E|_{\mathcal{F}})$, and \mathcal{F} denotes a connected fundamental domain of the action of Γ on \mathbb{H} .*

Definition. For fixed $t > 0$, define the idempotent

$$e_t(D^+) \in M_2(\mathcal{R}(\Gamma, \sigma) \otimes \widetilde{\Psi}^{-\infty})$$

as follows:

$$e_t(D^+) = \begin{pmatrix} e^{-tD^-D^+} & e^{-\frac{t}{2}D^-D^+} \frac{(1 - e^{-tD^-D^+})}{D^-D^+} D^+ \\ e^{-\frac{t}{2}D^+D^-} D^+ & 1 - e^{-tD^+D^-} \end{pmatrix},$$

where $\mathcal{R}(\Gamma, \sigma) \otimes \widetilde{\Psi}^{-\infty}$ denotes the unital algebra associated with $\mathcal{R}(\Gamma, \sigma) \otimes \Psi^{-\infty}$. It is the twisted analogue of the **Wasserman idempotent**.

Exercise. Verify that this is an idempotent!

Since K -theory is invariant under stable isomorphism, and using the fact that the algebra $\mathcal{R}(\Gamma, \sigma) \otimes \Psi^{-\infty}$ is closed under the holomorphic functional calculus, we see that

$$K_{\bullet}(\mathcal{R}(\Gamma, \sigma)) \cong K_{\bullet}(\mathcal{R}(\Gamma, \sigma) \otimes \Psi^{-\infty}) \cong K_{\bullet}(C_r^*(\Gamma, \sigma)).$$

Using this isomorphism, the **twisted analytic index** is defined as

$$\text{index}_{\sigma}^a(D^+) = [e_t(D^+)] - [E_0] \in K_0(C_r^*(\Gamma, \sigma)),$$

where $t > 0$ and E_0 is the idempotent

$$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathcal{R}(\Gamma, \sigma) \otimes \Psi^{-\infty}).$$

Since the difference $e_t(D^+) - E_0$ is in $M_2(C_r^*(\Gamma, \sigma) \otimes \mathcal{K})$, we see that the right hand side of equation is in $K_0(C_r^*(\Gamma, \sigma))$ as asserted.

Unravelling the notation, we get a homomorphism,

$$\mu_\sigma : K_0(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) \rightarrow K_0(C_r^*(\Gamma, \sigma))$$

called the **twisted Baum-Connes map**, and which is defined by

$$\mu_\sigma([E]) = \text{index}_\sigma^a(\not{D}_{E \otimes \mathcal{L}}^+)$$

where we have used the identification,

$$K_0(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) \cong K_\Gamma^0(P(g, \nu_1, \dots, \nu_n)).$$

Next time, I'll explain why the map μ_σ is an isomorphism. Assuming for now that this true, we calculate the range of the trace on the K -theory of the twisted group C^* -algebra (here $D^\pm = \not{D}_{E \otimes \mathcal{L}}^\pm$)

$$[\text{tr}](\mu_\sigma([E])) = \text{tr}(e^{-tD^-D^+}) - \text{tr}(e^{-tD^+D^-})$$

Taking the limit on the RHS as $t \rightarrow 0$, and using the local index theorem,

$$\begin{aligned} [\text{tr}](\mu_\sigma([E])) &= \frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} \text{ch}(E) e^B \\ &= \frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} c_1(E) + \text{rank}(E) \frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} B \end{aligned}$$

This is well defined since the integrand is Γ -invariant.

By the index theorem for S^1 -orbifolds, due to Atiyah-Singer, we see that

$$\frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} \text{ch}(E) + \frac{1}{2\pi} \sum_{i=1}^n \beta_i / \nu_i = \text{index}(\not{D}_E^+) \in \mathbb{Z},$$

Therefore we see that

$$\frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} c_1(E) \in \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$$

But the flux θ is

$$\theta = \frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} B$$

It follows that the range of the trace map on K_0 is

$$\mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i).$$

Therefore, assuming for the moment that the twisted Baum-Connes map is an isomorphism, we obtain a proof of the the first main result that I mentioned:

Theorem [MM] The range of the trace is

$$[\text{tr}](K_0(C_r^*(\Gamma, \sigma))) = \mathbb{Z} + \theta\mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$$

where $\Gamma = \Gamma(g : \nu_1, \dots, \nu_n)$ is the Fuchsian group of signature $(g; \nu_1, \dots, \nu_n)$, σ is the multiplier obtained from the Γ -invariant 2-form B (which is the magnetic field) on \mathbb{H} , and $\theta = \int_{\Sigma(g, \nu_1, \dots, \nu_n)} B$ is the flux.

As an immediate consequence, we get

Corollary When the flux $\theta \in \mathbb{Q}$, then the Kadison constant, $C_\sigma(\Gamma)$ is positive. In fact, if $\theta = p/q$, then $C_\sigma(\Gamma) \geq \frac{1}{q \prod_{i=1}^n \nu_i}$. Therefore there can only be finitely many gaps in the spectrum of the discrete magnetic Schrodinger operator $\Delta_\sigma + V$. In any finite interval, there are only a finite number of gaps in the spectrum of Hamiltonian in the continuous model, etc, etc.

Twisted Baum-Connes isomorphism

There are many proofs of this, but I will choose the original one. For other proofs, see my paper [math.KT/0606790], which uses V. Lafforgue's technology, which is more efficient.

The original proof uses the **Wasserman-Packer-Raeburn trick**: let A be a Γ - C^* -algebra, then the twisted crossed product $A \rtimes_{\sigma,r} \Gamma$, when stabilized, becomes an ordinary crossed product, i.e.

$$A \rtimes_{\sigma,r} \Gamma \otimes \mathcal{K} \cong (A \otimes \mathcal{K}) \rtimes_r \Gamma$$

where $\mathcal{K} = \mathcal{K}(\ell^2(\Gamma))$, equipped with the action of Γ given by the Adjoint of L^σ -action (which is an ordinary action) and the diagonal action on the RHS.

The proof also uses **Dixmier-Douady theory**, namely, any algebra bundle over a space X , with fibre isomorphic to \mathcal{K} , is classified upto isomorphism by $H^3(X, \mathbb{Z})$, and more generally, with the action of a compact group K on X , it is equivariantly classified by $H_K^3(X, \mathbb{Z})$.

We apply this to the case when $A = C_0(\mathrm{PSL}(2, \mathbb{R}))$. Then by the W-P-R trick,

$$\begin{aligned} (C_0(\mathrm{PSL}(2, \mathbb{R})) \rtimes_{\sigma, r} \Gamma) \otimes \mathcal{K} &\cong (C_0(\mathrm{PSL}(2, \mathbb{R})) \otimes \mathcal{K}) \rtimes_r \Gamma \\ &= C(P(g; \nu_1, \dots, \nu_n), \mathcal{E}) \end{aligned}$$

where $\mathcal{E} = \mathrm{PSL}(2, \mathbb{R}) \times_{\Gamma} \mathcal{K}$ is a $\mathrm{SO}(2)$ -equivariant algebra bundle over the smooth manifold $P(g; \nu_1, \dots, \nu_n) = \Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$, with fibre isomorphic to \mathcal{K} . Since it is flat, by Dixmier-Douady theory, it is classified by a torsion class in $H^3(P(g; \nu_1, \dots, \nu_n), \mathbb{Z}) \cong \mathbb{Z}$, ie it has trivial DD-invariant, so

$$C(P(g; \nu_1, \dots, \nu_n), \mathcal{E}) \cong C(P(g; \nu_1, \dots, \nu_n)) \otimes \mathcal{K}$$

With a bit more work, this is also true $\mathrm{SO}(2)$ -equivariantly. Combining,

$$(C_0(\mathrm{PSL}(2, \mathbb{R})) \rtimes_{\sigma, r} \Gamma) \otimes \mathcal{K} \cong C(P(g; \nu_1, \dots, \nu_n)) \otimes \mathcal{K}.$$

Next we use the fact that the Baum-Connes conjecture, with coefficients, is true for Γ - this is due to Kasparov '83. It says in this particular instance that the assembly map

$$\begin{aligned} \mu_{\mathcal{K}} : K_{\bullet}^K(C(P(g; \nu_1, \dots, \nu_n), \mathcal{E})) &\cong K_{\bullet}(\mathcal{K} \rtimes_r \Gamma) \\ &\cong K_{\bullet}(C_r^*(\Gamma, \sigma)), \end{aligned}$$

where we have used the W-P-R trick again to see that

$$\begin{aligned} C_r^*(\Gamma, \sigma) \otimes \mathcal{K} &= (\mathbb{C} \rtimes_{\sigma, r} \Gamma) \otimes \mathcal{K} \\ &\cong \mathcal{K} \rtimes_r \Gamma. \end{aligned}$$

It will turn out that $\mu_{\mathcal{K}} = \mu_{\sigma}$, and in this way one deduces from Farsi's result that

$$\begin{aligned} K_0(C_r^*(\Gamma, \sigma)) &\cong \mathbb{Z}^{2-n + \sum_{j=1}^n \nu_j} \\ K_1(C_r^*(\Gamma, \sigma)) &\cong \mathbb{Z}^{2g} \end{aligned}$$

This completes the proof that the range of the trace on K-theory is as stated before.

Assume that $\Gamma = \Gamma_g$ i.e. Γ is a torsion-free surface group. Then since Γ is an ICC group, i.e every non-trivial conjugacy class in Γ is infinite, it follows that the twisted group von Neumann algebra is a type II_1 factor, so it has a unique normalized trace. It follows that $C_r^*(\Gamma_g, \sigma)$ has a unique normalized trace.

Proposition [CHMM][Isomorphism classification of twisted group C^* -algebras]
 $C_r^*(\Gamma_g, \sigma) \cong C_r^*(\Gamma_{g'}, \sigma')$ if and only if $g = g'$ and $\theta' \in \{\theta, 1 - \theta\}$.

Proof. Let tr and tr' denote the canonical traces on $C_r^*(\Gamma_g, \sigma)$ and $C_r^*(\Gamma_g, \sigma')$ respectively. Let

$$\phi : C_r^*(\Gamma_g, \sigma) \cong C_r^*(\Gamma_g, \sigma')$$

be an isomorphism, and let

$$\phi_* : K_0(C_r^*(\Gamma_g, \sigma)) \cong K_0(C_r^*(\Gamma_g, \sigma'))$$

denote the induced map on K_0 . But

$$[\text{tr}](K_0(C_r^*(\Gamma_g, \sigma))) = \mathbb{Z}\theta + \mathbb{Z}, \quad [\text{tr}'](K_0(C_r^*(\Gamma_g, \sigma'))) = \mathbb{Z}\theta' + \mathbb{Z}.$$

So there are elements $[P] \in K_0(C_r^*(\Gamma_g, \sigma))$ and $[P'] \in K_0(C_r^*(\Gamma_g, \sigma'))$ such that $[\text{tr}]([P]) = \theta$ and $[\text{tr}']([P']) = \theta'$. By uniqueness of the trace, one has $\text{tr} \circ \phi = \text{tr}'$, which induces the identity $[\text{tr}] \circ \phi_* = [\text{tr}']$ in $K_0(C_r^*(\Gamma_g, \sigma'))$. We have proved earlier that $K_0(C_r^*(\Gamma_g, \sigma)) \cong \mathbb{Z}^2 \cong K_0(C_r^*(\Gamma_g, \sigma'))$. In the basis above, one has

$$\phi_* : \mathbb{Z}[P] \oplus \mathbb{Z} \cong K_0(C_r^*(\Gamma_g, \sigma)) \rightarrow K_0(C_r^*(\Gamma_g, \sigma')) \cong \mathbb{Z}[P'] \oplus \mathbb{Z}.$$

Since $\phi_*[1] = [1]$ and $\phi_* \in \mathbf{GL}(2, \mathbb{Z})$, one sees that there is an integer n such that

$$\phi_* = \begin{pmatrix} 1 & n \\ 0 & \pm 1 \end{pmatrix}.$$

Assembling these results, one has $\theta = [\text{tr}]([P]) = [\text{tr}](\phi_*[P]) = [\text{tr}'](n[1] \pm [P']) = n \pm \theta'$. Since $\theta, \theta' \in (0, 1]$, one deduces that $\theta' \in \{\theta, 1 - \theta\}$.

Now let $\alpha \in \text{Aut}(\Gamma_g)$ be defined as

$$\begin{aligned} \alpha(A_j) &= B_{g+1-j} \\ \alpha(B_j) &= A_{g+1-j} \end{aligned}$$

Then α determines an isomorphism between $C^*(\Gamma_g, \sigma)$ and $C^*(\Gamma_g, \bar{\sigma})$ □

To determine the values of the conductance 2-cycle, we need to compute

$$[\mathrm{tr}_{c_\tau}](K_0(C^*(C_r^*(\Gamma, \sigma)))$$

where c_τ is the hyperbolic area cocycle on Γ , and tr_{c_τ} the associated cyclic 2-trace. By the previous isomorphism, it is sufficient to compute

$$[\mathrm{tr}_{c_\tau}](\mu_\sigma([E])) = \mathrm{tr}_{c_\tau}(e^{-tD^-D^+}, e^{-tD^-D^+}, e^{-tD^-D^+}) - \mathrm{tr}_{c_\tau}(e^{-tD^+D^-}, e^{-tD^+D^-}, e^{-tD^+D^-})$$

Taking the limit on the RHS as $t \rightarrow 0$, and using the local index theorem,

$$\begin{aligned} [\mathrm{tr}_{c_\tau}](\mu_\sigma([E])) &= \frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} \omega \wedge \mathrm{ch}(E) e^B \\ &= \frac{\mathrm{rank}(E)}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} \omega, \end{aligned}$$

where ω is the volume form on hyperbolic space, which corresponds to c_τ . This is well defined since the integrand is Γ -invariant. But $\frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} \omega$ is just the normalized orbifold volume, or the orbifold Euler characteristic. Therefore $[\mathrm{tr}_{c_\tau}](K_0(C^*(C_r^*(\Gamma, \sigma)))$ is equal to integer multiples of the orbifold Euler characteristic, which is a fraction $(2(g-1) + (n - \sum_j 1/\nu_j))$, completing the proof of the second assertion.

Some questions/conjectures:

1) Let Γ be a discrete group with bounded torsion, (for example if Γ is a torsion-free group), and suppose that σ is a **rational** multiplier on Γ , i.e. $[\sigma] \in H^2(\Gamma, \mathbb{Q}/\mathbb{Z})$. Then the Kadison constant $C_\sigma(\Gamma) > 0$ is positive.

This was proved for a wide class of groups under additional assumptions, in my paper in Tohoku Math.J. (2001)

2) Let $\Gamma = \Gamma_g$ be a surface group of genus g . Then $C_r^*(\Gamma_g, \sigma)$ is Morita equivalent to $C_r^*(\Gamma_g, \sigma')$ if and only if θ' is in the $GL(2, \mathbb{Z})$ orbit of θ . (Here θ and θ' are the flux of σ and σ' respectively)

This is true when the genus $g = 1$, and due to Rieffel.

3) Is the analog of the conjectures in Prof. A. Thom's lecture also true for the twisted group ring and rational multiplier, e.g. the Atiyah conjecture, the Bass conjecture, the analytic zero divisor conjecture?