Equivariant cohomology and localization for Lie algebroids and applications

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(w/ L. Cirio, P. Rossi, V. Roubtsov)

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(Topological supersymmetric gauge theories)

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$\Rightarrow$ equivariant cohomology $\boxed{H_G^\bullet(M)}$.

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where $L_p : T_pM \to T_pM, \qquad v \mapsto [\xi^*, v]$

(L_p is the Jacobian of ξ^* at p)

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Morphisms are defined in the obvious way

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$$\begin{aligned}(\delta\xi)(\alpha_1, \ldots, \alpha_{p+1}) = \\ &\textstyle\sum_{i=1}^{p+1} (-1)^{i-1} a(\alpha_i)(\xi(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{p+1})) \\ &+ \textstyle\sum_{i<j} (-1)^{i+j} \xi(\{\alpha_i, \alpha_j\}, \ldots, \hat{\alpha}_i, \ldots, \\ &\qquad \hat{\alpha}_j, \ldots, \alpha_{p+1})\end{aligned}$$

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$H_A^\bullet$ is called the cohomology of the Lie algebroid A.

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$$\int_M \gamma(\xi) = (-2\pi)^{n/2} \sum_{p \in M_\xi} R_{\gamma,\xi}(p)$$

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where the sum runs on the zeroes of $a(\alpha)$

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Computation done using ADHM data:
the moduli space is

$$\mathcal{M}(r,n) = \{(B_1, B_2, I, J) \in \mathrm{Mat}_{\mathbb{C}}(n,n) \times \mathrm{Mat}_{\mathbb{C}}(n,n) \times \mathrm{Mat}_{\mathbb{C}}(r,n) \times \mathrm{Mat}_{\mathbb{C}}(n,r) \text{ satisfying constraints (and stability condition)}\} \,/\, U(n)$$

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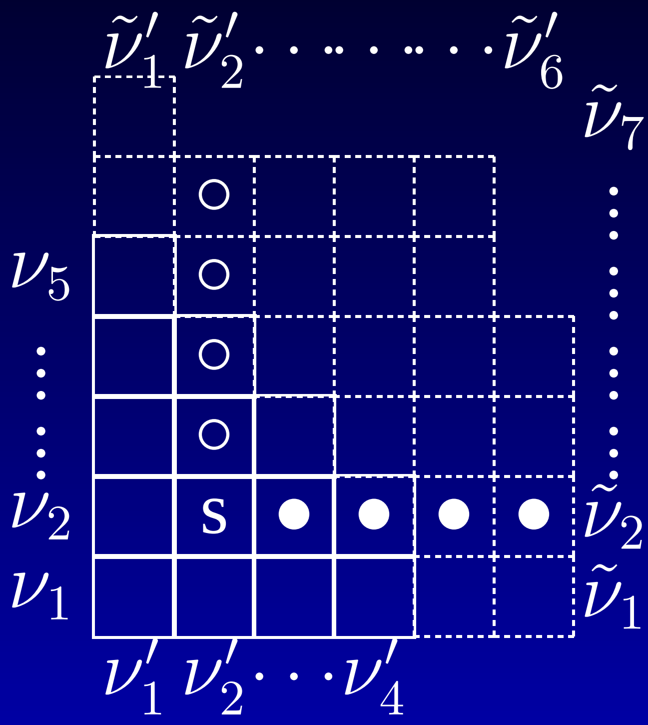
$\mathcal{M}(r,n)$ carries $U(1) \times U(1)$ action

$$(B_1, B_2, I, J) \mapsto (e^{i\varepsilon_1} B_1,\ e^{i\varepsilon_2} B_2,\ e^{i(\varepsilon_1+\varepsilon_2)} I,\ J)$$



$$\mathcal{Z}_{r,n} = \sum_{x_0} \frac{1}{\mathrm{Sdet}\hat{\mathcal{L}}_{x_0}} = \sum_{\{Y_\lambda\}} \prod_{\lambda,\tilde{\lambda}}^{N} \prod_{s \in Y_\lambda} \frac{1}{E(s)(E(s)-\epsilon)}$$

$$E(s) = a_{\lambda\tilde{\lambda}} - \epsilon_1 h(s) + \epsilon_2 (v(s)+1)$$

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