

Modular Bootstrap of $\mathcal{N} = 2$ Liouville Theory

T.Eguchi and Y.Sugawara
hep-th/0311141,040319,0411041

♠ $\mathcal{N} = 0$ Liouville Theory

Liouville theory is characterized by the exponential interaction $\mu e^{2b\phi}$ and a coupling to worldsheet curvature $\mathcal{Q}R\phi$. A typical irrational CFT ($c = 1 + 6\mathcal{Q}^2$). Difficult theory solved in late 90's using the method conformal bootstrap.

We consider a simplified version of modular bootstrap and determine boundary states/D-branes.

continuous rep.

$$\chi_p\left(-\frac{1}{\tau}\right) = \int_0^\infty dp \cos(2\pi pp') \chi_{p'}(\tau),$$
$$\chi_p(\tau) = \frac{q^{\frac{p^2}{2}}}{\eta(\tau)}$$

identity rep.

$$\chi_{h=0}\left(-\frac{1}{\tau}\right) = \int_0^\infty dp \sinh(2\pi bp) \sinh\left(\frac{2\pi p}{b}\right) \chi_p(\tau)$$
$$\mathcal{Q} = b + \frac{1}{b} : \text{background charge}$$

Identity LHS as open and RHS as closed string channel.

Spectrum:

open

closed

{ continuous rep.
identity rep.

continuous rep.

There is no identity representation in closed string channel. This is consistent with the presence of mass gap and decoupling of gravity

$$\begin{aligned} h(e^{2\alpha\phi}) &= -\alpha^2 + \alpha Q = -\left(\alpha - \frac{1}{2}Q\right)^2 + \frac{Q^2}{4} \\ &= p^2 + \frac{Q^2}{4} \geq \frac{Q^2}{4} \text{ for } \alpha = ip + \frac{1}{2}Q \end{aligned}$$

D-branes and boundary wave functions:

Zamolodchikov-Zamolodchikov
Fateev-Zamolodchikov-Zamolodchikov-Teschner

Introduce ZZ and FZZT branes and identify the characters as

$$\chi_0\left(-\frac{1}{\tau}\right) = \langle ZZ | e^{i\pi\tau H^{(c)}} | ZZ \rangle$$

$$\chi_p\left(-\frac{1}{\tau}\right) = \langle FZZT; p | e^{i\pi\tau H^{(c)}} | ZZ \rangle$$

$H^{(c)}$ denotes the closed string Hamiltonian.

Using Ishibashi states $|p\rangle\rangle$ with momentum p

$$\langle\langle p | e^{i\pi\tau H^{(c)}} | p' \rangle\rangle = \delta(p - p') \chi_p(\tau)$$

boundary states are expanded as

$$|ZZ\rangle = \int_0^\infty dp \Psi_0(p) |p\rangle\rangle$$

$$|FZZT; p\rangle = \int_0^\infty dp' \Psi_p(p') |p'\rangle\rangle$$

One finds that the disk wave-functions

$$\Psi_0(p) = \frac{2\pi i p \mu^{\frac{ip}{b}}}{\Gamma(1 + i2pb) \Gamma(1 + \frac{i2p}{b})}$$

$$\Psi_p(p') = \frac{-\mu^{\frac{ip'}{b}}}{2\pi i p'} \Gamma(1 - 2ibp') \Gamma(1 - \frac{2ip'}{b}) \cos(2\pi p p')$$

reproduce the modular transformation law.

♠ $\mathcal{N} = 2$ Liouville Theory

$\mathcal{N} = 2$ Liouville theory consists of fields $(\phi, Y; \psi^+, \psi^-)$. ϕ is the Liouville field coupled to the background charge Q while Y is a compact boson not coupled to Q . ψ^\pm are a pair of complex fermions.

$\mathcal{N} = 2$ Liouville theory is interesting from various points of view:

- Applications to string theory compactified on singular space-time, i.e. NS5 branes and singular CY manifolds.
- T-dual to $SL(2; R)/U(1)$ supercoset theory which describes 2 dim. black hole.
- Closely related to various bosonic theories with $c = 1$ matter coupled to Liouville.

Consider the case with central charge

$$c = 3(1 + Q^2) = 3\left(1 + \frac{2K}{N}\right)$$

T-dual to 2-dim black hole with an asymptotic radius of the cigar $\sqrt{2N/K}$.

Unitary representations

$$\left\{ \begin{array}{ll} \text{identity rep.} & j = 0 \\ \text{continuous rep.} & j = \frac{1}{2} + i\frac{p}{Q} \\ \text{discrete reps.} & 1 \leq s \leq N + 2K, j = \frac{s}{2K} \end{array} \right.$$

Representations are in one to one correspondence with those of $SL(2; R)/U(1)$ coset theory with level $k = N/K$.

For applications to string theory we want integral $U(1)_R$ charges. In order to impose charge integrality we consider the sum over spectral flows of irreducible characters of $\mathcal{N} = 2$ SCA

$$\chi(m; \tau, z) = \sum_{n \in m + NZ} q^{\frac{\hat{c}}{2} n^2} e^{\pi i \hat{c} z n} ch(\tau, z + n\tau)$$

(z denotes the angle coupled to the $U(1)$ charge).

We obtain extended characters

1. Identity representations :

$$\chi_{id}(m; \tau); \quad m = 2Kr, \quad r \in \mathbb{Z}_N,$$

2. Continuous representations :

$$\chi_{cont}(p, m; \tau); \quad p \geq 0, \quad m = 2Kr, \quad r \in \mathbb{Z}_N,$$

3. Discrete representations :

$$\chi_{dis}(s, r; \tau); \quad m = s + 2Kr, \quad r \in \mathbb{Z}_N, \\ 0 \leq s \leq N + 2K - 1$$

S transformations of these characters are given by

Continuous representations:

$$\begin{aligned} & \chi_{cont}(p, m; -\frac{1}{\tau}) \\ = & \sqrt{\frac{2}{NK}} \sum_{m' \in \mathbb{Z}_{2NK}} e^{-2\pi i \frac{mm'}{2NK}} \int_0^\infty dp' \cos(2\pi pp') \chi_{cont}(p', m'; \tau) \end{aligned}$$

Identity representations:

$$\begin{aligned} & \chi_{id}(m; -\frac{1}{\tau}) = \\ & \frac{1}{\sqrt{2NK}} \sum_{m' \in \mathbb{Z}_{2NK}} e^{-2\pi i \frac{mm'}{2KN}} \int_0^\infty dp' \frac{\sinh(\pi Q p') \sinh(2\pi \frac{p'}{Q})}{|\cosh \pi(\frac{p'}{Q} + i \frac{m'}{2K})|^2} \chi_{cont}(p', m'; \tau) \\ & + \frac{2}{N} \sum_{r' \in \mathbb{Z}_N} \sum_{s'=K+1}^{N+K-1} \sin\left(\frac{\pi(s' - K)}{N}\right) e^{-2\pi i \frac{m(s'+2Kr')}{2KN}} \chi_{dis}(s', r'; \tau) \end{aligned}$$

Discrete representations:

$$\begin{aligned} & \chi_{dis}(s, r; -\frac{1}{\tau}) \\ = & \frac{1}{\sqrt{2NK}} \sum_{m' \in \mathbb{Z}_{2NK}} e^{-2\pi i \frac{(s+2Kr)m'}{2NK}} e^{\frac{i\pi m'}{2K}} \int_{-\infty}^\infty dp' \frac{e^{-2\pi(\frac{s-K}{N} - \frac{1}{2})\frac{p'}{Q}}}{2 \cosh \pi(\frac{p'}{Q} + \frac{im'}{2K})} \chi_{cont}(p', m'; \tau) \\ & + \frac{i}{N} \sum_{r' \in \mathbb{Z}_N} \sum_{s'=K+1}^{N+K-1} e^{-2\pi i \frac{(s+2Kr)(s'+2Kr') - (s-K)(s'-K)}{2NK}} \chi_{dis}(s', r'; \tau) \\ & + \frac{i}{2N} \sum_{r' \in \mathbb{Z}_N} e^{-2\pi i \frac{(s+2Kr)(s'+2Kr')}{2KN}} \{ \chi_{dis}(s' = K, r'; \tau) - \chi_{dis}(s' = N + K, r'; \tau) \} \end{aligned}$$

This is a rather peculiar transformation law with

the pattern

$$(\text{continuous rep}) \xrightarrow{S} (\text{continuous rep})$$

$$(\text{discrete rep}) \xrightarrow{S} (\text{continuous rep}) + (\text{discrete rep})$$

Such a pattern was first observed in $\mathcal{N} = 4$ rep theory (**T.E.-Taormina**)

1. There are no identity reps in the RHS of above formulas.

2. Only the discrete reps in the range

$K + 1 \leq s \leq N + K - 1$ appear in the RHS. This corresponds exactly to the known spectrum

$$\frac{1}{2} \leq j \leq \frac{k+1}{2}$$

of $SL(2; R)/(U(1))$ theory.

3. It is still possible to show that $S^2 = C$.

Spectrum of the theory

• open $\left\{ \begin{array}{l} \text{identity rep.} \\ \text{continuous rep.} \\ \text{discrete reps. } 0 \leq s \leq N + 2K - 1 \end{array} \right.$

• closed $\left\{ \begin{array}{l} \text{continuous rep.} \\ \text{discrete reps. } K \leq s \leq N + K \end{array} \right.$

Identity rep. is missing in the closed string sector. Also only a part of discrete reps. appear. These appear to be general features of modular properties of irrational CFT's.

Boundary states

We have three types of boundary states of $\mathcal{N} = 2$ theory corresponding to each representation. Boundary wave functions are again given by the elements of the modular S matrix. We can compare our expressions with known results of $SL(2; R)/U(1)$ theory obtained by semi-classical method using DBI action. We can also test against conformal bootstrap calculation.

$\mathcal{N} = 2$ Liouville	$SL(2; R)/U(1)$
type A class 1 identity rep	type B D0 brane localized at tip of cigar
type B class 2' continuous rep	type A D1 brane extended along radial direction
type A class 3 discrete rep	type B D2 brane wraps the whole of cigar
type A class 2 continuous rep	type B D2 brane wraps a part of cigar

Ribault-Schomerus

Israel-Pakman-Troost
Fotopoulos-Niarchos-Prezas

There is a delicate discrepancy of class 2 wave function with that of D1 brane of $SL(2; R)/U(1)$ theory .

$$\Psi_{p,m}^{class2}(p', m') \approx f(p', m') \cos(2\pi pp')$$

$$\Psi_{p,m}^{D1}(p', m') \approx \frac{1}{2} f(p', m') (e^{2\pi i pp'} + (-1)^{m'} e^{-2\pi i pp'})$$

Class 2 does not seem to have a good semi-classical limit. On the other hand conformal bootstrap analysis reproduces class 1 and 2 wave functions (Ahn-Stanishkov-Yamamoto). We find an agreement of class 3 and D2 brane with the identification

$$\sigma = \pi \left(\frac{s-1}{k} - \frac{1}{2} \right)$$

where σ is related to the gauge field strength on D2 brane. Consistency of the overlap of two D2 branes with σ, σ' gives

$$\sigma - \sigma' = 2\pi m \frac{1}{k}$$

where m is an integer related to the D0 brane charges. This condition immediately follows from the above identification.

Geometry of $\mathcal{N} = 2$ Liouville fields:

We consider models of the following type:

Liouville theories \times minimal models

$$\prod_i L_{(N_i, K_i)} \otimes \prod_j M_{k_j}$$

These describe various singular geometries

- **NS5 brane**
- **ALE spaces**
- **ALE spaces fibered on P^1**
- **singular CY_3, CY_4 etc**

Vanishing cycles in these spaces are described by

- **Boundary state description**
ZZ brane of Liouville sectors \otimes boundary states of minimal models
- **Closed string description**
discrete states of Liouville \otimes discrete states of minimal

Elliptic genus:

$$Z(\tau; z) = \text{Tr}(-1)^{F_L+F_R} e^{2\pi i z J_o^L} q^{L_0} \bar{q}^{\bar{L}_0}$$

$$\begin{aligned} Z(\tau; z = 0) &= \chi; & \text{Euler number} \\ Z(\tau; z = \frac{1}{2}) &= \sigma + \mathcal{O}(q); & \text{signature} \\ &\equiv Z_\sigma(\tau) \end{aligned}$$

Results:

1. A_1 type ALE space

$$Z(\tau; z) = \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)} z^{n+\frac{1}{2}} \theta_2(\tau; z)}{1 - zq^n \eta(\tau)^3}$$

This coincides with massless character of $\mathcal{N} = 4$ algebra.

2. A_{n-1} type ALE space

$$\begin{aligned} Z(\tau; z) &= -\frac{1}{n} \sum_{a,b} (-1)^b \frac{\theta_1(\tau; \frac{n-1}{n}(z + a\tau + b))}{\theta_1(\tau; \frac{1}{n}(\frac{1}{2} + a\tau + b))} \\ &\times \mathcal{K}_{2n}(\tau, \frac{1}{n}(z + a\tau + b), 0) \frac{i\theta_2(\tau; z)}{\eta(\tau)^3} \end{aligned}$$

Here $\mathcal{K}_n(\tau, \nu, \mu)$ denotes the level n Appell function

$$\mathcal{K}_n(\tau, \nu, \mu) = \sum_m \frac{e^{i\pi m^2 n \tau + 2\pi i m n \nu}}{1 - e^{2\pi i(\nu + \mu + m\tau)}}$$

Recall the formula of CY/LG correspondence:

Consider a LG theory with a superpotential

$$W(X) = gX^n$$

where X is a chiral field $X = \varphi + \theta\psi + \dots$

$$U(1) \text{ charge of field } \begin{aligned} \varphi &= \frac{1}{n}, \\ \psi &= \frac{1}{n} - 1 \end{aligned}$$

When one gradually switches off the interaction $g \rightarrow 0$, topological invariants remain unchanged. Thus one may evaluate elliptic genus using free field theory. One finds

$$\begin{aligned} Z(\tau; z) &\approx \frac{\prod_n (1 - y^{\frac{1}{n}-1} q^n) (1 - y^{-\frac{1}{n}+1} q^n)}{\prod_n (1 - y^{\frac{1}{n}} q^n) (1 - y^{-\frac{1}{n}} q^n)} \\ &= \frac{\theta_1(\tau; (1 - \frac{1}{n})z)}{\theta_1(\tau; \frac{1}{n}z)} \end{aligned}$$

On the other hand from the rep. theory

$Z(\tau; z) =$ sum of characters of Ramond ground states

Thus we find a remarkable identity

$$\sum_{\ell=0}^{n-2} ch_{\ell, \ell+1}^{\tilde{R}}(\tau; z) = \frac{\theta_1(\tau; (1 - \frac{1}{n})z)}{\theta_1(\tau; \frac{1}{n}z)}$$

Witten

We have an analogous formula for Appell function

$$\sum_{s=K}^{N+K} \chi_{dis(N,K)}^{\tilde{R}}(s, s - K; \tau, z) = i\mathcal{K}_{2NK}(\tau, \frac{z}{N}) \frac{\theta_1(\tau, z)}{\eta(\tau)^3}$$

Free field interpretation of $\mathcal{N} = 2$ Liouville field?
Geometric computation of elliptic genus needed.