Functional relations
and Bethe Ansatz
for the open XXZ chain

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Introduction

Closed XXZ chain Hamiltonian:

$$H = \frac{1}{2} \sum_{n=1}^{N} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \mathrm{ch}\,\eta\ \sigma_n^z \sigma_{n+1}^z \right)$$

where

$$\sigma_n^i = \underbrace{\mathbb{I}}_{1} \otimes \cdots \otimes \mathbb{I} \otimes \underbrace{\sigma^i}_{n} \otimes \mathbb{I} \otimes \cdots \otimes \underbrace{\mathbb{I}}_{N}$$

$\sigma^i \quad (i = x\,, y\,, z)$ Pauli matrices

$$\mathbb{I} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

$\vec{\sigma}_{N+1} = \vec{\sigma}_1$ periodic boundary conditions

$\eta\,:$ bulk anisotropy parameter

$$[H\,, S^z] = 0\,, \qquad S^z = \tfrac{1}{2} \textstyle\sum_{n=1}^{N} \sigma_n^z$$

Eigenvalues ?

In particular, low-lying eigenvalues for $N \to \infty$?

- Fundamental model, like harmonic oscillator and hydrogen atom, but many-body!
- condensed matter
- statistical mechanics
- string/gauge theory

A priori, seems hopeless: # of eigenvalues $= 2^N$

However, “miracle” : Integrable!

i.e., there exists “transfer matrix” $t(u)$ such that

$$[t(u)\,,t(v)]=0$$

$$H \propto \frac{\partial}{\partial u}\ln t(u)\Big|_{u=0} + \text{ const}$$

Algebraic Bethe Ansatz:

(Faddeev, Kulish, Sklyanin, Takhtajan, . . .)

$$|\Omega\rangle \equiv \underbrace{\begin{pmatrix}1\\0\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1\\0\end{pmatrix}}_{N} \qquad \text{pseudovacuum}$$

$B(u)$: certain parameter-dependent creation operator

The state

$$B(u_1)\cdots B(u_M)|\Omega\rangle$$

is an eigenstate of $t(u)$ if $\{u_1\,,\cdots\,,u_M\}$ satisfy

$$\left(\frac{\text{sh}(u_j+\eta)}{\text{sh}(u_j)}\right)^N = -\prod_{k=1}^{M}\frac{\text{sh}(u_j-u_k+\eta)}{\text{sh}(u_j-u_k-\eta)} \qquad \text{Bethe Ansatz Eqs}$$

with eigenvalue

$$\Lambda(u)=\text{sh}^N(u+\eta)\prod_{k=1}^{M}\frac{\text{sh}(u-u_k-\eta)}{\text{sh}(u-u_k)}+\text{sh}^N(u)\prod_{k=1}^{M}\frac{\text{sh}(u-u_k+\eta)}{\text{sh}(u-u_k)}$$

- Particle-like excitations (“spinons”)
- Discretized form of sine-Gordon model, . . .

Open XXZ chain Hamiltonian:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\Big\{ \sum_{n=1}^{N-1} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \mathrm{ch}\,\eta\ \sigma_n^z \sigma_{n+1}^z\right) \\
&+ \mathrm{sh}\,\eta \Big[\mathrm{cth}\,\alpha_- \,\mathrm{th}\,\beta_- \sigma_1^z \\
&+ \mathrm{csch}\,\alpha_- \,\mathrm{sech}\,\beta_- (\,\mathrm{ch}\,\theta_- \sigma_1^x + i\,\mathrm{sh}\,\theta_- \sigma_1^y) \\
&- \mathrm{cth}\,\alpha_+ \,\mathrm{th}\,\beta_+ \sigma_N^z \\
&+ \mathrm{csch}\,\alpha_+ \,\mathrm{sech}\,\beta_+ (\,\mathrm{ch}\,\theta_+ \sigma_N^x + i\,\mathrm{sh}\,\theta_+ \sigma_N^y) \Big] \Big\}
\end{aligned}$$

$\alpha_\pm\,, \beta_\pm\,, \theta_\pm$: boundary parameters

Nondiagonal boundary terms!

$[\mathcal{H}\,, S^z] \neq 0$

Eigenvalues ?

- Fundamental many-body model with boundary
- condensed matter
- statistical mechanics
- string/gauge theory (?)

Good news: also integrable; i.e.,

there exists transfer matrix $t(u)$ such that

$$[t(u)\,,t(v)]=0$$

$$\mathcal{H}\propto\frac{\partial}{\partial u}t(u)\Big|_{u=0}+\text{ const}$$

Bad news: $|\Omega\rangle=\binom{1}{0}^{\otimes N}$ is not an eigenstate

⇒ cannot use algebraic Bethe Ansatz to determine eigenvalues

Main points of this talk:

- A 3-step procedure for solving integrable spin chain models which does not require the existence of a pseudovacuum state
- Solution of the open XXZ chain if the boundary parameters obey a certain constraint
- New solutions with 2 free boundary parameters; generalized $T-Q$ relations

Diagonal boundary terms: $\alpha_\pm$ or $\beta_\pm\to\pm\infty$

Gaudin (1971, 1983)
Alcaraz, Barber, Batchelor, Baxter & Quispel (1987)
Sklyanin (1988)

Outline

I. Closed XXZ chain – “warm up”
   re-derive standard Bethe Ansatz solution by new method

II. Open XXZ chain

I. Closed XXZ chain

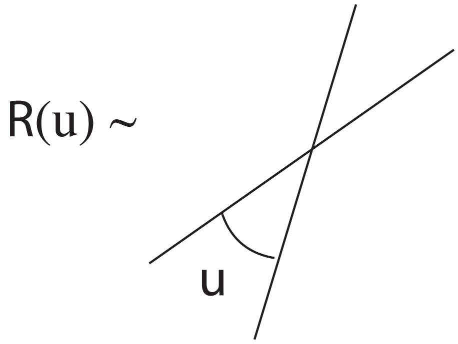
$$H = \frac{1}{2}\sum_{n=1}^{N}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \mathrm{ch}\,\eta\ \sigma_n^z\sigma_{n+1}^z\right)\,, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1$$

The transfer matrix is constructed from R matrix:

$$R(u) = \begin{pmatrix} \mathrm{sh}(u+\eta) & 0 & 0 & 0 \\ 0 & \mathrm{sh}\,u & \mathrm{sh}\,\eta & 0 \\ 0 & \mathrm{sh}\,\eta & \mathrm{sh}\,u & 0 \\ 0 & 0 & 0 & \mathrm{sh}(u+\eta) \end{pmatrix}$$

- acts on $C^2 \otimes C^2$;

i.e., spin $\frac{1}{2} \otimes$ spin $\frac{1}{2}$

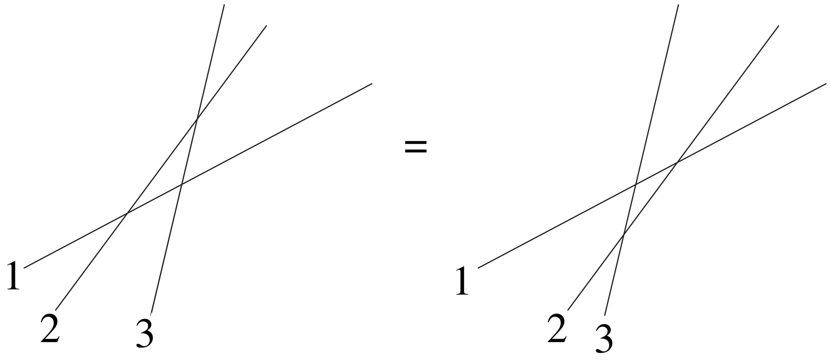


- satisfies Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)$$

acts on $C^2 \otimes C^2 \otimes C^2$

$R_{12} = R \otimes \mathbb{I}\,,\qquad R_{23} = \mathbb{I} \otimes R$, etc.



monodromy matrix

$$T_0(u) = R_{0N}(u) \dots R_{01}(u) \qquad \sim$$

0 : "auxiliary" space

1 , . . . , N : "quantum" spaces

acts on $\underbrace{C^2}_{0} \otimes \underbrace{C^2}_{1} \cdots \otimes \otimes \underbrace{C^2}_{N}$

transfer matrix

$$t(u) \equiv t^{(\frac{1}{2})}(u) = \mathrm{tr}_0\, T_0(u) \sim$$

spin $\frac{1}{2}$ auxiliary space

Yang-Baxter equation ⇒ commutativity property

$$[t(u)\,, t(v)] = 0$$

Further property:

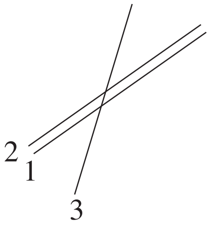
$$t(u) = t(u + i\pi) \qquad \text{periodicity}$$

Fusion for R matrix

Karowski (1979)
Kulish, Reshetikhin & Sklyanin (1981)
Kulish & Sklyanin (1982)

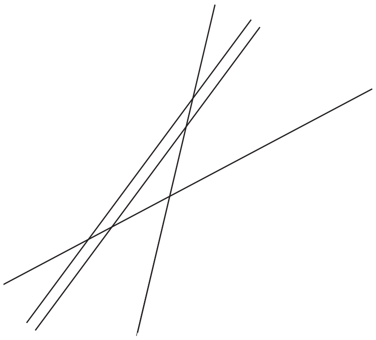
Can prove that the "fused" R matrix

$$R_{\langle 12\rangle 3}(u) \equiv P^{+}_{12}\ R_{13}(u)R_{23}(u+\eta)\ P^{+}_{12} \qquad \sim$$



which acts on $C^3 \otimes C^2$, i.e., spin 1 $\otimes$ spin $\frac{1}{2}$

satisfies *generalized* Yang-Baxter eqn:



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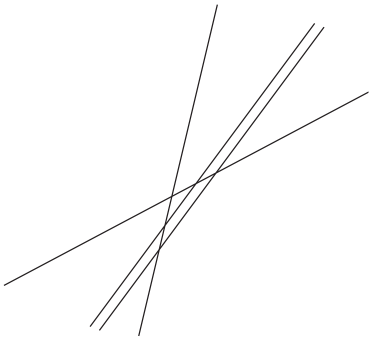
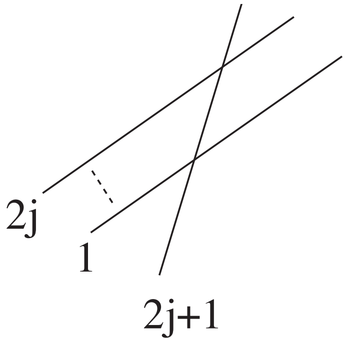


Generalization:

$$R_{\langle 1\cdots 2j\rangle 2j+1}(u) \qquad \sim$$



acts on $C^{2j+1} \otimes C^2$,

i.e., spin j $\otimes$ spin $\frac{1}{2}$ $\qquad\qquad j = \frac{1}{2}\,, 1\,, \frac{3}{2}\,, \ldots$

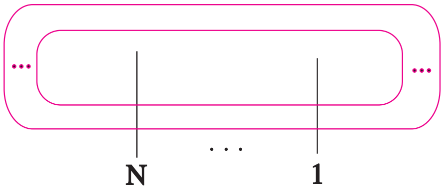
Denote also by $R^{(j\,,\frac{1}{2})}(u)$

3-step procedure for determining eigenvalues $\Lambda(u)$ of $t(u)$

Step 1: Obtain fusion hierarchy

Using “fused” R matrices, construct transfer matrices with spin j auxiliary space

$$t^{(j)}(u) = \mathrm{tr}_{1\cdots 2j}\, R_{\langle 1\cdots 2j\rangle N}(u) \cdots R_{\langle 1\cdots 2j\rangle 1}(u) \sim$$



Obey infinite “fusion hierarchy”: Kulish & Sklyanin (1982)
Kirillov & Reshetikhin (1986)

$$\boxed{\begin{array}{c} t^{(j)}(u)\ t^{(\frac{1}{2})}(u+2j\eta) = \delta(u+(2j-1))t^{(j-\frac{1}{2})}(u) + t^{(j+\frac{1}{2})}(u) \\ j \ = \ \frac{1}{2}, 1, \frac{3}{2}, \ldots \end{array}}$$

$$\delta(u) = (\mathrm{sh}(u+\eta)\,\mathrm{sh}(u-\eta))^{N}$$ “quantum determinant”

Izergin & Korepin (1981)

Similar to Clebsch-Gordon theorem for $su(2)$ irred reps:

$$\underline{j \otimes \frac{1}{2}} = \underline{j - \frac{1}{2}} \oplus \underline{j + \frac{1}{2}}$$

Step 2: Observe that for anisotropy values

$$\boxed{\eta = \frac{i\pi}{p+1}\,, \qquad p = 1\,, 2\,, \ldots\,,}$$

(thus, $q \equiv e^{\eta}$ is a root of unity, $q^{p+1} = -1$),

the infinite fusion hierarchy becomes truncated:

$$\boxed{t^{(\frac{p+1}{2})}(u) = a(u) t^{(\frac{p-1}{2})}(u+\eta) + b(u) F}$$

where $F = \prod_{k=1}^{N} \sigma_k^z$

Bazhanov, Lukyanov & Zamolodchikov (1997, 1999)
Kuniba, Sakai & Suzuki (1998)

Sketch: quantum group symmetry ⇒

explicit formula for higher-spin R matrices:

$$R^{(\frac{1}{2},j)}(u) = \begin{pmatrix} \text{sh}\left(u + (\frac{1}{2} + H)\eta\right) & \text{sh}\,\eta\ E_- \\ \text{sh}\,\eta\ E_+ & \text{sh}\left(u + (\frac{1}{2} - H)\eta\right) \end{pmatrix}$$

$E_\pm$, H form a $(2j+1)$-dim rep of $U_q(su(2))$

Kulish & Reshetikhin (1983)

⇒

$$R^{(\frac{p+1}{2},\frac{1}{2})}(u)\Big|_{\eta=\frac{i\pi}{p+1}} \sim \begin{pmatrix} \sigma^z & 0 & 0 \\ 0 & R^{(\frac{p-1}{2},\frac{1}{2})}(u+\eta) & * \\ 0 & 0 & \sigma^z \end{pmatrix}$$

block triangular!

Steps 1 & 2 $\Rightarrow$

functional relation for the fundamental transfer matrix $t(u)$ of order $p+1$

Example: $\eta = \frac{i\pi}{3} \qquad (p=2)$

$$\begin{aligned} & t(u)t(u+\eta)t(u+2\eta) \\ & \quad - \quad \delta(u)t(u+2\eta) - \delta(u+\eta)t(u) - a(u)t(u+\eta) \\ & \quad - \quad b(u)F = 0 \end{aligned}$$

Similar relations known for RSOS models

Baxter & Pearce (1982)
Bazhanov & Reshetikhin (1989)

Commutativity property $\Rightarrow$

eigenvalues $\Lambda(u)$ obey same functional relations

Step 3: Observe that the functional relations for the eigenvalues $\Lambda(u)$ can be written as

$$\det \mathcal{M}(u) = 0$$

where $\mathcal{M}(u)$ is a $(p+1) \times (p+1)$ matrix:

$$\mathcal{M} = \begin{pmatrix} \Lambda(u) & -h(u-\eta) & 0 & \dots & 0 & -Fh(u) \\ -h(u+\eta) & \Lambda(u+\eta) & -h(u) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -Fh(u+(p-1)\eta) & 0 & 0 & \dots & -h(u+p\eta) & \Lambda(u+p\eta) \end{pmatrix}$$

where

$$h(u) = \mathrm{sh}^N(u+\eta)\,, \qquad F = \pm 1\,.$$

$\Rightarrow$ $\mathcal{M}(u)$ has a null eigenvector $v(u)$:

$$\mathcal{M}(u)\ v(u) = 0\,, \qquad (*)$$

Note

$$S\mathcal{M}(u)S^{-1} = \mathcal{M}(u+\eta)\,,$$

where

$$S = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ F & 0 & 0 & \dots & 0 & 0 \end{pmatrix}\,, \qquad S^{p+1} = F\mathbb{I}\,.$$

$\Rightarrow$ $v(u)$ satisfies $Sv(u) = v(u+\eta)$. Thus,

$$v(u) = (Q(u)\,, Q(u+\eta)\,, \dots\,, Q(u+p\eta))\,, \quad Q(u+i\pi) = F\ Q(u)$$

Ansatz: $$\boxed{Q(u) = \prod_{j=1}^{M} \mathrm{sh}(u-u_j)}$$

zeros $\{u_1\,, \dots\,, u_M\}$ still to be determined, $\quad F = (-1)^M$

$(*) \Rightarrow$

$$\boxed{\Lambda(u) = h(u)\frac{Q(u-\eta)}{Q(u)} + h(u-\eta)\frac{Q(u+\eta)}{Q(u)}}$$

$\Lambda(u)$ must be analytic at $u = u_j$ $\Rightarrow$

$$\boxed{\frac{h(u_j)}{h(u_j-\eta)} = -\frac{Q(u_j+\eta)}{Q(u_j-\eta)}\,, \qquad j = 1, \ldots, M}$$

Bethe Ansatz Eqs

- Re-derived familiar Bethe Ansatz result for eigenvalues of transfer matrix of closed XXZ chain
- Assumed $\eta = \frac{i\pi}{p+1}$; but result known to be true for general η
- M not fixed $(= \frac{N}{2} - S_z)$
- Did not rely on existence of pseudovacuum state!

N. hep-th/0211001

II. Open XXZ chain

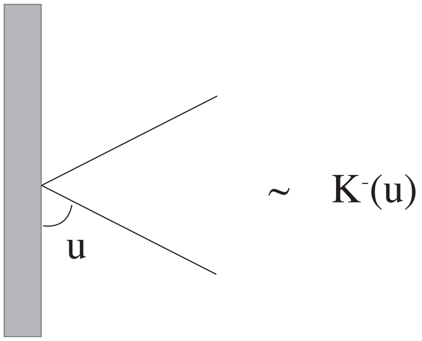
To construct transfer matrix, need also K matrices:

$$K^{-}(u) = \begin{pmatrix} \mathrm{sh}\,\alpha_- \,\mathrm{ch}\,\beta_- \,\mathrm{ch}\,u + \mathrm{ch}\,\alpha_- \,\mathrm{sh}\,\beta_- \,\mathrm{sh}\,u & e^{\theta_-}\,\mathrm{sh}\,u\,\mathrm{ch}\,u \\ e^{-\theta_-}\,\mathrm{sh}\,u\,\mathrm{ch}\,u & \mathrm{sh}\,\alpha_- \,\mathrm{ch}\,\beta_- \,\mathrm{ch}\,u - \mathrm{ch}\,\alpha_- \,\mathrm{sh}\,\beta_- \,\mathrm{sh}\,u \end{pmatrix}$$

de Vega & González-Ruiz (1993)
Ghoshal & Zamolodchikov (1994)

- acts on C^2;

i.e., spin $\frac{1}{2}$

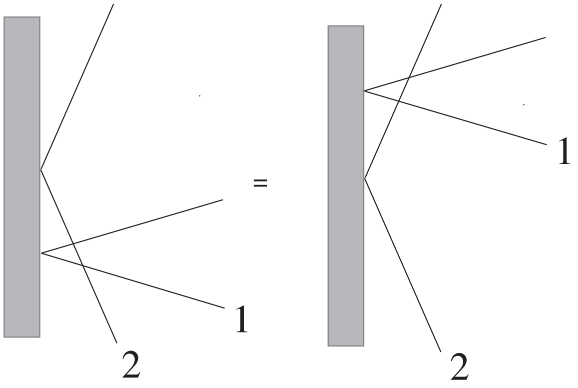


- satisfies boundary Yang-Baxter equation

$$R_{12}(u-v)K_1^-(u)R_{12}(u+v)K_2^-(v)$$
$$= K_2^-(v)R_{12}(u+v)K_1^-(u)R_{12}(u-v)$$

acts on $C^2 \otimes C^2$ Cherednik (1984)

$K_1^- = K^- \otimes \mathbb{I}\,, \qquad K_2^- = \mathbb{I} \otimes K^-$



$$K^{+}(u) = K^{-}(-u-\eta)\Big|_{\substack{\alpha_- \to -\alpha_+ \\ \beta_- \to -\beta_+ \\ \theta_- \to \theta_+}}$$

transfer matrix Sklyanin (1988)

$$t(u) \equiv t^{(\frac{1}{2})}(u) = \mathrm{tr}_0\, K_0^{+}(u)\ T_0(u)\ K_0^{-}(u)\ \hat{T}_0(u)$$

where

$$\begin{aligned} T_0(u) &= R_{0N}(u)\cdots R_{01}(u) \\ \hat{T}_0(u) &= R_{01}(u)\cdots R_{0N}(u) \end{aligned}$$

monodromy matrices

boundary Yang-Baxter, etc. ⇒ commutativity property

$$[t(u)\,,t(v)] = 0$$

Further properties:

$$t(u) = t(u+i\pi) \qquad \text{periodicity}$$

$$t(u) = t(-u-\eta) \qquad \text{crossing symmetry}$$

$$t(u) \underset{u\to\infty}{\sim} -\mathrm{ch}(\theta_- - \theta_+)\frac{e^{u(2N+4)+\eta(N+2)}}{2^{2N+1}}\mathbb{I} + \dots$$

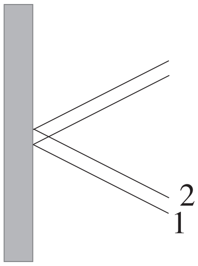
asymptotic behavior

Fusion for K matrix

Mezincescu, N. & Rittenberg (1990)
Mezincescu & N. (1992)

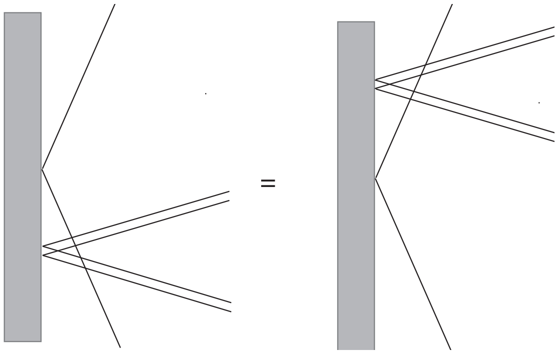
Can prove that the “fused” K^- matrix

$$K^-_{\langle 12\rangle}(u) \equiv P^+_{12}\; K^-_1(u) R_{12}(2u+\eta) K^-_2(u+\eta)\; P^+_{12} \sim$$



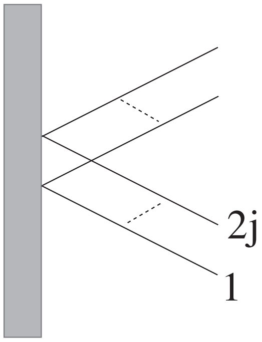
which acts on C^3 (spin 1)

satisfies *generalized* boundary Yang-Baxter eqn:



Generalization:

$$K^-_{\langle 1\cdots 2j\rangle}(u) \qquad \sim$$



acts on C^{2j+1}, i.e., spin j $\qquad j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$

We now try to determine eigenvalues of $t(u)$ by same 3-step procedure

Step 1: Obtain fusion hierarchy

$t^{(j)}(u)$: transfer matrix with spin j auxiliary space

$$t^{(j)}(u)\ t^{(\frac{1}{2})}(u+2j\eta) = c(u)\ t^{(j-\frac{1}{2})}(u) + d(u)\ t^{(j+\frac{1}{2})}(u)$$

$$j = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

Mezincescu & N. (1992)
Zhou (1996)

Step 2: Observe that for

$$\eta = \frac{i\pi}{p+1}\,, \qquad p = 1\,, 2\,, \ldots\,,$$

the infinite fusion hierarchy becomes truncated:

$$t^{(\frac{p+1}{2})}(u) = a(u) t^{(\frac{p-1}{2})}(u+\eta) + b(u)\mathbb{I}$$

N. hep-th/0110116

boundary quantum group symmetry

⇒ Mezincescu & N. (1998)

higher-spin K matrices Delius & N. hep-th/0204076

Steps 1 & 2 ⇒

functional relation for the fundamental transfer matrix $t(u)$ of order $p+1$

Example: $\eta = \frac{i\pi}{4} \qquad (p = 3)$

$$\begin{aligned} & t(u)\ t(u+\eta)\ t(u+2\eta)\ t(u+3\eta) \\ - & \ \delta(u-\eta)\ t(u+\eta)\ t(u+2\eta) - \delta(u)\ t(u+2\eta)\ t(u+3\eta) \\ - & \ \delta(u+\eta)\ t(u)\ t(u+3\eta) - \delta(u+2\eta)\ t(u)\ t(u+\eta) \\ + & \ \delta(u)\ \delta(u+2\eta) + \delta(u-\eta)\ \delta(u+\eta) \\ = & \ f(u) \end{aligned}$$

where $\delta(u)$ and $f(u)$ are known functions.

Again, commutativity property ⇒

eigenvalues $\Lambda(u)$ obey same functional relations

$$\delta(u) = \delta_0(u)\delta_1(u)\,, \qquad f(u) = f_0(u) f_1(u)\,,$$

where

$$\begin{aligned}
\delta_0(u) &= (\mathrm{sh}\, u\, \mathrm{sh}(u+2\eta))^{2N} \frac{\mathrm{sh}\, 2u\, \mathrm{sh}(2u+4\eta)}{\mathrm{sh}(2u+\eta)\, \mathrm{sh}(2u+3\eta)} \\
\delta_1(u) &= 2^4\, \mathrm{sh}(u+\eta+\alpha_-)\, \mathrm{sh}(u+\eta-\alpha_-)\, \mathrm{ch}(u+\eta+\beta_-)\, \mathrm{ch}(u+\eta-\beta_-) \\
&\times\ \mathrm{sh}(u+\eta+\alpha_+)\, \mathrm{sh}(u+\eta-\alpha_+)\, \mathrm{ch}(u+\eta+\beta_+)\, \mathrm{ch}(u+\eta-\beta_+)
\end{aligned}$$

For p even,

$$\begin{aligned}
f_0(u) &= (-1)^{N+1} 2^{-2pN}\, \mathrm{sh}^{2N}\left((p+1)u\right) \\
f_1(u) &= (-1)^{N+1} 2^{3-2p} \Big(
\end{aligned}$$

$$\begin{aligned}
&\mathrm{sh}\left((p+1)\alpha_-\right) \mathrm{ch}\left((p+1)\beta_-\right) \mathrm{sh}\left((p+1)\alpha_+\right) \mathrm{ch}\left((p+1)\beta_+\right) \mathrm{ch}^2\left((p+1)u\right) \\
&- \mathrm{ch}\left((p+1)\alpha_-\right) \mathrm{sh}\left((p+1)\beta_-\right) \mathrm{ch}\left((p+1)\alpha_+\right) \mathrm{sh}\left((p+1)\beta_+\right) \mathrm{sh}^2\left((p+1)u\right) \\
&-(-1)^N \mathrm{ch}\left((p+1)(\theta_- - \theta_+)\right) \mathrm{sh}^2\left((p+1)u\right) \mathrm{ch}^2\left((p+1)u\right) \Big)
\end{aligned}$$

For p odd,

$$\begin{aligned}
f_0(u) &= (-1)^{N+1} 2^{-2pN}\, \mathrm{sh}^{2N}\left((p+1)u\right) \tanh^2\left((p+1)u\right)\,, \\
f_1(u) &= -2^{3-2p} \Big(
\end{aligned}$$

$$\begin{aligned}
&\mathrm{ch}\left((p+1)\alpha_-\right) \mathrm{ch}\left((p+1)\beta_-\right) \mathrm{ch}\left((p+1)\alpha_+\right) \mathrm{ch}\left((p+1)\beta_+\right) \mathrm{sh}^2\left((p+1)u\right) \\
&- \mathrm{sh}\left((p+1)\alpha_-\right) \mathrm{sh}\left((p+1)\beta_-\right) \mathrm{sh}\left((p+1)\alpha_+\right) \mathrm{sh}\left((p+1)\beta_+\right) \mathrm{ch}^2\left((p+1)u\right) \\
&+(-1)^N \mathrm{ch}\left((p+1)(\theta_- - \theta_+)\right) \mathrm{sh}^2\left((p+1)u\right) \mathrm{ch}^2\left((p+1)u\right) \Big)
\end{aligned}$$

Step 3: The functional relations for the eigenvalues $\Lambda(u)$ CAN be written as

$$\det \mathcal{M}(u) = 0$$

where $\mathcal{M}(u)$ is the $(p+1) \times (p+1)$ matrix

$$\begin{pmatrix} \Lambda(u) & -\frac{\delta(u)}{h(u+\eta)} & 0 & \dots & 0 & -h(u) \\ -h(u+\eta) & \Lambda(u+\eta) & -\frac{\delta(u+\eta)}{h(u+2\eta)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\delta(u-\eta)}{h(u)} & 0 & 0 & \dots & -h(u+p\eta) & \Lambda(u+p\eta) \end{pmatrix}$$

IF there exists an $i\pi$-periodic function $h(u)$ such that

$$f(u) = \prod_{j=0}^{p} h(u+j\eta) + \prod_{j=0}^{p} \frac{\delta(u+j\eta)}{h(u+j\eta)}$$

Bad news: Have not (yet?) found solution for general values of boundary parameters.

Good news: There are solutions

$$\boxed{\begin{aligned} h^{(\pm)}(u) &= -4\,\mathrm{sh}^{2N}(u+\eta)\frac{\mathrm{sh}(2u+2\eta)}{\mathrm{sh}(2u+\eta)} \\ &\times \mathrm{sh}(u\pm\alpha_-)\,\mathrm{ch}(u\pm\beta_-)\,\mathrm{sh}(u\pm\alpha_+)\,\mathrm{ch}(u\pm\beta_+) \end{aligned}}$$

if the boundary parameters satisfy the constraint

$$\boxed{\alpha_- + \beta_- + \alpha_+ + \beta_+ = \pm(\theta_- - \theta_+) + \eta k}$$

where k is an integer with

$$-(N+1) \le k \le N+1 \quad \text{and} \quad k = \begin{cases} \text{even if } N \text{ odd} \\ \text{odd if } N \text{ even} \end{cases}$$

$\Rightarrow$ $\mathcal{M}(u)$ has a null eigenvector $v(u)$:

$$\mathcal{M}(u)\ v(u) = 0\,, \qquad (*)$$

Note

$$S\mathcal{M}(u)S^{-1} = \mathcal{M}(u+\eta)\,,$$

where

$$S = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}\,, \qquad S^{p+1} = \mathbb{I}\,.$$

$\Rightarrow$ $v(u)$ satisfies $Sv(u) = v(u+\eta)$. Thus,

$$v(u) = (Q(u)\,, Q(u+\eta)\,, \dots\,, Q(u+p\eta))\,, \qquad Q(u+i\pi) = Q(u)$$

Ansatz:

$$\boxed{Q^{(\pm)}(u) = \prod_{j=1}^{M^{(\pm)}} \mathrm{sh}(u - u_j^{(\pm)})\, \mathrm{sh}(u + u_j^{(\pm)} + \eta)}$$

$$Q^{(\pm)}(u) = Q^{(\pm)}(-u-\eta)$$

$(*) \Rightarrow$

$$\boxed{\Lambda^{(\pm)}(u) = h^{(\pm)}(u)\frac{Q^{(\pm)}(u-\eta)}{Q^{(\pm)}(u)} + h^{(\pm)}(-u-\eta)\frac{Q^{(\pm)}(u+\eta)}{Q^{(\pm)}(u)}}$$

Asymptotic behavior $\Rightarrow$

$$\boxed{M^{(\pm)} = \frac{1}{2}(N - 1 \pm k) \qquad \text{fixed!}}$$

Analyticity $\Rightarrow$ Bethe Ansatz equations

$$\boxed{\frac{h^{(\pm)}(u_j^{(\pm)})}{h^{(\pm)}(-u_j^{(\pm)} - \eta)} = -\frac{Q^{(\pm)}(u_j^{(\pm)} + \eta)}{Q^{(\pm)}(u_j^{(\pm)} - \eta)}\,, \quad j = 1\,, \ldots\,, M^{(\pm)}}$$

N. hep-th/0211001, hep-th/0304092

N. & Ravanini, hep-th/0307095

Similar results obtained using different method:

Cao, Lin, Shi & Wang, con-mat/0212163; (2003)

Conjecture 1: This solution holds for generic η

- true for closed chain
- verified numerically

Conjecture 2: This solution gives the complete set of 2^N eigenvalues

- verified numerically

Example: $N = 4 \quad \Rightarrow \quad k$ odd and $-5 \leq k \leq 5$

k	# eigenvalues given by $\Lambda^{(+)}(u)$	# eigenvalues given by $\Lambda^{(-)}(u)$
5	16	0
3	15	1
1	11	5
-1	5	11
-3	1	15
-5	0	16

Summary:

There is a more-or-less conventional Bethe Ansatz solution of the open XXZ chain, if the constraint

$$\alpha_- + \beta_- + \alpha_+ + \beta_+ = \pm(\theta_- - \theta_+) + \eta k$$

is satisfied.

Relates left and right boundary parameters (?!)

Applications:

Solution used to compute finite-size effects for XXZ chain and sine-Gordon model on a finite interval

Ahn & N., hep-th/0309261
Ahn, Bajnok, N., Palla & Takács, hep-th/0501047

Overcoming constraint?

Recall: need an $i\pi$-periodic function $h(u)$ such that

$$f(u) = \prod_{j=0}^{p} h(u+j\eta) + \prod_{j=0}^{p} \frac{\delta(u+j\eta)}{h(u+j\eta)}$$

Set $z(u) \equiv \prod_{j=0}^{p} h(u+j\eta)$

$\Rightarrow$

$$z(u) = \frac{1}{2}\left(f(u) \pm \sqrt{\Delta(u)}\right) , \qquad \Delta(u) \equiv f(u)^2 - 4 \prod_{j=0}^{p} \delta(u+j\eta)$$

Difficulties:

(1) For general values of boundary parameters,
$\Delta(u)$ is *not* a perfect square;
$z(u)$ is hard to factorize

(2) For some cases that $\Delta(u)$ *is* a perfect square,
find contradiction $z(u) \neq z(u+\eta)$

We focus on (2)

p even (i.e., $\eta = \frac{i\pi}{3}, \frac{i\pi}{5}, \ldots$)

Murgan & N., hep-th/0504124 + Addendum

$\Delta(u)$ is a perfect square if any 2 of the parameters $\{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ are arbitrary, and the others are either η or $i\pi/2$ $\quad (\theta_- = \theta_+)$

Example: $\alpha_\pm$ arbitrary and $\beta_\pm = \eta$

Use different $\mathcal{M}(u)$: $\quad S\mathcal{M}(u)S^{-1} = \mathcal{M}(u + p\eta)$

Find

$$\Lambda(u) = h(u)\frac{Q(u+p\eta)}{Q(u)} + h(-u+p\eta)\frac{Q(u-p\eta)}{Q(u)} \,,$$

$$Q(u) = \prod_{j=1}^{M} \mathrm{sh}\left(\frac{1}{2}(u-u_j)\right) \mathrm{sh}\left(\frac{1}{2}(u+u_j-p\eta)\right) \,,$$

$$M = N + 2p + 1$$

$$\frac{h(u_j)}{h(-u_j+p\eta)} = -\frac{Q(u_j-p\eta)}{Q(u_j+p\eta)} \,, \qquad j = 1, \ldots, M \,.$$

$$\begin{aligned} h(u) &= 4\,\mathrm{sh}^{2N}(u+\eta)\frac{\mathrm{sh}(2u+2\eta)}{\mathrm{sh}(2u+\eta)}\,\mathrm{ch}^2(u-\eta) \\ &\times \mathrm{sh}(u-\alpha_-)\,\mathrm{sh}(u+\alpha_+)\frac{\mathrm{ch}\left(\frac{1}{2}(u+\alpha_-+\eta)\right)}{\mathrm{ch}\left(\frac{1}{2}(u-\alpha_--\eta)\right)}\frac{\mathrm{ch}\left(\frac{1}{2}(u-\alpha_++\eta)\right)}{\mathrm{ch}\left(\frac{1}{2}(u+\alpha_+-\eta)\right)} \,, \end{aligned}$$

Completeness verified numerically

p odd (i.e., $\eta = \frac{i\pi}{2}\,, \frac{i\pi}{4}\,, \ldots$)

Murgan & N., hep-th/0507139

$\Delta(u)$ is a perfect square if any 2 of the parameters $\{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ are arbitrary, and the others are 0

Use more than one $Q(u)$!

i.e., schematically, rather than the usual $T - Q$ relation

$$t(u)\ Q(u) = Q(u') + Q(u'')\,,$$

we have instead a pair of relations:

$$\begin{aligned} t(u)\ Q_1(u) &= Q_2(u') + Q_2(u'')\,, \\ t(u)\ Q_2(u) &= Q_1(u''') + Q_1(u'''')\,. \end{aligned}$$

$$T \mathcal{M}(u) T^{-1} = \mathcal{M}(u + 2\eta)\,, \qquad T \equiv S^2\,,$$

$\det \mathcal{M}(u) = 0 \Rightarrow \mathcal{M}(u)$ has a null eigenvector $v(u)$:

$$\mathcal{M}(u)\ v(u) = 0\,,$$

where $v(u)$ satisfies $T v(u) = v(u + 2\eta)$. Thus,

$$v(u) = (Q_1(u)\,, Q_2(u)\,, \ldots\,, Q_1(u - 2\eta)\,, Q_2(u - 2\eta))\,.$$

Two independent Q(u)'s!

$$
\begin{aligned}
\Lambda(u) &= \frac{\delta(u)}{h^{(1)}(u)}\frac{Q_2(u)}{Q_1(u)} + \frac{\delta(u-\eta)}{h^{(2)}(u-\eta)}\frac{Q_2(u-2\eta)}{Q_1(u)} \\
&= h^{(1)}(u-\eta)\frac{Q_1(u-\eta)}{Q_2(u-\eta)} + h^{(2)}(u)\frac{Q_1(u+\eta)}{Q_2(u-\eta)}
\end{aligned}
$$

$$
\begin{aligned}
Q_1(u) &= \prod_{j=1}^{M_1} \mathsf{sh}(u-u_j^{(1)})\,\mathsf{sh}(u+u_j^{(1)}+\eta) \\
Q_2(u) &= \prod_{j=1}^{M_2} \mathsf{sh}(u-u_j^{(2)})\,\mathsf{sh}(u+u_j^{(2)}+3\eta)
\end{aligned}
$$

Analyticity $\Rightarrow$ Bethe Ansatz equations

$$
\begin{aligned}
\frac{\delta(u_j^{(1)})\ h^{(2)}(u_j^{(1)}-\eta)}{\delta(u_j^{(1)}-\eta)\ h^{(1)}(u_j^{(1)})} &= -\frac{Q_2(u_j^{(1)}-2\eta)}{Q_2(u_j^{(1)})}\,, \qquad j=1\,,2\,,\ldots\,,M_1\,, \\
\frac{h^{(1)}(u_j^{(2)})}{h^{(2)}(u_j^{(2)}+\eta)} &= -\frac{Q_1(u_j^{(2)}+2\eta)}{Q_1(u_j^{(2)})}\,, \qquad j=1\,,2\,,\ldots\,,M_2\,.
\end{aligned}
$$

Expect that there are sufficiently many equations to determine all zeros $\{u_j^{(1)}\,, u_j^{(2)}\}$

Completeness verified numerically

Conclusions

- Bethe Ansatz solutions of the open XXZ chain are also available for special cases that $\eta = i\pi/(p+1)$ and

$$\Delta(u) \equiv f(u)^2 - 4\prod_{j=0}^{p} \delta(u + j\eta)$$

is a perfect square; 2 arbitrary boundary parameters

- Bethe Ansatz can involve more than one $Q(u)$

– novel structure, which should be further understood

– may appear in other integrable models

- The general case that $\Delta(u)$ is *not* a perfect square or $\eta \neq i\pi/(p+1)$ remains to be understood . . .