Fractional Analytic Index

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Main reference:

[MMS3] V. Mathai, R.B. Melrose and I.M. Singer,
Fractional Analytic Index,
preprint 26 pages **[math.DG/0402329]**

Related reference:

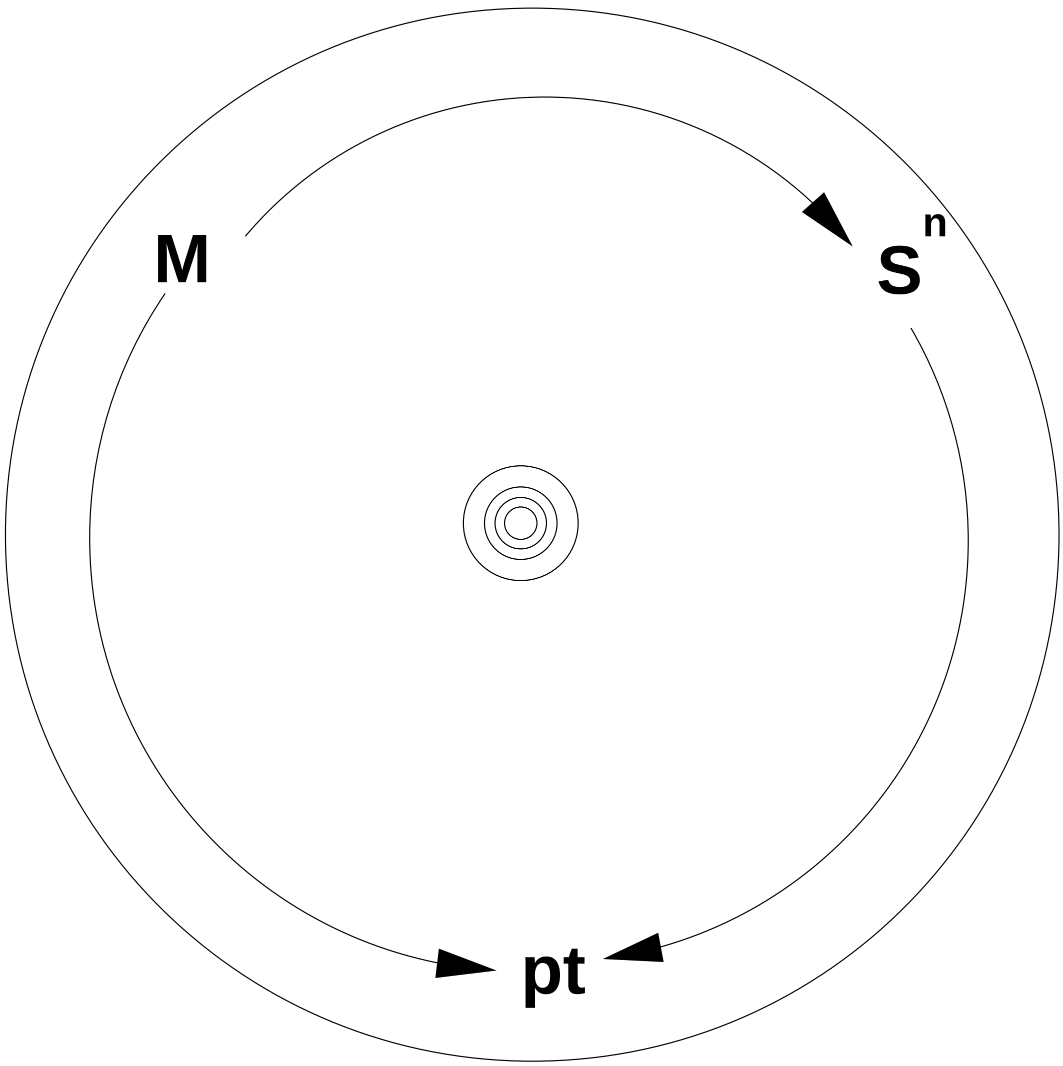
[MMS1] V. Mathai, R.B. Melrose and I.M. Singer,
The index of projective families of elliptic operators, *Geometry and Topology*, **9** (2005) 341-373.
[math.DG/0206002]

Milestones of the
Atiyah-Singer index theorem in 2004

Professor Sir Michael Atiyah turned 75

Professor Isadore Singer turned 80

were jointly awarded the **Abel Prize** in 2004, the Mathematical equivalent of the Nobel Prize, (US$900,000) presented by King Harald of Norway, ”for the Atiyah-Singer index theorem in 1962, bringing together topology, geometry and analysis, and their outstanding role in building new bridges between mathematics and theoretical physics.”



Sir Michael's Atiyah's surprise birthday cake.

Index theorem for Dirac operators

In 1962, Atiyah and Singer defined an elliptic operator $\eth^+$ called the Dirac operator, on any compact **spin** manifold M of even dimension, and computed the analytic index,

$$\text{Index}_a(\eth^+) = \int_M \widehat{A}(M) \in \mathbb{Z}$$

.

where $\widehat{A}(M)$ is the A-hat genus of the manifold M, which is expressed in terms of Pontrjagin classes.

The A-hat genus continues to make sense for **non-spin** manifolds, but it was a long-standing mystery as to what corresponded to the analytic index in this situation, as the Dirac operator does not exist?!

The **Chern character**, which is a characteristic class that was invented by the late Professor S.S. Chern, forms an essential part of the statement of the Atiyah-Singer index theorem for elliptic operators on compact manifolds. For instance, if $\eth_E^+$ denotes the Dirac operator twisted by a connection on a vector bundle $E \to M$, on any compact **spin** manifold M of even dimension, then the Atiyah-Singer index theorem is,

$$\boxed{\mathrm{Index}_a(\eth_E^+) = \int_M \widehat{A}(M) \wedge \mathrm{Ch}(E) \in \mathbb{Z}}$$

where $\mathrm{Ch}(E)$ denotes the Chern character of the vector bundle E, which is expressed in terms of the Chern classes (which are the fundamental integral characteristic classes for vector bundles also invented by Professor Chern).

Outline

• On any oriented even dimensional Riemannian manifold, we construct a global distributional section, supported on the diagonal, of the Clifford algebra bundle on the product space, conormal wrt the diagonal, and representing the **projective spin Dirac operator**, thereby finally clarifying the mystery concerning the analytic index for non-spin manifolds, which corresponds to the topological index given by the $\widehat{A}$-genus.

• More generally, for a finite rank, projective vector bundle E over a compact manifold Z, we similarly define the (graded) **ring of projective differential operators**, $\mathrm{Diff}^{\bullet}(Z, E)$.

• Even more generally the corresponding space of **pseudodifferential operators** $\Psi_{\epsilon}^{\bullet}(Z, E)$ is defined, with supports sufficiently close to the diagonal, i.e. close to the identity relation. It is **not** an algebra or a ring in general.

• For such pseudodifferential operators that are elliptic, we define the (numerical) **analytic index**, as the trace of the commutator of the operator and a parametrix.

• Using the residue trace and the regularized trace, we show that analytic index is **homotopy invariant**.

- Using the **heat kernel method**, the analogue of the McKean-Singer formula is established in this context, expressing the analytic index as the small time limit of the supertrace of a truncated heat kernel for the twisted projective spin Dirac operator. Then using the local index theorem, we show that this index is given by the usual formula, in terms of the twisted **Chern character** of the symbol, which defines an element of twisted K-theory, and the Todd class of the manifold.

- Explicit calculations show that the analytic index is only a **rational number** but in general it is not an integer. Worked out examples and applications will be given.

Projective vector bundles

Projective vector bundles are also known as gauge bundles in physics and are a special case of bundle gerbe modules.

It is **not** a global bundle on Z, but rather is a vector bundle E over Y satisfying some equivariance properties, where $\phi : Y \to Z$ is a principal $PU(n)$-bundle. More precisely,

$$\mathcal{L}_g \otimes E_y \cong E_{g.y}, \qquad g \in PU(n),\ y \in Y \quad (1)$$

where $\mathcal{L} = U(n) \times_{U(1)} \mathbb{C} \to PU(n)$ is the canonical primitive line bundle, i.e.

$$\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2} \cong \mathcal{L}_{g_1.g_2}, \qquad g_i \in PU(n)$$

The identification (1) gives a projective action of $PU(n)$ on E, i.e. an action of $U(n)$ on E such that the center $U(1)$ acts as scalars.

The transition functions of the principal $PU(n)$ bundle Y over Z define a class in $H^1(Z, \underline{PU(n)})$. The exact sequence of sheaves

$$0 \to \underline{U(1)} \to \underline{U(n)} \to \underline{PU(n)} \to 1$$

gives rise to a connecting homomorphism

$$\delta : H^1(Z, \underline{PU(n)}) \to H^2(Z, \underline{U(1)}) \cong H^3(Z, \mathbb{Z})$$

The **Dixmier-Douady invariant** of Y is,

$$DD(Y) = \delta(Y) \in \text{Torsion}(H^3(Z, \mathbb{Z}))$$

The associated bundle of matrix algebras $\mathcal{A} = Y \times_{PU(n)} M_n(\mathbb{C})$ is called the associated **Azumaya** bundle. The construction also works for any principal G bundle P over Z, together with a central extension of G.

Difficulties. Since a projective vector bundle E is not global on Z, **cannot** make sense of sections of E, let alone operators acting between sections of projective vector bundles!

The goal of this talk is to make sense of differential and ΨDOs associated to projective vector bundles, and moreover to prove an **Index Theorem** for such elliptic operators, which reduces to the Atiyah-Singer index theorem when the projective vector bundles are ordinary vector bundles, i.e. when the Dixmier-Douady invariant vanishes.

The motivation for studying this comes from natural examples of projective vector bundles in mathematics, and also in string theory.

Projective vector bundle of spinors

Let $E \to Z$ be a real Riemannian vector bundle, $\psi : SO(E) \to Z$ the principal bundle of oriented orthonormal frames on E:

$$U(1) \to Spin^{\mathbb{C}}(n) \to SO(n)$$

the central extension associated to the group $Spin^{\mathbb{C}}(n)$. The Dixmier-Douady invariant of $SO(E)$ can be identified with the 3^{rd} integral Stieffel-Whitney class $W_3(E) \in H^3(Z, \mathbb{Z})$. It is the obstruction to writing the complex bundle of Clifford algebras $Cl(E) \to Z$ as $End(\mathcal{S})$, for some complex vector bundle $\mathcal{S} \to Z$. $\mathcal{S}$ is generally called a **bundle of spinors** for E. If $W_3(E) \neq 0$, then there is **no** such vector bundle $\mathcal{S}$ over Z.

However, it is easy to see that $\mathcal{S}$ exists as a **projective vector bundle**, as follows.

The principal $SO(n)$ bundle $\psi^*SO(E) \to SO(E)$ has trivial W_3 invariant, since

$$\psi^*SO(E) = SO(E) \times SO(n)$$

is the trivial bundle.

The associated Clifford algebra bundle

$$Cl(\psi^*E) = \psi^*Cl(E)$$

can be written as $End(\mathcal{S})$, for some complex vector bundle $\mathcal{S} \to SO(E)$, which is routine to verify is a projective vector bundle.

Then the **twisted K-theory** $K^0(Z, \mathcal{A})$ is defined as differences of projective vector bundles over Z, under stable equivalence.

Since $\mathcal{L}^{\otimes n}$ is trivial, we see that

$$E_y^{\otimes n} \cong E_{g.y}^{\otimes n} \qquad g \in PU(n),\ y \in Y$$

$E^{\otimes n} = \phi^*(\xi)$, where $\xi \to Z$ is a vector bundle.

So informally, **a projective vector can be viewed as an n^{th}-root of a vector bundle.**

Then the Chern character satisfies

$$Ch(E^{\otimes n}) = Ch(E)^n = \phi^* Ch(\xi)$$

An easy induction argument shows that

$Ch(E) = \phi^*(\Lambda)$ for some $\Lambda \in H^{even}(Z, \mathbb{R})$.

Define the **twisted Chern character** as,

$$Ch_{\mathcal{A}}(E) = \Lambda \in H^{even}(Z, \mathbb{R}).$$

$Ch_{\mathcal{A}} : K^0(Z, \mathcal{A}) \to H^{even}(Z, \mathbb{R})$ is an isomorphism/$\otimes\mathbb{R}$.

Key idea

Recall that for a compact manifold, Z, and vector bundles E and F over Z, the

Schwartz kernel theorem

gives a one-to-one correspondence between

$$\Big\{\text{continuous linear operators, } C^\infty(Z,E) \longrightarrow C^{-\infty}(Z,F)\Big\}$$

$$\downarrow\uparrow$$

$$\Big\{\text{distributional sections, } C^{-\infty}(Z^2, \mathrm{Hom}(E,F)\otimes\Omega_R)\Big\}$$

.

where $\mathrm{Hom}(E,F)_{(z,z')} = F_z \otimes E^*_{z'}$ is the 'big' homomorphism bundle over Z^2 and Ω_R the density bundle from the right factor.

When restricted to **pseudodifferential operators** $\Psi^m(Z,E,F)$, get an isomorphism with the space of **conormal distributions** with respect to the diagonal, $I^m(Z^2,\Delta;\mathrm{Hom}(E,F))$. i.e.

$$\boxed{\Psi^m(Z,E,F) \longleftrightarrow I^m(Z^2,\Delta;\mathrm{Hom}(E,F))}$$

When further restricted to **differential operators** $\mathrm{Diff}^m(Z,E,F)$ (which by definition have the property of being local operators) this becomes an isomorphism with the space, $I^m_\Delta(Z^2,\Delta;\mathrm{Hom}(E,F))$, of conormal distributions with respect to the diagonal, **supported within the diagonal**, Δ. i.e.

$$\boxed{\mathrm{Diff}^m(Z,E,F) \longleftrightarrow I^m_\Delta(Z^2,\Delta;\mathrm{Hom}(E,F))}$$

Projective differential operators/ΨDOs

The previous facts motivates our definition of projective differential and pseudodifferential operators when E and F are only projective vector bundles associated to a fixed finite-dimensional Azumaya bundle $\mathcal{A}$. $\mathrm{Hom}(E, F)$ is then again a projective bundle on Z^2 associated to the tensor product $\mathcal{A}_L \times \mathcal{A}'_R$ of the pull-back of $\mathcal{A}$ from the left and the conjugate bundle from the right. In particular if E and F have DD invariant $\tau \in H^3(Z; \mathbb{Z})$ then $\mathrm{Hom}(E, F)$ has DD invariant $\pi_L^* \tau - \pi_R^* \tau \in H^3(Z^2; \mathbb{Z})$. Since this class is trivial in a tubular neighborhood of the diagonal it is reasonable to expect that $\mathrm{Hom}(E, F)$ may be realized as an ordinary vector bundle there.

In [MMS], it is shown that there is a **canonical choice**, $\text{Hom}^{\mathcal{A}}(E,F)$ of extension to N_ϵ, such that the composition properties also extend.

This allows us to identify the space of projective pseudodifferential operators with kernels supported in a sufficiently neighborhood N_ϵ of the diagonal $\Psi^\bullet_\epsilon(Z;E,F)$, with the space of conormal distributions $I^\bullet_\epsilon(N_\epsilon, \text{Diag}; \text{Hom}^{\mathcal{A}}(E,F))$. Despite **not** being a space of operators, this has precisely the same local structure as in the untwisted case and has similar composition properties provided supports are restricted to appropriate neighbourhoods of the diagonal. The space of **projective smoothing operators** $\Psi^{-\infty}_\epsilon(Z;E,F)$ is therefore identified with the space of smooth sections, $\mathcal{C}^\infty_{\mathsf{c}}(N_\epsilon; \text{Hom}^{\mathcal{A}}(E,F) \otimes \pi_R^*\Omega)$.

• As an example, surprisingly there is a **projective spin Dirac operator** on every oriented even-dimensional compact manifold!

• As discussed earlier, there is a projective bundle of spinors on any oriented manifold Z which we denote by S; since Z is oriented it splits globally as the direct sum of two projective bundles $S^{\pm}$. There are natural connections on $Cl(Z)$ and $S^{\pm}$ arising from the Levi-Civita connection on T^*Z.

• Also as discussed earlier, $\hom(S,S)$ which can be identified with the Clifford bundle $Cl(Z)$, has an extension to $\tilde{C}l(Z)$ in a neighbourhood of the diagonal Diag, and this extended bundle also has an induced connection ∇.

Projective spin Dirac operator

Choose a projective spin structure. Then the projective spin Dirac operator is defined as the distributional section

$$\not{\partial} = cl \cdot \nabla_L(\kappa_{Id}), \quad \kappa_{Id} = \delta(z - z')Id_S.$$

Here κ_{Id} is the Schwartz kernel of the identity operator in $\mathrm{Diff}^*(Z; S)$ and ∇_L is the connection ∇ restricted to the left variables with cl the contraction given by the Clifford action cl of T^*Z on the left.

As in the usual case, the projective spin Dirac operator $\not{\partial}$ is **elliptic** (next slide) and odd with respect to the $\mathbb{Z}_2$ grading of S and locally, this projective spin Dirac operator can be identified with the usual spin Dirac operator.

The **principal symbol map** is well defined for conormal distributions, leading directly to the symbol map on $\Psi_{\epsilon}^{m}(Z;E,F)$ with values in $C^{\infty}(T^{*}Z,\pi^{*}\operatorname{hom}(E,F))$ homogeneous of degree m; here $\operatorname{hom}(E,F)$, the 'diagonal' homomorphism bundle which **is** a vector bundle with fibre, $\operatorname{hom}(E,F)_{z}=F_{z}\otimes E_{z}^{*}$.

Thus **ellipticity** is well defined, as the invertibility of this symbol. In particular, if $A\in\Psi_{\epsilon/2}^{m}(Z;E,F)$ is elliptic, then there is $B\in\Psi_{\epsilon/2}^{-m}(Z;F,E)$ and smoothing operators $Q_{R}\in\Psi_{\epsilon}^{-\infty}(Z;E,E)$, $Q_{L}\in\Psi_{\epsilon}^{-\infty}(Z;F,F)$ satisfying

$$BA=I_{E}-Q_{R},\quad AB=I_{F}-Q_{L}$$

The **trace functional** is defined on smoothing operators $\mathrm{Tr} : \Psi_{\epsilon}^{-\infty}(Z;E) \to \mathbb{C}$ as

$$\mathrm{Tr}(Q) = \int_Z \mathrm{tr}\, Q(z,z),$$

vanishes on commutators $\mathrm{Tr}(QR - RQ) = 0$, if $R, Q \in \Psi_{\epsilon/2}^{-\infty}(Z;E)$, as follows from Fubini's theorem.

The **analytic index** of the projective elliptic operator $A \in \Psi_{\epsilon}^{\bullet}(Z;E,F)$ is then defined as

$$\mathrm{Ind}_a(A) = \mathrm{Tr}(I_E - BA) - \mathrm{Tr}(I_F - AB) = \mathrm{Tr}([A,B])$$

where B is a parametrix for A. NB. The commutator $[A,B]$ is a smoothing operator.

If B' is any other parametrix for A, then $B - B'$ is a smoothing operator. For any smoothing operator Q,

$$\mathrm{Tr}([A, Q]) = 0.$$

This is proved using an approximation property of ΨDOs by smoothing operators.

Therefore the analytic index

$$\mathrm{Ind}_a(A) = \mathrm{Tr}([A, B]) = \mathrm{Tr}([A, B'])$$

is **well defined**, and is independent of the choice of parametrix for A.

For $A \in \Psi^m_{\epsilon/4}(Z; E, F)$ of integral order, m, the Guillemin-Wodzicki **residue trace** is

$$\mathrm{Tr}_R(A) = \lim_{z \to 0} z \,\mathrm{Tr}(AD(z)) \qquad (1)$$

where $\mathrm{Tr}(AD(z))$ is known to be meromorphic with at most simple poles at $z = -k - \dim Z + \{0, 1, 2, \ldots\}$, and $D(z) \in \Psi^z_{\epsilon/4}(Z; E)$ is an entire family of ΨDOs of complex order z which is elliptic and has $D(0) = I$. It is independent of the choice of such a family.

- The residue trace Tr_R vanishes on all ΨDOs of sufficiently negative order.
- The residue trace Tr_R is also a trace functional, that is,

$$\mathrm{Tr}_R([A, B]) = 0,$$

for $A \in \Psi^m_{\epsilon/4}(Z; E, F),\ B \in \Psi^{m'}_{\epsilon/4}(Z; F, E)$.

The **regularized trace**, is defined to be

$$\mathrm{Tr}_D(A) = \lim_{z\to 0} \frac{1}{z}\left(z\,\mathrm{Tr}(AD(z)) - \mathrm{Tr}_R(A)\right).$$

For general A, it does depend on the regularizing family, but for smoothing operators it **coincides** with the standard trace, therefore

$$\boxed{\mathrm{Ind}_a(A) = \mathrm{Tr}_D([A,B])}$$

for an elliptic operator A, and B a parametrix. The regularized trace Tr_D is **not** a trace function but rather the 'trace defect' satisfies

$$\mathrm{Tr}_D([A,B]) = \mathrm{Tr}_R(B\delta_D A)$$

where δ_D is a **derivation** acting on the full symbol algebra and which also satisfies

$$\mathrm{Tr}_R(\delta_D a) = 0 \quad \forall\, a. \tag{2}$$

Homotopy invariance of the index

Let A_t be a smooth 1-parameter family of projective elliptic ΨDOs, and B_t a smooth 1-parameter family of parametrizes for A_t. Then we have

$$\begin{aligned}\frac{d}{dt}\mathrm{Ind}_a(A_t) =& \frac{d}{dt}\mathrm{Tr}_D([A_t, B_t])\\ =& \mathrm{Tr}_D([\dot{A}_t, B_t]) + \mathrm{Tr}_D([A_t, \dot{B}_t])\\ =& \mathrm{Tr}_R(a_t^{-1}\delta_D \dot{a}_t) + \mathrm{Tr}_R((\frac{d}{dt}a_t^{-1})\delta_D a_t,)\\ =& -\mathrm{Tr}_R(\dot{a}_t \delta_D a_t^{-1})\\ & -\mathrm{Tr}_R(a_t^{-1}\dot{a}_t a_t^{-1}\delta_D a_t)\\ =& 0\end{aligned}$$

Here, a_t is the image of A_t in the full symbol algebra such that the image of B_t is a_t^{-1}.

Multiplicativity property of the index

If A_i for $i = 1, 2$ are two elliptic projective operators with the image bundle of the first being the same as the domain bundle of the second, they can be composed if their supports are sufficiently small. Let B_i be corresponding parametrices, again with very small supports. Then $B_1 B_2$ is a parametrix for $A_2 A_1$ and the index of the product is given in terms of the 'full symbols' a_i of the A_i

$$\begin{aligned}
\mathrm{Ind}_a(A_2 A_1) &= \mathrm{Tr}_D([A_2 A_1, B_1 B_2]) \\
&= \mathrm{Tr}_R(a_1^{-1} a_2^{-1} \delta_D(a_2 a_1)) \\
&= \mathrm{Tr}_R(a_1^{-1} \delta_D a_1) + \mathrm{Tr}_R(a_2^{-1} \delta_D a_2) \\
&= \mathrm{Ind}_a(A_1) + \mathrm{Ind}_a(A_2).
\end{aligned}$$

Heat kernel method

The heat kernel H_t that formally represents $\exp(-t\partial\!\!\!/_E^2)$, is a well-defined smooth kernel near the submanifold $\{t = 0\} \times \text{Diag}$, with values in $\text{Hom}^{Cl\otimes\mathcal{A}}(S \otimes E) \otimes \Omega_R$. It follows that if χ is a smooth function on $[0, \infty) \times Z^2$, supported and $\chi \equiv 1$ near $\{t = 0\} \times \text{Diag}$, then $H(t) = \chi(\exp(H_t))$ is a globally defined, **truncated heat kernel**. Then the analogue of the **McKean-Singer formula** holds and

$$\text{Ind}_a(\partial\!\!\!/_E^+) = \lim_{t\downarrow 0} \text{Tr}_S(H(t)) \qquad (3)$$

where Tr_S is the supertrace, the difference of the traces on $S^+ \otimes E$ and $S^- \otimes E$. The local index formula, as a result of rescaling, asserts the existence of this limit and its evaluation.

In the standard case the McKean-Singer formula for the actual heat kernel, follows by comparison with the limit as $t \to \infty$, which explicitly gives the index. Indeed then the function $\mathrm{Tr}_S(\exp(-t \not\partial_E^2))$ is constant in t.

However in the projective case, since only an approximate heat kernel $H(t)$ makes global sense, **the result only holds when $t \downarrow 0$.**

Now if $B(z) = \int_0^\infty t^z H(t) dt$, then through analytic continuation to $z = 0$, $B(0) = \not\partial_E^{-2}$.
Therefore $B = B(0) \not\partial_E^-$ is a parametrix for the projective Dirac operator $\not\partial_E^+$.

Inserting this as the parametrix in the definition of the analytic index gives,

$$\begin{aligned}\mathrm{Ind}_a(\partial\!\!\!/_E^{+}) &= \mathrm{Tr}([\partial\!\!\!/_E^{+}, B]) \\ &= \mathrm{Tr}\left([\partial\!\!\!/_E^{+}, \int_0^\infty t^z H(t)dt\ \partial\!\!\!/_E^{-}\Big|_{z=0}]\right) \\ &= -\,\mathrm{Tr}_S\left(\int_0^\infty \left(t^z \frac{d}{dt} H(t)\right) dt\Big|_{z=0}\right) \\ &= \lim_{z\to 0} z \int_0^\infty t^{z-1}\,\mathrm{Tr}_S(H(t))dt.\end{aligned} \tag{4}$$

Here in the passage from the second to the third line, smoothing error term $(\frac{d}{dt}+\partial\!\!\!/_E^2)H(t)$ contributes nothing to the trace. The residue in the last line is just the value at $z=0$ of the supertrace of the approximate heat kernel so we arrive at the

McKean-Singer formula

$$\mathrm{Ind}_a(\partial\!\!\!/_E^{+}) = \lim_{t\downarrow 0} \mathrm{Tr}_S(H(t)) \tag{5}$$

Index of projective spin Dirac operators

The local index theorem can be applied, thanks to the McKean-Singer formula, to obtain the **index theorem** for projective spin Dirac operators twisted by projective vector bundles.

Theorem (MMS). *The projective spin Dirac operator twisted by a unitary projective vector bundle E, has index*

$$\mathrm{Ind}_a(\not{\partial}_E^+) = \int_Z \widehat{A}(Z) \wedge \mathrm{Ch}_{\mathcal{A}}(E) \tag{6}$$

where $\mathrm{Ch}_{\mathcal{A}} : K^0(Z;\mathcal{A}) \longrightarrow H^{even}(Z;\mathbb{Q})$ *is the Chern character in twisted K-theory.*

Index of projective elliptic ΨDOs

The previous theorem together with the analogue of Bott periodicity in this context yields,

Theorem (MMS). *Given an Azumaya bundle, $\mathcal{A}$, over an even-dimensional compact manifold Z the analytic index defines a map*

$$\mathrm{Ind}_a : K^0_c(T^*Z; \pi^*\mathcal{A}) \longrightarrow \mathbb{Q} \tag{7}$$

where $\mathrm{Ind}_a(A) = \mathrm{Ind}_a(\sigma(A))$ for elliptic elements of $\Psi^\bullet_\epsilon(Z; E, F)$ for projective vector bundles associated to $\mathcal{A}$ and for all

$b \in K_c(T^*Z; \pi^*(\mathcal{A}))$, one has

$$\mathrm{Ind}_a(b) = \int_{T^*Z} \mathrm{Td}(T^*Z) \wedge \mathrm{Ch}_{\mathcal{A}}(b) \tag{8}$$

Fractions and the index formula

On an oriented even-dimensional manifold Z, the vanishing of w_2 is equivalent to the existence of a Spin structure. Nevertheless, there is always a **projective spin Dirac operator**. The previously discussed Index Theorem applied in this case case gives the usual formula

$$\mathrm{Ind}_a(\partial\!\!\!/^+) = \int_Z \widehat{A}(Z).$$

Recall that $Z = \mathbb{C}P^{2n}$ is an oriented but **non- spin** manifold and $\int_Z \widehat{A}(Z)$ is a **fraction**, justifying the title of the talk. e.g.

If $Z = \mathbb{C}P^2$, then $\mathrm{Ind}_a(\partial\!\!\!/^+) = -\frac{1}{8}$.
If $Z = \mathbb{C}P^4$, then $\mathrm{Ind}_a(\partial\!\!\!/^+) = \frac{3}{128}$.
In fact, if $Z = \mathbb{C}P^{2n}$, then $\mathrm{Ind}_a(\partial\!\!\!/^+) \notin \mathbb{Z}$.

Applications/observations

By the Atiyah-Singer index theorem, one easily sees that the **topological index** of the Dirac operator on a Spin manifold does not depend on the Spin structure. But is there a way to see this analytically?

Examples such as Riemann surfaces show that the dimension of the **nullspace** of the Dirac operator on a Spin manifold **does** in general depend on the choice of Spin structure.

However, it is clear that the **Schwartz kernel** of the Dirac operator on a Spin manifold **does not** depend on the choice of Spin structure.

In the almost complex case with Hermitian metric, we have the Spin$^{\mathbb{C}}$ Dirac operator

$$\overline{\partial} + \overline{\partial}^* : \Lambda^{0,even} Z \longrightarrow \Lambda^{0,odd} Z. \qquad (9)$$

Its index is $\int_Z \widehat{A}(Z) \wedge e^{\frac{1}{2}c_1}$ where $c_1 = c_1(Z)$ is the Chern class of the canonical line bundle.

We can show that $\overline{\partial} + \overline{\partial}^*$ is actually equal to the projective spin Dirac operator coupled to the square root of the canonical bundle (which is not a line bundle on Z, but rather, on the frame bundle of Z). Previously this interpretation was only possible when Z was itself spin, when this square root bundle exists as an ordinary line bundle on Z. Applies also to general Spin$^{\mathbb{C}}$ Dirac operators.