

Conformal Triality and *AdS/CFT*³ Correspondence

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Outline

- ◇ The Beltrami model of dS and AdS ;
proposal of special relativity on them
- ◇ Conformal Mink-, dS - and AdS -spaces from a projective subspace
- ◇ Conformal triality and AdS/CFT^3 correspondence
 CFT^3 : three kinds of CFT s.
- ◇ Null geodesics in conformal Mink-, dS - and AdS -spaces

1 The Beltrami Model of dS/AdS and Special Relativity on Them

◇ References.

- ▶ H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, “On Special Relativity with Cosmological Constant”, *Phys. Lett.* **A331** (2004) 1–7, hep-th/0403171.
- ▶ H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, “On Beltrami Model of de Sitter Spacetime”, *Mod. Phys. Lett.* **A19** (2004) 1701, hep-th/0311156;
- ▶ Basic ideas traced back to:
K. H. Look (Qi-Keng Lu), “Why the Minkowski metric must be used?”, (1970), unpublished.
- ▶ The Beltrami coordinates:
H. S. Snyder, *Phys. Rev.* **71** (1947) 38.
W. de Sitter, “Einstein’s theory of gravitation III”, *Roy. Astr. Soc. Month. Not.* **78** (1917) 3.
W. Pauli (’20), *Theory of Relativity*, Pergamon, 1958.

1.1 Why could there be a special relativity on dS/AdS ?

1. Analogy in geometry.

Euclidean geometry	SR on Minkowski
Lobachevskian geometry	SR on AdS ?
Riemann's spherical geometry	SR on dS ?

- ▶ Three kinds of SR from inverse Wick rotation — [hep-th/0508094](#).

2. Experiments and observations.

- ▶ SR: supported by [local](#) experiments;
- ▶ the cosmos: asymptotically dS .

[Isn't it a good thing to have a theory supported by all these experiments and observations?](#)

3. Different features of SR and GR.

(1) In SR:

- ▶ existence of inertial reference frames;
- ▶ with the principle of SR **related to them**;
- ▶ metric as a **background only**;
- ▶ spacetime symmetry: the Poincaré group;
- ▶ symmetry group of dynamics: the Poincaré group;
- ▶ the Minkowski coordinates \Leftrightarrow inertial reference frames;

...

(2) In GR:

- ▶ no inertial reference frames, even no global frames in general;
- ▶ the principle of GR;
- ▶ metric — **background** and **dynamical**;
- ▶ spacetime symmetry: generally, no;
- ▶ dynamics: diffeomorphic-invariant;
- ▶ coordinate independent;

...

(3) A question: how the features of GR turn out to be those of SR when the curvature is zero?

(4) The correct relation of SR and GR:

SR is a theory which describes a special solution of GR.

$$\lim_{\mathbf{R} \rightarrow 0} \text{GR} \neq \text{SR}. \quad (1)$$

Then why there could not be theories of “SR” to describe other spacetimes?

1.2 dS and Its Geometry

◇ Usually viewed as a hypersurface in $\mathcal{M}^{1,4}$:

$$\eta_{AB} \xi^A \xi^B = -R^2. \quad (2)$$

$$(\eta_{AB})_{A,B=0,1,\dots,4} = \text{diag}(1, -1, -1, -1, -1).$$

◇ Equivalently, $dS \cong \mathcal{S} / \sim$

▶ $\mathcal{S} \subset \mathcal{M}^{1,4}$: the set of spacelike vectors;

▶ \sim : the equivalence relation **similar to** that in projective geometry:

$$\xi' \sim \xi \iff \exists c > 0 \text{ s.t. } \xi'^A = c \xi^A. \quad (3)$$

▶ dS : the set of spacelike rays from the origin of $\mathcal{M}^{2,4}$.

▶ Why antipodal points not identified?

Orientable and time orientable.

◇ The Beltrami coordinates — inhomogeneous coordinates.

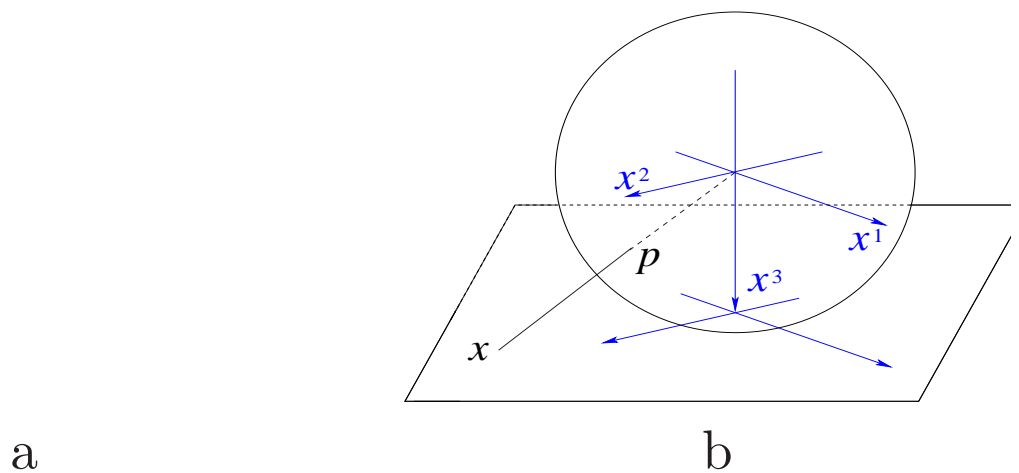
$$x^\mu := R \frac{\xi^\mu}{\xi^4}, \quad (\mu = 0, 1, 2, 3) \quad (4)$$

on the regions $U_{\pm 4}$ where $\xi^4 > 0$ or $\xi^4 < 0$.

▶ Other coordinate neighborhoods.

▶ $\sigma(x) > 0$ where

$$\sigma(x) := 1 - \frac{1}{R^2} \eta_{\mu\nu} x^\mu x^\nu. \quad (5)$$



◇ The $O(1, 4)$ transformations:

$$x'^{\mu} = \pm \frac{\sqrt{\sigma(a)} D^{\mu}_{\nu} (x^{\nu} - a^{\nu})}{\sigma(a, x)}, \quad (6)$$

$$D^{\mu}_{\nu} = L^{\mu}_{\nu} + \frac{R^{-2} L^{\mu}_{\rho} a^{\rho} a_{\nu}}{\sigma(a) + \sqrt{\sigma(a)}}, \quad (7)$$

$$\sigma(a, x) := 1 - \frac{1}{R^2} \eta_{\mu\nu} a^{\mu} x^{\nu}, \quad (8)$$
$$(L^{\mu}_{\nu}) \in O(1, 3), \quad \sigma(a) > 0$$

- ▶ Fractional linear.
- ▶ The same form as the $O(3)$ transformations on S^2 or $\mathbb{R}P^2$.
- ▶ Turn out to be Poincaré transformations up to the order of $1/R^2$.

◇ The $O(1, 4)$ -invariant metric:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \quad (9)$$

$$g_{\mu\nu}(x) := \frac{\eta_{\mu\nu}}{\sigma(x)} + \frac{\eta_{\mu\alpha} \eta_{\nu\beta} x^\alpha x^\beta}{R^2 \sigma(x)}. \quad (10)$$

- ▶ Induced from the hypersurface, or
- ▶ derived from the invariant cross ratio.
- ▶ The same form as that on S^2 or $\mathbb{R}P^2$.
- ▶ $O(1, 4)$ -invariant:

$$\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) = g_{\mu\nu}(x'). \quad (11)$$

- ▶ $g_{\mu\nu}(x) = \eta_{\mu\nu} + O(1/R^2)$

◇ Geodesics

▶ Geodesics are “projective” straight lines, and vice versa.

$$x^\mu = x_0^\mu + w(s) u^\mu, \quad (12)$$

$$\frac{dw}{ds} = \sigma(x) = 1 - \frac{1}{R^2} \eta_{\mu\nu} x^\mu x^\nu. \quad (13)$$

▶ $w \sim s$ if all $x^\mu \ll R$.

▶ Preserved quantities:

$$P^\mu = \frac{m}{\sigma(x)} \frac{dx^\mu}{ds}, \quad L^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu. \quad (14)$$

▶

$$\eta_{\mu\nu} P^\mu P^\nu - \frac{1}{2R^2} \eta_{\mu\rho} \eta_{\nu\sigma} L^{\mu\nu} L^{\rho\sigma} = m^2. \quad (15)$$

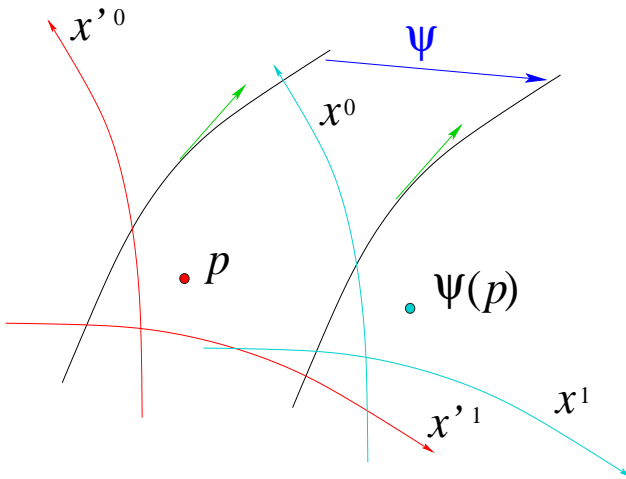
1.3 Proposal of SR on dS/AdS

- ▶ Method of projective geometry works, nearly without use of DG.
- ▶ Inertial motions and inertial reference frames (IRF) can be defined **intrinsically**.
- ▶ An IRF can be transformed to be another IRF by $O(1, 4)$ transformations.
- ▶ The Beltrami coordinates and the Minkowski coordinates cannot be distinguished in small scales.
- ▶ All the formulae can be degenerated to those in SR in small scales.
- ◇ If SR on dS/AdS is accepted, then **nonzero curvature does not necessarily mean gravitation**.

2 Conformal Transformations

◇ For a (M, \mathbf{g}) , $\psi : M \rightarrow M$ is a conformal transformation if

- (1) it is a diffeomorphism and
- (2) $\psi^* \mathbf{g} = \rho^2 \mathbf{g}$.



$x^\mu \rightarrow x'^\mu := \psi^* x^\mu = x'^\mu(x)$, then

$$\frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x') = \rho^2(x) g_{\mu\nu}(x), \quad (16)$$

$$ds'^2 = \rho^2(x) ds^2. \quad (17)$$

◇ The conformal Minkowski space: Special conformal transformations are not diffeomorphisms on the Minkowski space. Additional points (points at infinity, ideal points) must be added.

◇ Similar for conformal dS - and AdS -spaces.

3 Conformal dS/AdS -Spaces

◇ As hypersurfaces in $M^{1,4}$ and $M^{2,3}$, respectively,

$$H_{\theta}^{1,3} : \quad \eta_{\theta AB} \xi^A \xi^B = -\theta R^2. \quad (18)$$

$$\theta = \pm, \quad (\eta_{\theta AB})_{A,B=0,\dots,4} = \text{diag}(J, -\theta) \quad (19)$$

$$J = (\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1). \quad (20)$$

◇ In $\mathcal{M}^{2,4}$, if set

$$\zeta^A = \kappa \xi^A, \quad (A = 0, \dots, 4) \quad \zeta^5 = \kappa R, \quad (21)$$

then eq. (18) \Rightarrow

$$\mathcal{N}_{\theta} : \quad \eta_{\theta \hat{A}\hat{B}} \zeta^{\hat{A}} \zeta^{\hat{B}} = 0. \quad (22)$$

$$(\zeta^0, \dots, \zeta^5) \sim (\xi^0, \dots, \xi^4, R), \quad (23)$$

$$dS \cong H_+^{1,3} = \mathcal{N}_+ \cap \mathcal{P}, \quad (24)$$

$$AdS \cong H_-^{1,3} = \mathcal{N}_- \cap \mathcal{P}, \quad (25)$$

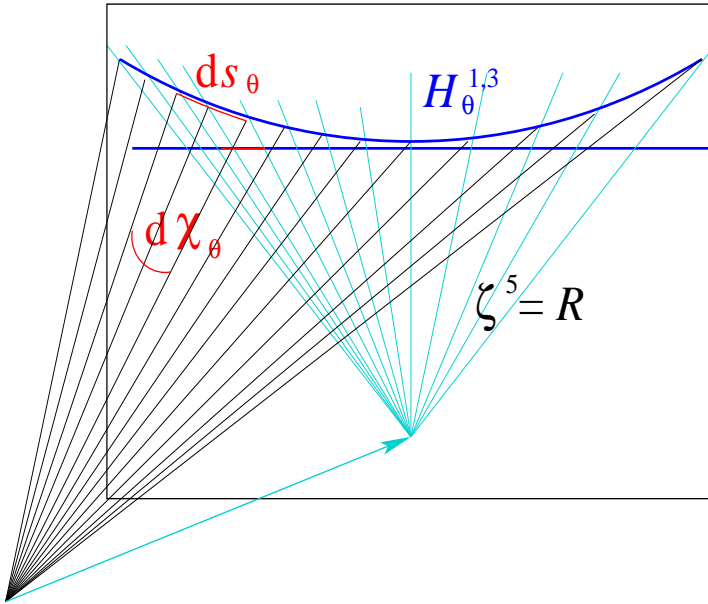
$$\mathcal{P} : \quad \zeta^5 = R. \quad (26)$$

◇ Induced metric:

$$d\chi_\theta^2 = \kappa^2 ds_\theta^2 = \left(\frac{\zeta^5}{R}\right)^2 ds_\theta^2, \quad (27)$$

where $d\chi_\theta^2 := \eta_{\hat{A}\hat{B}} d\zeta^{\hat{A}} d\zeta^{\hat{B}}, \quad (28)$

$$ds_\theta^2 := \eta_{AB} d\xi^A d\xi^B. \quad (29)$$



Each $O(2, 4)$ transformation induces a conformal transformation on $H_\theta^{1,3}$:

$$ds_\theta'^2 = \rho^2 ds_\theta^2, \quad \rho = \frac{\zeta^5}{\zeta'^5}. \quad (30)$$

◇ Conformal transformations on BdS and BAdS:

$$\zeta'^{\hat{A}} = C^{\hat{A}}_{\hat{B}} \zeta^{\hat{B}}, \quad (31)$$

$$\zeta'^{\mu} = C^{\mu}_{\nu} \zeta^{\nu} + C^{\mu}_4 \zeta^4 + C^{\mu}_5 \zeta^5, \quad (32)$$

$$\zeta'^4 = C^4_{\alpha} \zeta^{\alpha} + C^4_4 \zeta^4 + C^4_5 \zeta^5; \quad (33)$$

$$\zeta'^5 = C^5_{\alpha} \zeta^{\alpha} + C^5_4 \zeta^4 + C^5_5 \zeta^5. \quad (34)$$

$$x'^{\mu} = R \frac{\xi'^{\mu}}{\xi'^4} = R \frac{\zeta'^{\mu}}{\zeta'^4} \quad (35)$$

$$x'^{\mu} = \frac{C^{\mu}_{\nu} x^{\nu} + C^{\mu}_4 R \pm C^{\mu}_5 R \sqrt{\sigma(x)}}{C^4_4 + \frac{1}{R} C^4_{\alpha} x^{\alpha} \pm C^4_5 \sqrt{\sigma(x)}}, \quad (36)$$

$$\rho = \frac{\pm \sqrt{\sigma(x)}}{C^5_4 + \frac{1}{R} C^5_{\alpha} x^{\alpha} \pm C^5_5 \sqrt{\sigma(x)}}. \quad (37)$$

◇ Examples.

$$C = \begin{pmatrix} I & 0 & 0 \\ 0 & \gamma & \gamma\beta \\ 0 & \gamma\beta & \gamma \end{pmatrix}, \quad (38)$$

$$x'^{\mu} = \frac{x^{\mu} \sqrt{1 - \beta^2}}{1 \pm \beta \sqrt{\sigma(x)}}; \quad (39)$$

$$C = \begin{pmatrix} \delta_{\nu}^{\mu} - \frac{\theta}{R^2} \frac{b^{\mu} b_{\nu}}{1 + \sqrt{\sigma(b)}} & 0 & \frac{b^{\mu}}{R} \\ 0 & 1 & 0 \\ -\frac{b_{\nu}}{R} & 0 & \sqrt{\sigma(b)} \end{pmatrix}, \quad (40)$$

$$x'^{\mu} = x^{\mu} - \frac{\theta(b \cdot x)}{1 + \sqrt{\sigma(b)}} \frac{b^{\mu}}{R^2} \pm b^{\mu} \sqrt{\sigma(x)}. \quad (41)$$

4 Triality

4.1 Outline and review

- (1) A null-cone \mathcal{N} in $\mathcal{M}^{2,4}$: $\eta_{\hat{A}\hat{B}} \zeta^{\hat{A}} \zeta^{\hat{B}} = 0$.
(2) A 4-dim $[\mathcal{N}]$ is resulted in:

$$[\mathcal{N}] = \mathcal{N} - \{0\} / \sim, \quad [\mathcal{N}] \cong S^1 \times S^3.$$

- (3) An action of $O(2, 4)$ is induced on $[\mathcal{N}]$.
(4) The Minkowski space is $\mathcal{N} \cap \mathcal{P}$, with \mathcal{P} a null hyperplane

$$\zeta^- := \frac{1}{\sqrt{2}} (-\zeta^4 + \zeta^5) = R.$$

- (5) dS -space is $\mathcal{N} \cap \mathcal{P}$, with \mathcal{P} a hyperplane $\zeta^5 = R$. The normal vector of \mathcal{P} is timelike.
(6) For AdS -space, the normal vector of \mathcal{P} is spacelike.
(7) Relation of metrics: For Minkowski and dS/AdS , respectively,

$$d\chi_M^2 = \left(\frac{\zeta^-}{R} \right)^2 ds_M^2, \quad d\chi_M^2 = \kappa^2 ds_\theta^2.$$

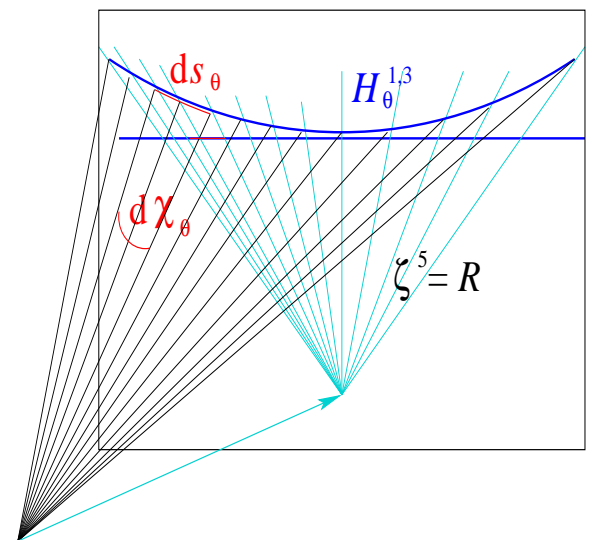
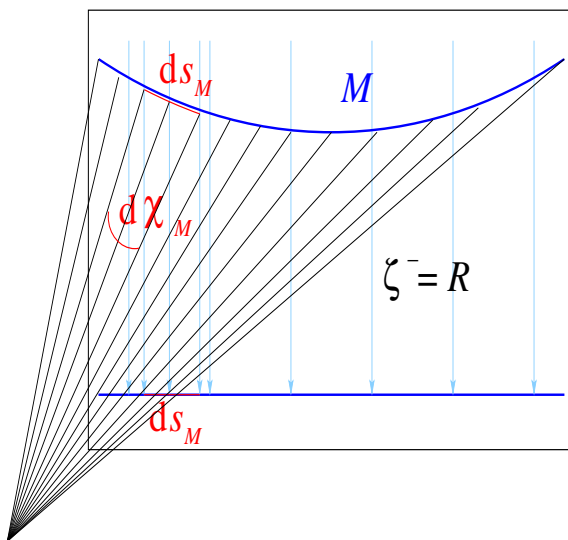
The induced $O(2, 4)$ transformations on $\mathcal{N} \cap \mathcal{P}$ are conformal:

$$\left(\frac{\zeta'^{-}}{R}\right)^2 ds_M'^2 = d\chi_M'^2 \equiv d\chi_M^2 = \left(\frac{\zeta^-}{R}\right)^2 ds_M^2 \Rightarrow$$

$$ds_M'^2 = \rho^2 ds_M^2, \quad \rho := \frac{\zeta^-}{\zeta'^{-}};$$

$$\kappa'^2 ds_\theta'^2 = d\chi_\theta'^2 \equiv d\chi_\theta^2 = \kappa^2 ds_\theta^2, \quad \Rightarrow$$

$$ds_\theta'^2 = \rho^2 ds_\theta^2, \quad \rho := \frac{\kappa}{\kappa'} = \frac{\zeta^5}{\zeta'^5}.$$



(8) Transformation law in terms of the Minkowski or Beltrami coordinates.

4.2 A generic description

(1) \mathcal{P} : a hyperplane, not passing through the origin, **orientation** induced.

\mathbf{n} : a normal vector of \mathcal{P} .

Items	Minkowski	dS	AdS
\mathbf{n}	null	timelike	spacelike
\mathcal{P}	null	spacelike	timelike
signature	0, +, -, -, -	1 + 4	2 + 3
$\mathcal{N} \cap \mathcal{P}$	Minkowski	dS	AdS
status of \mathbf{n}	up to a scalar	determined	determined

(2) For null \mathcal{P} ,
 \mathbf{l} : a null vector pointing to \mathcal{P} , and $\mathbf{l} \cdot \mathbf{n} = 1$. (**n determined by l.**)
 \mathbf{e}_μ ($\mu = 0, 1, 2, 3$): orthonormal, tangent to \mathcal{P} ,

$$\mathbf{l} \cdot \mathbf{e}_\mu = \mathbf{n} \cdot \mathbf{e}_\mu = 0. \quad (42)$$

$\{\mathbf{e}_\mu, \mathbf{n}, \mathbf{l}\}$ — orientation. \mathbf{e}_0 future pointed.

♣ $\zeta \in \mathcal{P}$ can be expressed as

$$\zeta = x^\mu \mathbf{e}_\mu + x^+ \mathbf{n} + R \mathbf{l}, \quad \zeta \in \mathcal{N} \cap \mathcal{P} \Leftrightarrow x^+ = -\frac{1}{2R} \eta_{\mu\nu} x^\mu x^\nu. \quad (43)$$

♣ $C \in O(2, 4)$ transforms $\zeta \in \mathcal{N} \cap \mathcal{P}$ to

$$\zeta' = x'^\mu \mathbf{e}_\mu + x'^+ \mathbf{n} + R \mathbf{l} = \rho C \zeta. \quad (44)$$

$$ds_M'^2 = \rho^2 ds_M^2. \quad (45)$$

♣ $\{\mathcal{P}, -\mathcal{P}\}$ C -invariant $\Leftrightarrow C$ induces a Poincaré transformation on $\mathcal{N} \cap \mathcal{P}$.

♣ Change of \mathbf{l} induces a Poincaré coordinate transformation:

$$R \mathbf{l}' = a^\mu \mathbf{e}_\mu + a^+ \mathbf{n} + R \mathbf{l} \quad \Rightarrow \quad (46)$$

$$\mathbf{e}'_\mu = \mathbf{e}_\mu - \frac{\eta_{\mu\nu}}{R} a^\nu \mathbf{n}, \quad x'^\mu = x^\mu - a^\mu. \quad (47)$$

(3) For a spacelike \mathcal{P} : ($\mathcal{N} \cap \mathcal{P} \cong dS$)
 $\{\mathbf{e}_A, \mathbf{n} \mid A = 0, 1, \dots, 4\}$ — oriented orthonormal basis,
 \mathbf{n} — unit timelike normal vector pointing to \mathcal{P} ,
 \mathbf{e}_0 — future pointed.

♣ $\zeta \in \mathcal{P}$:

$$\zeta = \xi^A \mathbf{e}_A + R \mathbf{n}, \quad \zeta \in \mathcal{N} \cap \mathcal{P} \quad \Leftrightarrow \quad \eta_{AB} \xi^A \xi^B = -R^2. \quad (48)$$

♣ Beltrami coordinates can be defined. E.g.,

$$x^\mu := R \frac{\xi^\mu}{\xi^4}. \quad \Rightarrow \quad \xi^4 = \pm \frac{R}{\sqrt{\sigma(x)}}. \quad (49)$$

$$\sigma(x) > 0, \quad \sigma(x) := 1 - \frac{1}{R^2} \eta_{\mu\nu} x^\mu x^\nu. \quad (50)$$

♣ $C \in O(2, 4)$ transforms $\zeta \in \mathcal{N} \cap \mathcal{P}$ to

$$\zeta' = \xi'^A \mathbf{e}_A + R \mathbf{n} = \rho C \zeta, \quad (51)$$

$$ds'_+{}^2 = \rho^2 ds_+{}^2. \quad (52)$$

♣ $\{\mathcal{P}, -\mathcal{P}\}$ C -invariant $\Leftrightarrow C$ induces a de Sitter transformation on $\mathcal{N} \cap \mathcal{P}$.

(4) For a timelike \mathcal{P} : ($\mathcal{N} \cap \mathcal{P} \cong AdS$)
 $\{\mathbf{e}_\mu, \mathbf{n}, \mathbf{e}_4\}$ — oriented orthonormal basis,
 \mathbf{n} — unit spacelike normal vector pointing to \mathcal{P} ,
 \mathbf{e}_0 — future pointed.

♣ $\zeta \in \mathcal{P}$:

$$\zeta = \xi^A \mathbf{e}_A + R \mathbf{n}, \quad \zeta \in \mathcal{N} \cap \mathcal{P} \Rightarrow \eta_{AB} \xi^A \xi^B = R^2. \quad (53)$$

♣ Beltrami-Hua-Lu coordinates can be defined. E.g.,

$$x^\mu := R \frac{\xi^\mu}{\xi^4}. \quad \Rightarrow \quad \xi^4 = \pm \frac{R}{\sqrt{\sigma(x)}}. \quad (54)$$

$$\sigma(x) > 0, \quad \sigma(x) := 1 + \frac{1}{R^2} \eta_{\mu\nu} x^\mu x^\nu. \quad (55)$$

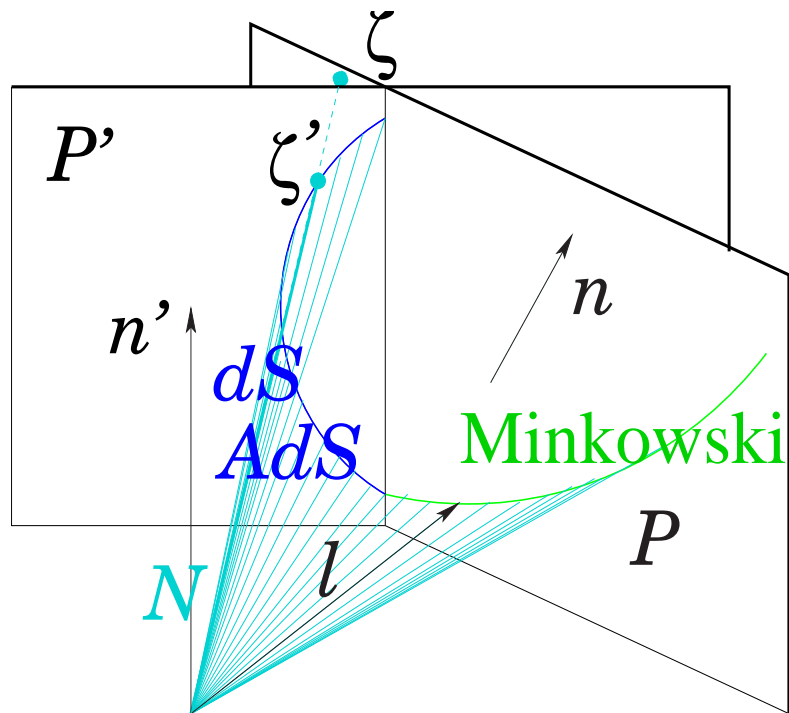
♣ $C \in O(2, 4)$ transforms $\zeta \in \mathcal{N} \cap \mathcal{P}$ to

$$\zeta' = \xi'^A \mathbf{e}_A + R \mathbf{n} = \rho C \zeta, \quad (56)$$

$$ds'^2 = \rho^2 ds^2. \quad (57)$$

♣ $\{\mathcal{P}, -\mathcal{P}\}$ is C -invariant $\Leftrightarrow C$ induces an anti-de Sitter transformation on $\mathcal{N} \cap \mathcal{P}$.

4.3 Conformal maps between conformal Minkowski, dS and AdS



(1) Conformal map from the Minkowski to dS/AdS .
 \mathcal{P} : null, $\mathcal{N} \cap \mathcal{P}$ Minkowski,
 \mathcal{P}' : spacelike or timelike, $\mathcal{N} \cap \mathcal{P}' \cong dS$ or AdS , respectively.

$$\clubsuit \zeta = x^\mu \mathbf{e}_\mu + x^+ \mathbf{n} + R \mathbf{l} \in \mathcal{N} \cap \mathcal{P} \quad \sim$$

$$\zeta' = \xi^A \mathbf{e}'_A + R' \mathbf{n}' \in \mathcal{N} \cap \mathcal{P}',$$

$$\zeta' = \rho \zeta \quad \Rightarrow$$

$$x'^{\mu} = R' \frac{C^{\mu}_{\nu} x^{\nu} - \frac{1}{2^{\frac{3}{2}} R} (C^{\mu}_4 + C^{\mu}_5) (x \cdot x) - \frac{1}{\sqrt{2}} (C^{\mu}_4 - C^{\mu}_5)}{C^4_{\alpha} x^{\alpha} - \frac{1}{2^{\frac{3}{2}} R} (C^4_4 + C^4_5) (x \cdot x) - \frac{1}{\sqrt{2}} (C^4_4 - C^4_5)} \quad (58)$$

$$\rho = \frac{R'}{C^5_{\alpha} x^{\alpha} - \frac{1}{2^{\frac{3}{2}} R} (C^5_4 + C^5_5) (x \cdot x) - \frac{1}{\sqrt{2}} (C^5_4 - C^5_5) R}, \quad (59)$$

$$ds'^2_{\theta} = \rho^2 ds^2_M, \quad \theta = \pm 1. \quad (60)$$

♣ A special example.

$$x'^{\mu} = -\frac{x^{\mu}}{1 + \frac{1}{4R'^2} (x \cdot x)}, \quad ds'^2_{\theta} = \frac{ds^2_M}{[1 - \frac{1}{4R'^2} (x \cdot x)]^2}. \quad (61)$$

(2) The conformal map from dS to AdS .

♣ Generic transformation.

$$x'^{\mu} = R' \frac{C^{\mu}_{\nu} x^{\nu} + C^{\mu}_4 R \pm C^{\mu}_5 R \sqrt{\sigma(x)}}{C^4_{\alpha} x^{\alpha} + C^4_4 R \pm C^4_5 R \sqrt{\sigma(x)}}, \quad (62)$$

$$\rho = \frac{\pm R' \sqrt{\sigma(x)}}{C^5_{\alpha} x^{\alpha} + C^5_4 R \pm C^5_5 R \sqrt{\sigma(x)}}, \quad (63)$$

$$ds'^2_{-} = \rho^2 ds^2_{+}. \quad (64)$$

♣ A special example:

$$x'^{\mu} = \pm \frac{R'}{R} \frac{x^{\mu}}{\sqrt{\sigma(x)}}, \quad \rho = \pm \frac{R'}{R} \sqrt{\sigma(x)}. \quad (65)$$

The inverse map:

$$x^{\mu} = \pm \frac{R}{R'} \frac{x'^{\mu}}{\sqrt{\sigma'(x')}}, \quad (66)$$

$$\rho' = \pm \frac{R}{R'} \sqrt{\sigma'(x')}. \quad (67)$$

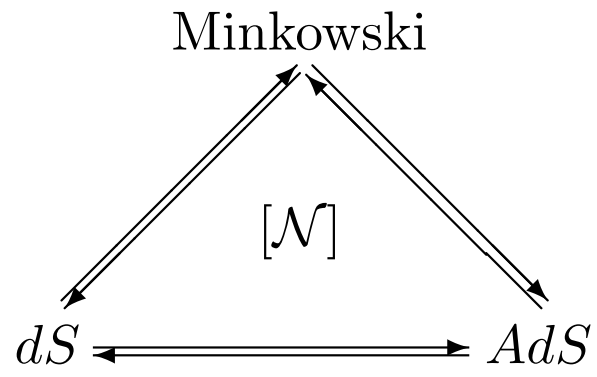
$$\sigma'(x') \sigma(x) = 1, \quad (68)$$



$$\sigma'(x') = 1 + \frac{1}{R'^2} \eta_{\mu\nu} x'^{\mu} x'^{\nu}, \quad (69)$$

$$\sigma(x) = 1 - \frac{1}{R^2} \eta_{\mu\nu} x^{\mu} x^{\nu}. \quad (70)$$

◇ Triality.



5 Null Geodesics, Massive Particles

◇ Null geodesics are conformally transformed to each other.

Timelike or spacelike geodesics cannot be.

Is there a clear picture?

◇ Momentum and angular momentum as conserved quantities in inertial motions. What if under conformal transformations/maps?

◇ For two linear independent $\zeta_0, \zeta_1 \in \mathcal{N}$,

$$[\zeta_0] \neq [\zeta_1]. \quad \Rightarrow$$

A 2-dim linear supspace Σ spanned by ζ_0 and ζ_1 , or a projective straight line in $\mathcal{M}^{2,4} - \{0\} / \sim$.

Is it in $[\mathcal{N}]$?

♣ $\zeta_0, \zeta_1 \in \mathcal{N} \Rightarrow \zeta_0 \cdot \zeta_0 = 0, \zeta_1 \cdot \zeta_1 = 0$.

$\forall \zeta \in \Sigma, \zeta = x\zeta_0 + y\zeta_1$. Since $\zeta \cdot \zeta = 2xy \zeta_0 \cdot \zeta_1$,

$$\Sigma \subset \mathcal{N} \Leftrightarrow \zeta_0 \cdot \zeta_1 = 0 \Leftrightarrow$$

$\Sigma \cap \mathcal{P}$ is a straight line in $\mathcal{N} \cap \mathcal{P}$.

◇ The equation of $\Sigma \cap \mathcal{P}$: Assume $\zeta_0, \zeta_1 \in \mathcal{P}$,

$$\zeta = (1 - t) \zeta_0 + t \zeta_1. \quad (71)$$

In terms of the Minkowski/Beltrami coordinates:

$$x^\mu = a^\mu + w(s) u^\mu, \quad \frac{ds}{dw} = \frac{1}{\sigma(x)}. \quad (72)$$

Conserved momentum and angular momentum along geodesics:

$$P^\mu := \frac{1}{\sigma(x)} \frac{dx^\mu}{ds}, \quad L^{\mu\nu} := x^\mu P^\nu - x^\nu P^\mu. \quad (73)$$

For BdS ,

$$\mathcal{L}^{\mu\nu} = L^{\mu\nu}, \quad \mathcal{L}^{4\nu} = -\mathcal{L}^{\nu 4} = RP^\nu. \quad (74)$$

◇ For Σ , define $\mathcal{L} = \zeta_0 \wedge \zeta_1$:

$$\mathcal{L}^{\hat{A}\hat{B}} := \zeta_0^{\hat{A}} \zeta_1^{\hat{B}} - \zeta_0^{\hat{B}} \zeta_1^{\hat{A}}. \quad (75)$$

When $\{\zeta_0, \zeta_1\} \rightarrow \{\zeta'_0, \zeta'_1\}$, $\zeta'_i = A^j{}_i \zeta_j$,

$$\mathcal{L}' = (\det A) \mathcal{L}. \quad (76)$$

Σ is specified by $\mathcal{L}^{\hat{A}\hat{B}}$ up to a nonzero scaling constant.

♣ When $\zeta_0, \zeta_1 \in \mathcal{P}$, $\Sigma \cap \mathcal{P}$ is

$$\zeta(t) = (1-t)\zeta_0 + t\zeta_1, \quad \frac{d\zeta}{dt} = \zeta_1 - \zeta_0.$$

\mathcal{L} is the conserved angular momentum:

$$\mathcal{L} = \zeta(t) \wedge \frac{d\zeta}{dt}. \quad (77)$$

♣ When $\zeta_0 \cdot \zeta_1 = 0$, $\Sigma \cap \mathcal{P}$ is a null geodesic in $\mathcal{N} \cap \mathcal{P}$.

♣ If $\mathcal{N} \cap \mathcal{P}$ is Minkowski, with \mathcal{P} being $\zeta^- = R$,

$$x^\mu(t) = x_0^\mu + tv^\mu, \quad v^\mu = x_1^\mu - x_0^\mu. \quad (78)$$

Define

$$L^{\mu\nu} := x^\mu(t)v^\nu - x^\nu(t)v^\mu, \quad P^\mu = v^\mu. \quad (79)$$

Then

$$\mathcal{L}^{\mu\nu} = L^{\mu\nu}, \quad (80)$$

$$\mathcal{L}^{-\mu} = RP^\mu, \quad (81)$$

$$\mathcal{L}^{+\mu} = -\frac{\eta_{\rho\sigma}}{2R} (x_0^\rho x_0^\sigma v^\mu - 2x_0^\rho v^\sigma x_0^\mu), \quad (82)$$

$$\mathcal{L}^{+-} = \eta_{\mu\nu} x_0^\mu v^\nu. \quad (83)$$

♣ When $\mathcal{N} \cap \mathcal{P}$ is dS/AdS ,

$$x^\mu(t) = x_0^\mu + w(t)v^\mu, \quad v^\mu := x_1^\mu - x_0^\mu, \quad (84)$$

$$w(t) := \frac{t\xi_1^4}{\xi_0^4 + t(\xi_1^4 - \xi_0^4)}. \quad (85)$$

$$P^\mu := \frac{1}{\sigma(x)} \frac{dx^\mu}{dt} = v^\mu, \quad L^{\mu\nu} := x^\mu P^\nu - x^\nu P^\mu. \quad (86)$$

$$\mathcal{L}^{\mu\nu} = \frac{L^{\mu\nu}}{\sqrt{\sigma(x_0)\sigma(x_1)}}, \quad \mathcal{L}^{5\nu} = \pm \left(\frac{x_1^\nu}{\sqrt{\sigma(x_1)}} - \frac{x_0^\nu}{\sqrt{\sigma(x_0)}} \right), \quad (87)$$

$$\mathcal{L}^{4\nu} = \frac{RP^\nu}{\sqrt{\sigma(x_0)\sigma(x_1)}}, \quad \mathcal{L}^{45} = \mp \left(\frac{R^2}{\sqrt{\sigma(x_1)}} - \frac{R^2}{\sqrt{\sigma(x_0)}} \right). \quad (88)$$

◇ Massive case.

(1) Δ : $(1 + 2)$ -dim or $(2 + 1)$ -dim.

(2) $(2 + 1)$ -dim: timelike straight line ($\mathbf{n} \in \Delta$) or hyperbola.

(3) $(1 + 2)$ -dim: spacelike straight line ($\mathbf{n} \in \Delta$) or conic curve.

6 AdS_5/CFT^3 Correspondence

- ◇ AdS_5 : a “projective” subspace
- ◇ $[\mathcal{N}] = \partial AdS_5$
- ◇ $[\mathcal{N}]$: the conformal Minkowski, dS - or AdS -space.
- ◇ Conformal maps, conformal triality.
- ◇ CFT on Minkowski = CFT on dS = CFT on AdS .
- ◇ Three kinds of AdS_5/CFT correspondence.
- ◇ Can be generalized to AdS_{d+1}/CFT_d^3 correspondence.

7 Conclusion

- ◇ Conformal Minkowski, dS - and AdS -spaces are the same thing.
- ◇ They are three landscapes from different point of view.
- ◇ They can be linked mutually to each other and to themselves by conformal maps.
- ◇ Minkowski coordinates and Beltrami coordinates are at the equal footing.
- ◇ Physical contents can be moved around from these spaces, provided they are conformally “covariant”.
- ◇ Projective might be “dead” in mathematics, but it is finding a new life in physics.

Thanks!