

3D integrable lattice models and the Bazhanov-Stroganov model

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XXIII DGMPN Nankai, Aug.22, 2005

joint work with S. Pakuliak, S. Sergeev

J.Phys.A36 (2003) 975–998; Int.J.Mod.Phys.A19 (2004) 179–204;

J.Phys.A37 (2004) 917–935; J.Phys.A38 (2005) 7269–7298.

Many 2D integrable lattice spin models known: Ising, XXZ, RSOS, etc.
Systematic construction of solutions of Y-B equations: quantum groups.

Not many integrable 3D models: almost all generalizations of \mathbb{Z}_N -
Zamolodchikov-Baxter-Bazhanov (ZBB) model A.B.Z. 1981, B-B 1992

Tetrahedron eqs. (3D analog of Y-B-eqs.) very restrictive, no analog to
quantum group construction of solutions known.

Baxter-Bazhanov (1992) recognized intimate relation of ZBB model to
2D integrable chiral Potts model.

Elegant formulation of generalized BZZ-model:

Vertex formulation in terms of Sergeev mapping, satisfying modified
tetrahedron equation (MTE).

Mangazeev, Stroganov 1993, Sergeev, Mangazeev, Stroganov 1995, Sergeev 1999

Aim of this talk: Use 3D Sergeev mapping to find new results for 2D
BS-model.

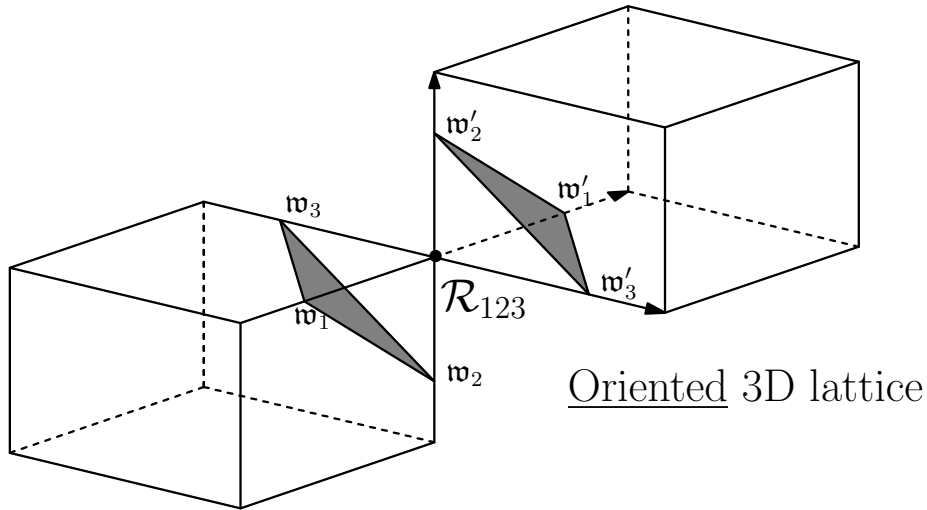
1. ZBB-model, Sergeev mapping
2. BS-model
3. BS-L-operator from Sergeev Linear Problem
4. BS-S-operator from 3D mapping \mathbf{R}
5. Intertwining of classical BS-L-operator from 3D $\mathcal{R}^{(f)}$
6. Isospectral transformation of BS-L-operator

1 Sergeev formulation of the 3D vertex ZBB-model

Quantum variables: elements of ultralocal Weyl algebra at root of unity:

$$\mathbf{u}_j \cdot \mathbf{w}_j = \omega \mathbf{w}_j \cdot \mathbf{u}_j; \quad \omega = e^{2\pi i/N}; \quad \mathbf{u}_i \cdot \mathbf{w}_j = \mathbf{w}_j \cdot \mathbf{u}_i \text{ for } i \neq j.$$

Attach also scalar κ_j to each link, together: $\mathfrak{w}_j = (\mathbf{u}_j, \mathbf{w}_j, \kappa_j)$.

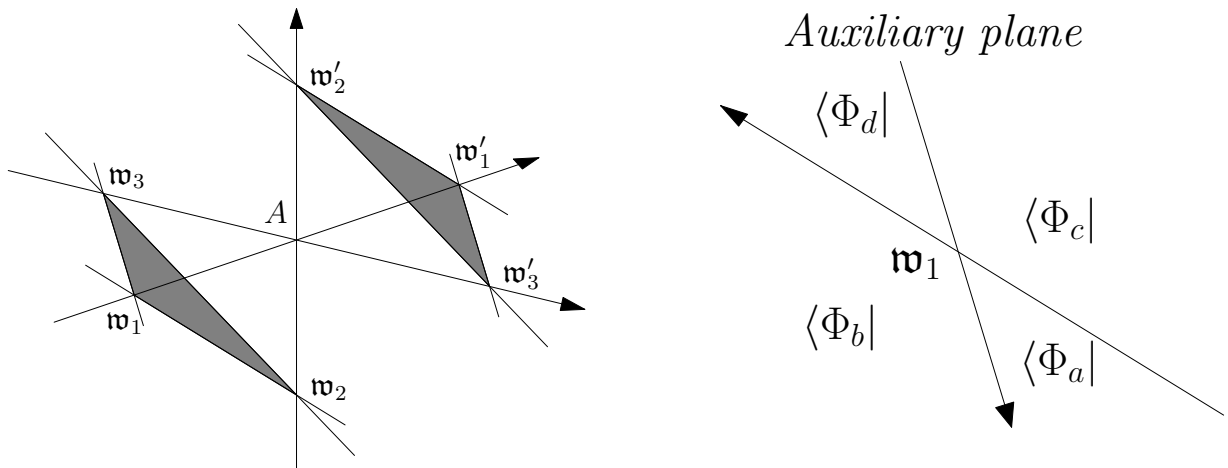


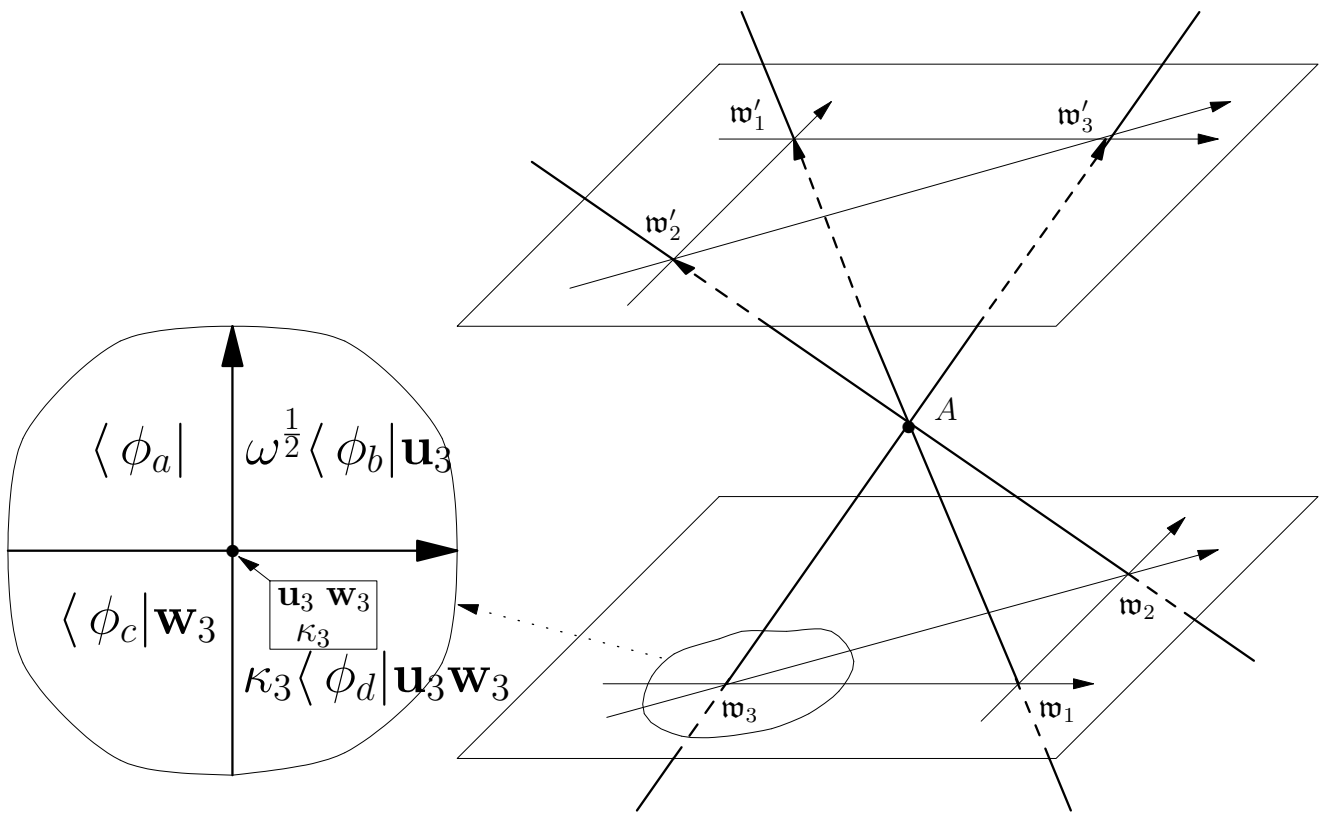
Key object: Canonical invertible rational mapping operator \mathcal{R}_{123}

$$(\mathcal{R}_{123} \circ \Psi)(\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \dots, \mathbf{w}_3) = \Psi(\mathbf{u}'_1, \mathbf{w}'_1, \mathbf{u}'_2, \dots, \mathbf{w}'_3).$$

\mathcal{R}_{123} is uniquely determined from postulates:

- 1: Baxter Z-invariance (lines may be shifted respect to each other)
- 2: Linear Problem: $0 = \langle \Phi_a | + \omega^{1/2} \langle \Phi_b | \mathbf{u}_1 + \langle \Phi_c | \mathbf{w}_1 + \kappa_1 \langle \Phi_d | \mathbf{u}_1 \mathbf{w}_1$.



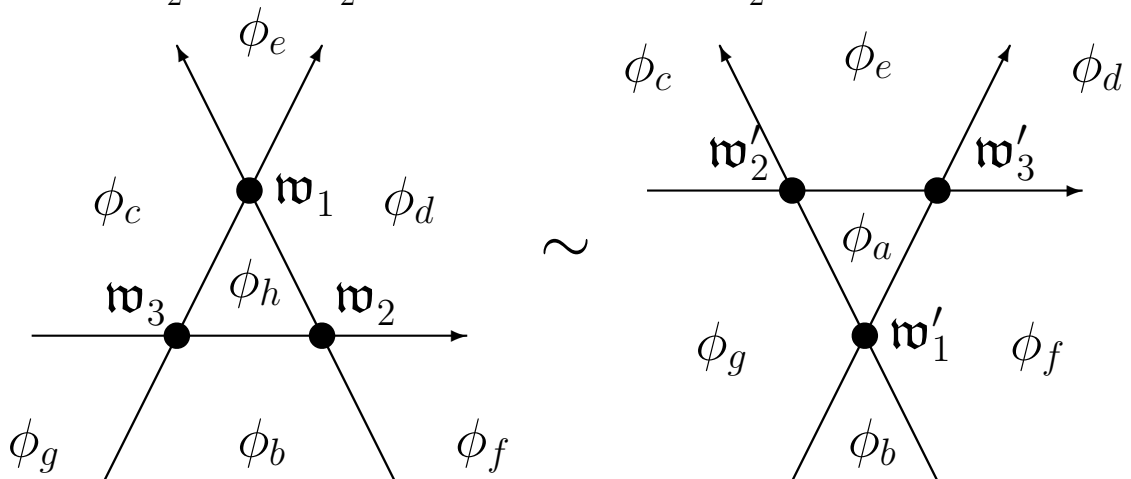


Linear problem: 2D-analog to QIS L -operator relation

$$0 = \langle \Phi^{(k)} | L^{(k)}(x) - \langle \Phi^{(k+1)} | \ell^{(k)}.$$

$\nabla = \Delta$ gives six equations: eliminate co-currents obtain unique solution:

$$\mathbf{u}'_1{}^{-1} = \frac{\kappa_1}{\kappa_2} \mathbf{u}_2^{-1} + \frac{\kappa_3}{\kappa_2} \mathbf{u}_1^{-1} \mathbf{w}_2^{-1} \mathbf{w}_3 - \omega^{1/2} \frac{\kappa_1 \kappa_3}{\kappa_2} \mathbf{u}_2^{-1} \mathbf{w}_2^{-1} \mathbf{w}_3, \text{ etc.}$$



Rational mapping \mathcal{R}_{123} looks complicated in terms of Weyl operators:

$$\begin{aligned}\kappa_2 \mathbf{u}'_1{}^{-1} &= \kappa_1 \mathbf{u}_2^{-1} + \kappa_3 \mathbf{u}_1^{-1} \mathbf{w}_2^{-1} \mathbf{w}_3 - \omega^{1/2} \kappa_1 \kappa_3 \mathbf{u}_2^{-1} \mathbf{w}_2^{-1} \mathbf{w}_3, \\ \mathbf{u}'_2{}^{-1} &= \mathbf{u}_1^{-1} - \omega^{1/2} \mathbf{u}_1^{-1} \mathbf{w}_1 \mathbf{u}_3^{-1} + \kappa_1 \mathbf{w}_1 \mathbf{u}_2^{-1} \mathbf{u}_3^{-1}; \\ \mathbf{w}'_1 &= \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3^{-1} - \omega^{1/2} \mathbf{w}_2 \mathbf{u}_3 \mathbf{w}_3^{-1} + \kappa_3 \mathbf{u}_3 \\ \mathbf{w}'_1 \mathbf{w}'_2 &= \mathbf{w}_2 \mathbf{w}_1; \quad \mathbf{u}'_3 \mathbf{u}'_2 = \mathbf{u}_2 \mathbf{u}_3; \quad \mathbf{w}'_3{}^{-1} \mathbf{u}'_1 = \mathbf{u}_1 \mathbf{w}_3^{-1}.\end{aligned}$$

At root of unity, affine Weyl operators have $N \times N$ -matrix representation:

$$\begin{aligned}\mathbf{u} &\equiv u \mathbf{X}; & \mathbf{w} &\equiv w \mathbf{Z}; & u, w &\in \mathbb{C}; & \mathbf{X} \mathbf{Z} &= \omega \mathbf{Z} \mathbf{X} \\ \mathbf{X} |\beta\rangle &= \omega^\beta |\beta\rangle; & \mathbf{Z} |\beta\rangle &= |\beta+1\rangle; & \langle \alpha | \beta \rangle &= \delta_{\alpha, \beta}; & \mathbf{X}^N &= \mathbf{Z}^N = 1.\end{aligned}$$

N -th powers of Weyl are centers: $\mathbf{u}_j^N = u_j^N \equiv U_j$; $\mathbf{w}_j^N = w_j^N \equiv W_j$.

Now if we take N -th powers of the mapping relations:

$$(\mathbf{u} + \mathbf{w})^N = U + W,$$

\mathcal{R}_{123} induces *functional mapping* $\mathcal{R}_{123}^{(f)}$ of the centers ($K_j \equiv \kappa_j^N$):

$$\begin{aligned}\frac{U'_1}{U_1} &= \frac{W'_3}{W_3} = \frac{K_2 U_2 W_2}{K_1 U_1 W_2 + K_3 U_2 W_3 + K_1 K_3 U_1 W_3}; \\ \frac{W_1}{W'_1} &= \frac{W'_2}{W_2} = \frac{W_1 W_3}{W_1 W_2 + U_3 W_2 + K_3 U_3 W_3}; \\ \frac{U'_2}{U_2} &= \frac{U_3}{U'_3} = \frac{U_1 U_3}{U_2 U_3 + U_2 W_1 + K_1 U_1 W_1}.\end{aligned}$$

Remarkable feature: \mathcal{R}_{123} decomposes into a matrix conjugation \mathbf{R}_{123}

(\mathbf{R}_{123} is $N^3 \times N^3$ -matrix) and $\mathcal{R}_{123}^{(f)}$: (Baxter, Bazhanov, Reshetikhin, Bobenko, Sergeev, Mangazeev, Stroganov 1995)

$$\mathcal{R}_{123} \circ \Psi = \mathbf{R}_{123} \left(\mathcal{R}_{123}^{(f)} \circ \Psi \right) \mathbf{R}_{123}^{-1}.$$

\mathbf{R}_{123} has compact expression in of root-of-unity q -Gamma-functions $w_p(n)$:

$$\mathbf{R}_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \delta_{i_2+i_3, j_2+j_3} \omega^{(j_1-i_1)j_3} \frac{w_{p_1}(i_2 - i_1) w_{p_2}(j_2 - j_1)}{w_{p_3}(j_2 - i_1) w_{p_4}(i_2 - j_1)}.$$

Each $w_p(n)$ ($n \in \mathbb{Z}_N$) depends on point $p = (x, y)$ on Fermat curve

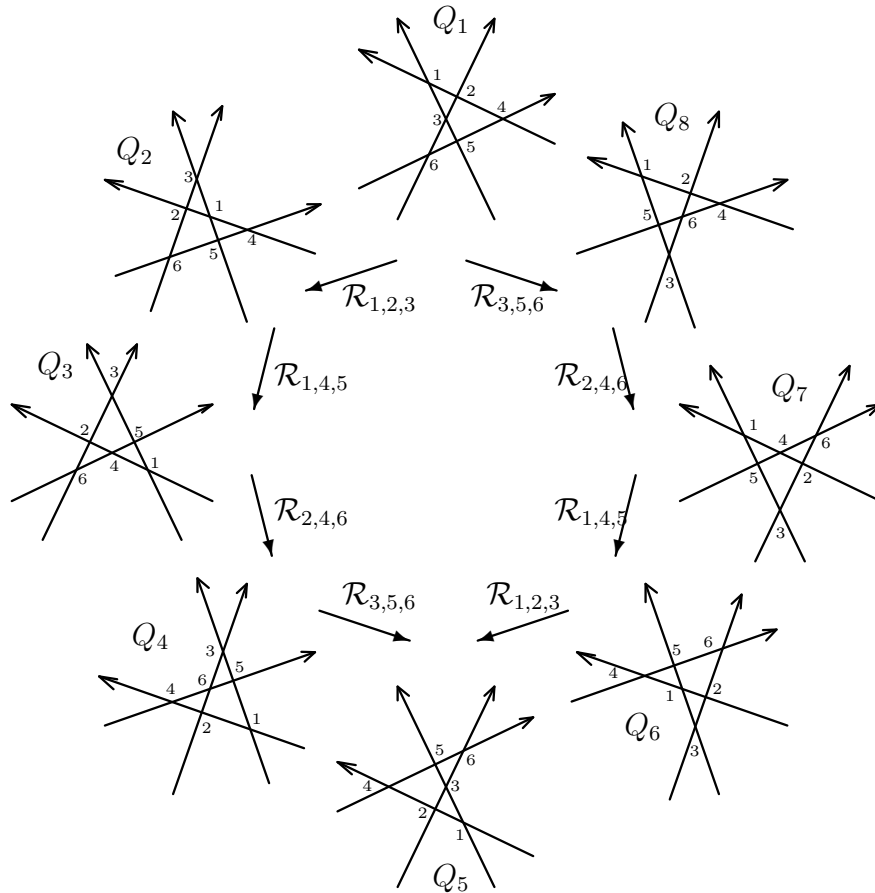
$$x^N + y^N = 1 : \\ w_p(n) = \prod_{j=1}^n \frac{y}{1 - \omega^j x}, \quad w_p(0) = 1; \quad w_p(n + N) = w_p(n).$$

\mathbf{R}_{123} is a weighted cross-ratio of four w_{p_i} , only three Fermat points are independent: $x_1 x_2 = \omega x_3 x_4$.

The Fermat points are related to initial and final Weyl variables $\mathfrak{w}_j, \mathfrak{w}'_j$ by

$$x_1 = \frac{1}{\omega^{1/2}} \frac{u_2}{\kappa_1 u_1}; \quad x_2 = \frac{1}{\omega^{1/2}} \frac{\kappa_2 u'_2}{u'_1}; \quad x_3 = \frac{1}{\omega} \frac{u'_2}{u_1}; \quad \frac{y_3}{y_1} = \frac{\kappa_1 w_1}{u'_3}.$$

The Tetrahedron equation follows *without calculation* considering four successive transformations \mathcal{R}_{ijk} of quadrangles Q_m in different order:



$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} = \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123}.$$

6 spaces, each \mathcal{R}_{ijk} depends on 3 param., there are 5 different param. in the TE

2 The Bazhanov-Stroganov model

Bazhanov-Stroganov 1990 (Korepanov 1985): standard 6-v R -matrix

$$R(\lambda, \nu) = \begin{pmatrix} \lambda - \omega\nu & 0 & 0 & 0 \\ 0 & \omega(\lambda - \nu) & \lambda(1 - \omega) & 0 \\ 0 & \nu(1 - \omega) & \lambda - \nu & 0 \\ 0 & 0 & 0 & \lambda - \omega\nu \end{pmatrix}.$$

intertwines not only the 6-vertex L -operators, but at $\omega = e^{2\pi i/N}$ also

$$L(\lambda, \mathbf{a}) = \begin{pmatrix} 1 + \lambda b_1 \mathbf{Z}; & \lambda \mathbf{X}^{-1}(a_1 - b_2 \mathbf{Z}) \\ \mathbf{X}(a_2 - b_3 \mathbf{Z}); & \lambda a_1 a_2 + b_2 b_3 b_1^{-1} \mathbf{Z} \end{pmatrix}; \quad \mathbf{X} \mathbf{Z} = \omega \mathbf{Z} \mathbf{X},$$

with entries \mathbf{X}, \mathbf{Z} : Weyl elements, $\mathbf{a} = (a_1, \dots, b_3) \in \mathbb{C}^5$.

Their new intertwining relation is:

$$\begin{aligned} \sum_{j_1, j_2, \beta} R_{i_1 j_1, i_2 j_2}(\lambda, \nu) L_{j_1 k_1}^{\alpha_1 \beta}(\lambda, \mathbf{a}) L_{j_2 k_2}^{\beta \alpha_2}(\nu, \mathbf{a}) \\ = \sum_{j_1, j_2, \beta} L_{i_2 j_2}^{\alpha_1 \beta}(\nu, \mathbf{a}) L_{i_1 j_1}^{\beta \alpha_2}(\lambda, \mathbf{a}) R_{j_1 k_1, j_2 k_2}(\lambda, \nu). \end{aligned}$$

$\alpha_1, \alpha_2, \beta = 0, 1, \dots, N - 1$ and $i_1, i_2, j_1, \dots = 0, 1$.

Further they noticed: for special choice of \mathbf{a} there exists \mathbf{S} intertwining two L in the Weyl (quantum) space:

$$\begin{aligned} \sum_{\beta_1, \beta_2, k} \mathbf{S}_{\alpha_1 \alpha_2; \beta_1 \beta_2}(p, p', q, q') L_{i_1 k}^{\beta_1 \gamma_1}(\lambda; p, p') L_{k i_2}^{\beta_2 \gamma_2}(\lambda; q, q') \\ = \sum_{\beta_1, \beta_2, k} L_{i_1 k}^{\alpha_2 \beta_2}(\lambda; q, q') L_{k i_2}^{\alpha_1 \beta_1}(\lambda; p, p') \mathbf{S}_{\beta_1, \beta_2; \gamma_1, \gamma_2}(p, p', q, q'). \end{aligned}$$

$SLL = LLS$ if

$$L(\lambda; q, q') = \begin{pmatrix} 1 + \lambda \frac{y_q y_{q'}}{\mu_q \mu_{q'}} \mathbf{Z} & \lambda \mathbf{X}^{-1} \left(x_q - \frac{y_{q'}}{\mu_q \mu_{q'}} \mathbf{Z} \right) \\ \mathbf{X} \left(\omega x_{q'} - \frac{y_q}{\mu_q \mu_{q'}} \mathbf{Z} \right) & \lambda \omega x_q x_{q'} + \frac{1}{\mu_q \mu_{q'}} \mathbf{Z} \end{pmatrix}$$

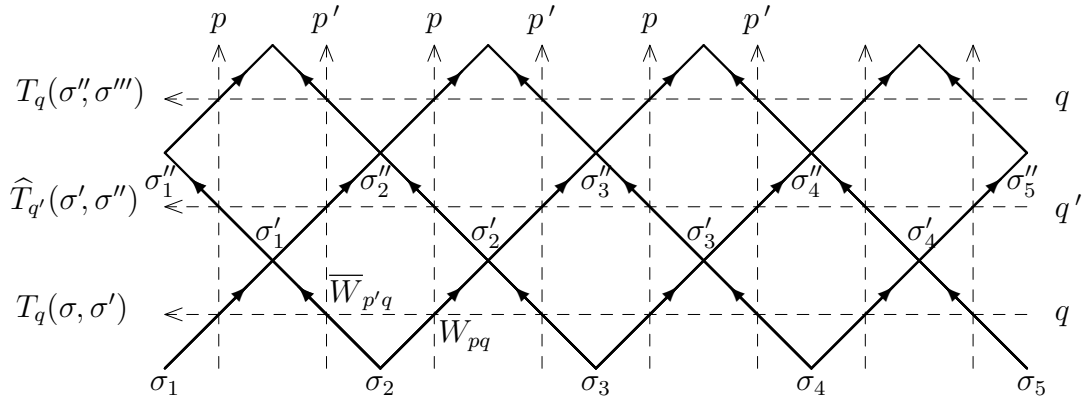
where $x_q, y_q, \mu_q, x_{q'}, y_{q'}, \mu_{q'}$ fulfil the high-genus chiral Potts conditions:

$$\begin{aligned} x_q^N + y_q^N &= k(1 + x_q^N y_q^N); & kx_q^N &= 1 - k' \mu_q^{-N}; & ky_q^N &= 1 - k' \mu_q^N; & (*) \\ k^2 + k'^2 &= 1, & k &: \text{fixed temperature-like parameter.} \end{aligned}$$

Intertwining matrix S is product of four *chiral Potts-Boltzmann* weights:

$$\begin{aligned} S_{\alpha_1 \alpha_2, \beta_1 \beta_2}(p, p', q, q') \\ = W_{pq}(\alpha_1 - \alpha_2) W_{p'q}(\beta_2 - \beta_1) \overline{W}_{pq}(\beta_2 - \alpha_1) \overline{W}_{p'q}(\beta_1 - \alpha_2). \end{aligned}$$

Integrable CP-model: (Baxter, Perk, Au-Yang 1988)



different left $\overline{W}_{pq}(\sigma_2 - \sigma'_1)$ and right $W_{pq}(\sigma_2 - \sigma'_2)$ Boltzmann weights:

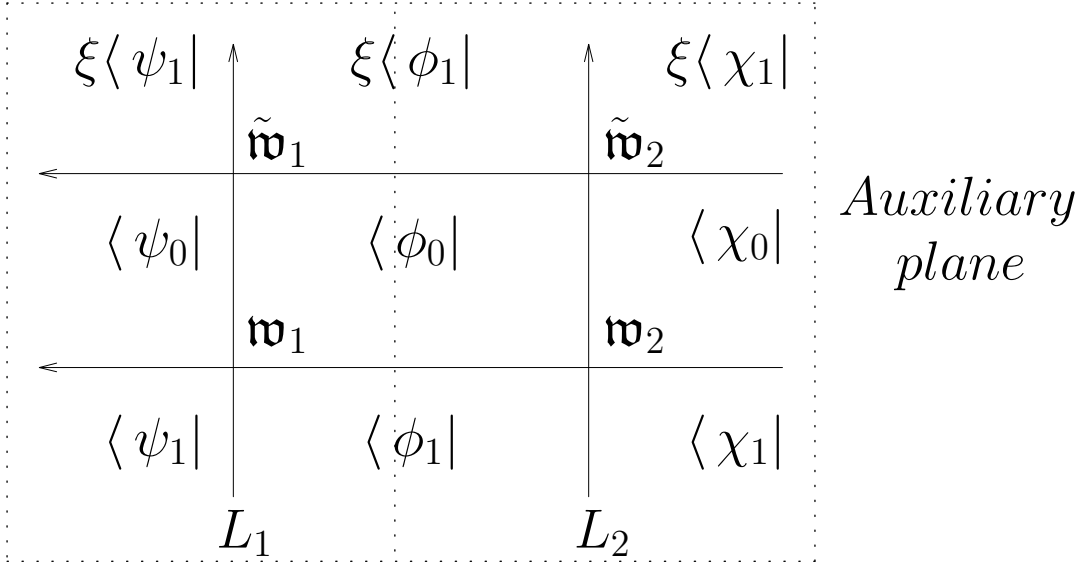
$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q} \right)^n \prod_{j=1}^n \frac{y_q - x_p \omega^j}{y_p - x_q \omega^j}; \quad \overline{W}_{pq}(n) = (\mu_p \mu_q)^n \prod_{j=1}^n \frac{\omega x_p - x_q \omega^j}{y_q - y_p \omega^j},$$

where x_q, y_q, μ_q , etc. satisfy (*).

$$W, \overline{W} \text{ are cyclic:} \quad W_{pq}(n+N) = W_{pq}(n); \quad \overline{W}_{pq}(n+N) = \overline{W}_{pq}(n),$$

$$\text{Special Y-B-eq.:} \quad \sum_d \overline{W}_{qr}(d-b) W_{pr}(d-a) \overline{W}_{pq}(c-d) = R_{pqr} W_{pq}(b-a) \overline{W}_{pr}(c-b) W_{qr}(c-a).$$

3 BS-L-operator from Sergeev Linear Problem



Consider four quantum variables $\mathfrak{w}_1, \tilde{\mathfrak{w}}_1, \mathfrak{w}_2, \tilde{\mathfrak{w}}_2$ on links of the basic 3D-lattice, in the auxiliary plane through these variables.

Assume lattice *periodic* in vertical direction, quasi-momentum ξ .

Linear problem at \mathfrak{w}_1 and $\tilde{\mathfrak{w}}_1$ (direction of lines important!):

$$\begin{aligned} 0 &= \langle \psi_0 | + \xi \omega^{1/2} \langle \psi_1 | \tilde{\mathbf{u}}_1 + \langle \phi_0 | \tilde{\mathbf{w}}_1 + \xi \tilde{\kappa}_1 \langle \phi_1 | \tilde{\mathbf{u}}_1 \tilde{\mathbf{w}}_1, \\ 0 &= \langle \psi_1 | + \omega^{1/2} \langle \psi_0 | \mathbf{u}_1 + \langle \phi_1 | \mathbf{w}_1 + \kappa_1 \langle \phi_0 | \mathbf{u}_1 \mathbf{w}_1. \end{aligned}$$

In matrix form: $\langle \psi | (\omega \xi \mathbf{u}_1 \tilde{\mathbf{u}}_1 - 1) \tilde{\mathbf{w}}_1^{-1} = \langle \phi | \cdot L_1(\xi),$

$$\text{where} \quad \langle \phi | = (\langle \phi_0 |, \langle \phi_1 |); \quad \langle \psi | = (\langle \psi_0 |, \langle \psi_1 |),$$

$L_1(\xi)$ has operator-valued elements:

$$L_1(\xi) = \begin{pmatrix} 1 - \omega^{1/2} \xi \mathbf{u}_1 \tilde{\mathbf{u}}_1 \kappa_1 \mathbf{w}_1 \tilde{\mathbf{w}}_1^{-1} & -\mathbf{u}_1 \left(\omega^{1/2} - \kappa_1 \mathbf{w}_1 \tilde{\mathbf{w}}_1^{-1} \right) \\ \xi \tilde{\mathbf{u}}_1 \left(\tilde{\kappa}_1 - \omega^{1/2} \mathbf{w}_1 \tilde{\mathbf{w}}_1^{-1} \right) & -\omega^{1/2} \xi \mathbf{u}_1 \tilde{\mathbf{u}}_1 \tilde{\kappa}_1 + \mathbf{w}_1 \tilde{\mathbf{w}}_1^{-1} \end{pmatrix}.$$

Only $\mathbf{w}_1 \tilde{\mathbf{w}}_1^{-1}, \mathbf{u}_1, \tilde{\mathbf{u}}_1$ appear: we can use $N \times N$ -rep for the operators:

$$\mathbf{w}_1 \tilde{\mathbf{w}}_1^{-1} = \frac{w_1}{\tilde{w}_1} \mathbf{Z}_1; \quad \mathbf{u}_1 = u_1 \mathbf{X}_1; \quad \tilde{\mathbf{u}}_1 = \tilde{u}_1 \mathbf{X}_1^{-1}.$$

Inserting representation and diagonal gauge transformation gives

$$\tilde{L}_1(\tilde{\xi}) = \begin{pmatrix} 1 - \omega^{1/2} \tilde{\xi} \kappa_1 w_1 \tilde{w}_1^{-1} \mathbf{Z}_1 & -\mathbf{X}_1 (\omega^{1/2} - \kappa_1 w_1 \tilde{w}_1^{-1} \mathbf{Z}_1) \\ \tilde{\xi} \mathbf{X}_1^{-1} (\tilde{\kappa}_1 - \omega^{1/2} w_1 \tilde{w}_1^{-1} \mathbf{Z}_1) & -\omega^{1/2} \tilde{\xi} \tilde{\kappa}_1 + w_1 \tilde{w}_1^{-1} \mathbf{Z}_1 \end{pmatrix}.$$

$\tilde{L}_1(\tilde{\xi})$: depends on spectral param. $\tilde{\xi}$ and 3 variables $\kappa_1, \tilde{\kappa}_1, w_1/\tilde{w}_1$.

\implies Bazhanov-Stroganov L -operator if we identify

$$\kappa_1 = \omega^{1/2} \frac{x_q}{y_{q'}}; \quad \tilde{\kappa}_1 = \frac{1}{\omega^{1/2}} \frac{y_q}{x_{q'}}; \quad \frac{w_1}{\tilde{w}_1} = \frac{1}{\omega} \frac{y_q y_{q'}}{x_q x_{q'} \mu_q \mu_{q'}}.$$

Recall:

$$L^{BS}(\lambda; q, q') = \begin{pmatrix} 1 + \lambda \frac{y_q y_{q'}}{\mu_q \mu_{q'}} \mathbf{Z} & \lambda \mathbf{X}^{-1} \left(x_q - \frac{y_{q'}}{\mu_q \mu_{q'}} \mathbf{Z} \right) \\ \mathbf{X} \left(\omega x_{q'} - \frac{y_q}{\mu_q \mu_{q'}} \mathbf{Z} \right) & \lambda \omega x_q x_{q'} + \frac{1}{\mu_q \mu_{q'}} \mathbf{Z} \end{pmatrix}$$

We shall soon get these relations from \mathbf{S} -operator too.

Remark: If we demand periodicity after *third* vertical step:

get L -operator of cyclic rep of $U_q(\widehat{\mathfrak{sl}}_3)$ etc. Isaev, Sergeev 2003,
Bazhanov et al, Date et al, Tarasov 1990

Successive action of two L -operators simply

$$\langle \psi | \cdot \tilde{\mathbf{w}}_1^{-1} \tilde{\mathbf{w}}_2^{-1} (\lambda \mathbf{u}_1 \tilde{\mathbf{u}}_1 - 1) (\lambda \mathbf{u}_2 \tilde{\mathbf{u}}_2 - 1) = \langle \chi | \cdot L_2(\lambda) L_1(\lambda).$$

4 BS-S-operator from matrix conjugation operators \mathbf{R}_{123}

We consider two $N^3 \times N^3$ matrices \mathbf{R} and $\tilde{\mathbf{R}}$ of the 3-D model:

$$\mathbf{R}_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \delta_{i_2+i_3, j_2+j_3} \omega^{(j_1-i_1)j_3} \frac{w_{p_1}(i_2-i_1)w_{p_2}(j_2-j_1)}{w_{p_3}(j_2-i_1)w_{p_4}(i_2-j_1)}.$$

with Fermat parameters p_i resp. \tilde{p}_i ($i = 1, 2, 3, 4$).

In this Section the Sergeev functional transformation $\mathcal{R}_{123}^{(f)}$ is chosen to be trivial.

Proposition:

The BS-intertwining $N^2 \times N^2$ -matrix \mathbf{S} can be written as

$$\mathbf{S}_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \sum_{m, n \in \mathbb{Z}_N} \mathbf{R}_{\alpha_1, \alpha_2, m}^{\beta_1 \beta_2, n} \tilde{\mathbf{R}}_{-\alpha_1, -\alpha_2, n}^{-\beta_1, -\beta_2, m}$$

if the Fermat parameters are related to the CP-parameters by:

$$\begin{aligned} x_1 &= \frac{y_{q'}}{\omega x_p}; & x_2 &= \frac{x_q}{y_{p'}}; & x_3 &= \frac{x_q}{\omega x_p}; & x_4 &= \frac{y_{q'}}{\omega y_{p'}}; \\ \tilde{x}_1 &= \frac{x_{q'}}{y_p}; & \tilde{x}_2 &= \frac{y_q}{\omega x_{p'}}; & \tilde{x}_3 &= \frac{y_q}{\omega y_p}; & \tilde{x}_4 &= \frac{x_{q'}}{\omega x_{p'}}. \end{aligned}$$

$$\frac{\tilde{y}_1}{y_1} = \omega^{1/2} \frac{\mu_p x_p}{\mu_{q'} y_p}; \quad \frac{\tilde{y}_2}{y_2} = \frac{1}{\omega^{1/2}} \frac{\mu_q y_{p'}}{\mu_{p'} x_{p'}}; \quad \frac{\tilde{y}_3}{y_3} = \omega^{1/2} \frac{\mu_p \mu_q x_p}{y_p}; \quad \frac{\tilde{y}_4}{y_4} = \frac{1}{\omega^{1/2}} \frac{y_{p'}}{\mu_{p'} \mu_{q'} x_{p'}}.$$

Proof: use property of $w_p(n)$ (q -analog of $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$):

$$w_p(n) = \frac{(-1)^n}{w_{Op}(-n) \omega^{n^2/2}}, \quad \text{where} \quad Op = (\omega^{-1}x^{-1}, \omega^{-1/2}x^{-1}y).$$

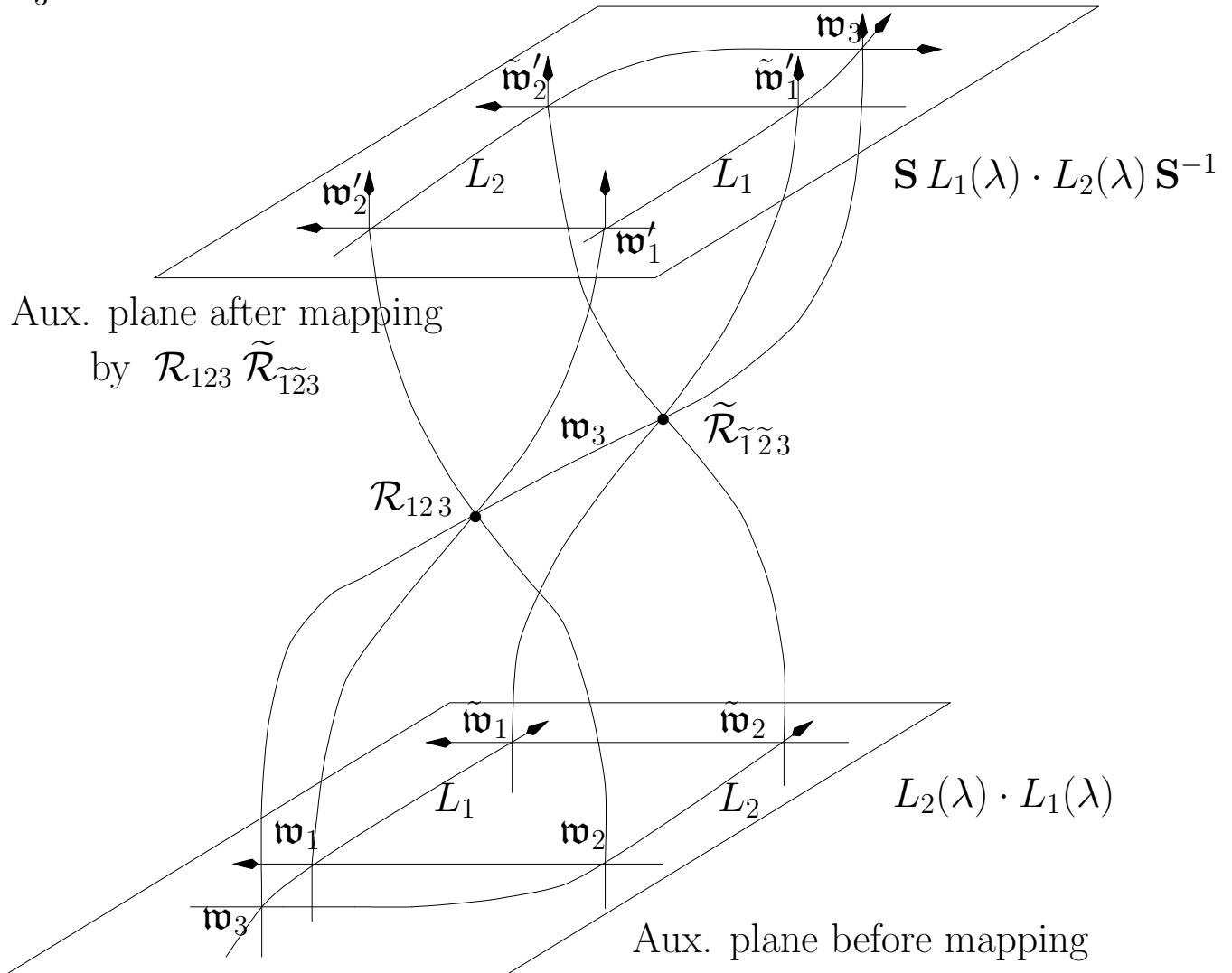
to get

$$\mathbf{S}_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \frac{w_{\tilde{p}_1}(\alpha_1 - \alpha_2)}{w_{Op_1}(\alpha_1 - \alpha_2)} \frac{w_{p_2}(\beta_2 - \beta_1)}{w_{Op_2}(\beta_2 - \beta_1)} \frac{w_{Op_3}(\beta_2 - \alpha_1)}{w_{p_3}(\beta_2 - \alpha_1)} \frac{w_{Op_4}(\beta_1 - \alpha_2)}{w_{\tilde{p}_4}(\beta_1 - \alpha_2)}.$$

Now identify

$$W_{p'q}(\alpha_1 - \alpha_2) \equiv \frac{w_{\tilde{p}_1}(\alpha_1 - \alpha_2)}{w_{Op_1}(\alpha_1 - \alpha_2)}; \quad W_{p'q}(\beta_2 - \beta_1) \equiv \frac{w_{p_2}(\beta_2 - \beta_1)}{w_{Op_2}(\beta_2 - \beta_1)} \text{ etc.}$$

Intertwining of $L_1 L_2$ visualized in 3D using auxiliary periodic variable \mathfrak{w}_3 :



$$\mathcal{R}_{123} : (\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3) \mapsto (\mathfrak{w}'_1, \mathfrak{w}'_2, \mathfrak{w}_3);$$

$$\tilde{\mathcal{R}}_{\tilde{1}\tilde{2}3} : (\tilde{\mathfrak{w}}_1, \tilde{\mathfrak{w}}_2, \mathfrak{w}_3) \mapsto (\tilde{\mathfrak{w}}'_1, \tilde{\mathfrak{w}}'_2, \mathfrak{w}_3);$$

From $\mathbf{S} = \mathbf{R} \tilde{\mathbf{R}}$ we get new parametrizations of \mathbf{S} : (5 contin. param.):

1. Chiral Potts: q, q', p, p', k .
2. Fermat: $x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ with $\frac{y_1 y_2}{y_3 y_4} = \frac{\tilde{y}_1 \tilde{y}_2}{\tilde{y}_3 \tilde{y}_4}$.
3. Weyl: $\kappa_1, \tilde{\kappa}_1, \kappa_2, \tilde{\kappa}_2, w_1/\tilde{w}_1, w_2/\tilde{w}_2, + \text{constraint}$.
4. *New*: Cross-ratios of 8 points in \mathbb{C} : 5 independent

Trivial $\mathcal{R}_{123}^{(f)}$ means

$$(K_2 U_2 - K_1 U_1) W_2 = (K_1 U_1 + U_2) K_3 W_3;$$

$$(W_1 - K_3 U_3) W_3 = (W_1 + U_3) W_2; \quad (U_1 - U_2) U_3 = (K_1 U_1 + U_2) W_1$$

The solution in terms of 6 points $X', X, Y', Y; Z'_0, Z_0 \in \mathbb{C}$ is:

$$\begin{aligned} U_1 &= -\varepsilon \frac{Y - Z'_0}{Y - Z_0}; & U_2 &= -\varepsilon \frac{X - Z'_0}{X - Z_0}; & U_3 &= -\varepsilon \frac{X - Y'}{X - Y}; \\ W_1 &= \varepsilon \frac{Y' - Z_0}{Y - Z_0}; & W_2 &= \varepsilon \frac{X' - Z_0}{X - Z_0}; & W_3 &= \varepsilon \frac{X' - Y}{X - Y}; \\ K_1 &= - \begin{bmatrix} Y' & Y \\ Z'_0 & Z_0 \end{bmatrix}; & K_2 &= - \begin{bmatrix} X' & X \\ Z'_0 & Z_0 \end{bmatrix}; & K_3 &= - \begin{bmatrix} X' & X \\ Y' & Y \end{bmatrix}, \end{aligned}$$

where $\varepsilon = (-1)^N$ and for cross-ratios we write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \frac{(A - C)(B - D)}{(A - D)(B - C)}.$$

Same eqs. with tilde-variables: replace Z'_0, Z_0 by Z'_1, Z_1 : then 8 points.

Nice result: N th powers of the Fermat points are simple cross-ratios:

$$\begin{aligned} x_1^N &= \begin{bmatrix} X & Y' \\ Z'_0 & Z_0 \end{bmatrix}; & x_2^N &= \begin{bmatrix} Y & X' \\ Z_0 & Z'_0 \end{bmatrix}; & x_3^N &= \begin{bmatrix} Y & X \\ Z_0 & Z'_0 \end{bmatrix}; & x_4^N &= \begin{bmatrix} X' & Y' \\ Z'_0 & Z_0 \end{bmatrix}; \\ y_1^N &= \begin{bmatrix} X & Z'_0 \\ Y' & Z_0 \end{bmatrix}; & y_2^N &= \begin{bmatrix} Y & Z_0 \\ X' & Z'_0 \end{bmatrix}; & y_3^N &= \begin{bmatrix} Y & Z_0 \\ X & Z'_0 \end{bmatrix}; & y_4^N &= \begin{bmatrix} X' & Z'_0 \\ Y' & Z_0 \end{bmatrix}, \end{aligned}$$

Also N th powers of chiral Potts variables are cross-ratios:

$$\begin{aligned} k x_p^N &= \begin{bmatrix} X & Z'_1 \\ Z_0 & Z'_0 \end{bmatrix}; & k y_p^N &= \begin{bmatrix} X & Z'_0 \\ Z_1 & Z'_1 \end{bmatrix}; & k' \mu_p^N &= \begin{bmatrix} X & Z_1 \\ Z'_0 & Z'_1 \end{bmatrix}; \\ k x_{p'}^N &= \begin{bmatrix} X' & Z'_0 \\ Z_1 & Z'_1 \end{bmatrix}; & k y_{p'}^N &= \begin{bmatrix} X' & Z'_1 \\ Z_0 & Z'_0 \end{bmatrix}; & k' \mu_{p'}^N &= \begin{bmatrix} X' & Z_0 \\ Z'_1 & Z'_0 \end{bmatrix}; \\ k x_q^N &= \begin{bmatrix} Y & Z'_1 \\ Z_0 & Z'_0 \end{bmatrix}; & k y_q^N &= \begin{bmatrix} Y & Z'_0 \\ Z_1 & Z'_1 \end{bmatrix}; & k' \mu_q^N &= \begin{bmatrix} Y & Z_1 \\ Z'_0 & Z'_1 \end{bmatrix}; \\ k x_{q'}^N &= \begin{bmatrix} Y' & Z'_0 \\ Z_1 & Z'_1 \end{bmatrix}; & k y_{q'}^N &= \begin{bmatrix} Y' & Z'_1 \\ Z_0 & Z'_0 \end{bmatrix}; & k' \mu_{q'}^N &= \begin{bmatrix} Y' & Z_0 \\ Z'_1 & Z'_0 \end{bmatrix}. \end{aligned}$$

with

$$k^2 = \begin{bmatrix} Z_0 & Z'_0 \\ Z_1 & Z'_1 \end{bmatrix} \quad \text{or} \quad k'^2 = \begin{bmatrix} Z'_0 & Z'_1 \\ Z_0 & Z_1 \end{bmatrix}.$$

In this cross-ratio parametriz. all constraints automatically fulfilled.

5 Intertwining of classical BS-L-operators from $\mathcal{R}_{123}^{(f)}$

Classical Linear Problem:

$$\begin{aligned} 0 &= \Psi_0 - \Psi_1 \Lambda \tilde{U}_1 + \Phi_0 \tilde{W}_1 + \Phi_1 \Lambda \tilde{K}_1 \tilde{U}_1 \tilde{W}_1 \\ 0 &= \Psi_1 - \Psi_0 U_1 + \Phi_1 W_1 + \Phi_0 K_1 U_1 W_1 \end{aligned}$$

in matrix form:

$$\Psi (\Lambda U_1 \tilde{U}_1 - 1) = \Phi \cdot \mathcal{L}_1(\Lambda)$$

defining the classical BS- \mathcal{L} -operator

$$\mathcal{L}_1(\Lambda) = \begin{pmatrix} \tilde{W}_1 + \Lambda U_1 \tilde{U}_1 K_1 W_1 & U_1 (\tilde{W}_1 + K_1 W_1) \\ \Lambda \tilde{U}_1 (W_1 + \tilde{K}_1 \tilde{W}_1) & W_1 + \Lambda \tilde{K}_1 U_1 \tilde{U}_1 \tilde{W}_1 \end{pmatrix}$$

acting in $\Psi = (\Psi_0, \Psi_1)$ and $\Phi = (\Phi_0, \Phi_1)$.

We take $\mathcal{L}_1(\Lambda)$ to define a discrete classical Bazhanov-Stroganov model.

Goal: describe solitonic solutions and soliton creation.

Define $\mathcal{L}_2(\Lambda)$ as above, but with $U_2, \tilde{U}_2, W_2, \tilde{W}_2, K_2, \tilde{K}_2,$

$\mathcal{L}_1^*(\Lambda)$ with $U_1^*, \tilde{U}_1, W_1^*, \tilde{W}_1^*, K_1, \tilde{K}_1,$ analogously $\mathcal{L}_2^*(\Lambda)$.

We are looking for $(U_1, \tilde{U}_1, W_1, \dots, \tilde{W}_2) \mapsto (U_1^*, \tilde{U}_1^*, W_1^*, \dots, \tilde{W}_2^*)$

which solves the intertwining relation

$$\mathcal{L}_2(\Lambda) \mathcal{L}_1(\Lambda) = \mathcal{L}_1^*(\Lambda) \mathcal{L}_2^*(\Lambda).$$

Direct brute force solution is quite hopeless.

However, the 3D Sergeev functional mapping $\mathcal{R}_{123}^{(f)}$

$$\begin{aligned} \frac{U_1'}{U_1} = \frac{W_3'}{W_3} &= \frac{K_2 U_2 W_2}{K_1 U_1 W_2 + K_3 U_2 W_3 + K_1 K_3 U_1 W_3}; \\ \frac{W_1}{W_1'} = \frac{W_2'}{W_2} &= \frac{W_1 W_3}{W_1 W_2 + U_3 W_2 + K_3 U_3 W_3}; \quad \frac{U_2'}{U_2} = \frac{U_3}{U_3'} = \frac{U_1 U_3}{U_2 U_3 + U_2 W_1 + K_1 U_1 W_1}. \end{aligned}$$

suggests the solution: Introduce additional variables U_3, W_3

and constant $K_3,$ demand periodicity.

We write 3D Sergeev functional mapping $\mathcal{R}_{123}^{(f)}$ as

$$\mathcal{R}_{123}^{(f)} : U_1, W_1, U_2, W_2, U_3, W_3 \mapsto U'_1, W'_1, U'_2, W'_2, U'_3, W'_3$$

Consider the *composition* of two of these rational transformations

$$\mathcal{R}_{123}^{(f)} : U_1, W_1, U_2, W_2, U_3, W_3 \mapsto U_1^*, W_1^*, U_2^*, W_2^*, U'_3, W'_3,$$

$$\mathcal{R}_{123}^{(f)} : \tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2, U'_3, W'_3 \mapsto \tilde{U}_1^*, \tilde{W}_1^*, \tilde{U}_2^*, \tilde{W}_2^*, U_3^*, W_3^*,$$

together with a periodic condition

$$U_3^* = U_3, \quad W_3^* = W_3$$

and denote this composition by

$$S_{12}^{(f)} : U_1, W_1, U_2, W_2, \tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2 \mapsto U_1^*, W_1^*, U_2^*, W_2^*, \tilde{U}_1^*, \tilde{W}_1^*, \tilde{U}_2^*, \tilde{W}_2^*.$$

With these definitions we have

Proposition: The rational transformation $S_{12}^{(f)}$ solves the intertwining relations $\mathcal{L}_2(\Lambda) \mathcal{L}_1(\Lambda) = \mathcal{L}_1^*(\Lambda) \mathcal{L}_2^*(\Lambda)$.

Proof: Solve first using the periodic b.c. for U_3, W_3 , which gives

$$U_3 = \frac{U_1(\tilde{K}_1 \tilde{U}_1 \tilde{W}_1 + K_1 \tilde{U}_2 W_1) + \tilde{U}_2(U_2 W_1 + U_1 \tilde{W}_1)}{U_1 \tilde{U}_1 - U_2 \tilde{U}_2}$$

$$W_3 = \frac{W_2 \tilde{W}_2 (\tilde{K}_2 K_2 \tilde{U}_2 U_2 - \tilde{K}_1 K_1 \tilde{U}_1 U_1)}{K_3 (U_2 (\tilde{K}_1 \tilde{U}_1 \tilde{W}_2 + K_2 \tilde{U}_2 W_2) + \tilde{K}_1 \tilde{U}_1 (K_1 U_1 \tilde{W}_2 + K_2 U_2 W_2))},$$

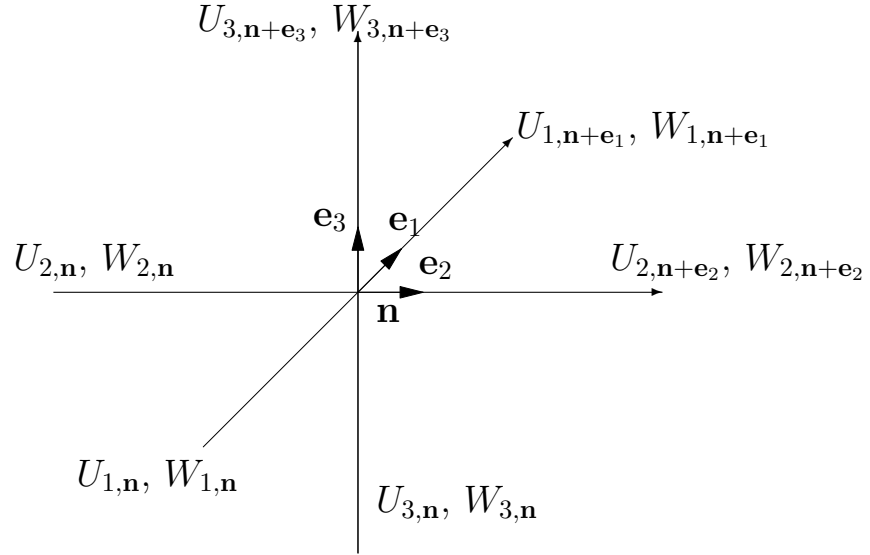
then just insert.

6 Uniformization of the bulk functional mapping

Up to now we discussed *local* features of the models. Now bulk properties:

First we consider the classical part.

Functional mapping, written for a general vertex $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$:



$$\frac{U_{1,n+e_1}}{U_{1,n}} = \frac{W_{3,n+e_3}}{W_{3,n}} = \frac{K_{2:n_1,n_3} U_{2,n} W_{2,n}}{K_{1:\dots} U_{1,n} W_{2,n} + K_{3:\dots} U_{2,n} W_{3,n} + K_{1:\dots} K_{3:\dots} U_{1,n} W_{3,n}},$$

$$\frac{W_{1,n}}{W_{1,n+e_1}} = \frac{W_{2,n+e_2}}{W_{2,n}} = \frac{W_{1,n} W_{3,n}}{W_{1,n} W_{2,n} + U_{3,n} W_{2,n} + K_{3:n_1,n_2} U_{3,n} W_{3,n}},$$

$$\frac{U_{2,n+e_2}}{U_{2,n}} = \frac{U_{3,n}}{U_{3,n+e_3}} = \frac{U_{1,n} U_{3,n}}{U_{2,n} U_{3,n} + U_{2,n} W_{1,n} + K_{1:n_2,n_3} U_{1,n} W_{1,n}}.$$

Legendre transf. from the 12 variables at each vertex $U_{1,n+e_1}, \dots, U_{3,n}$ to ratios of 9 fcts $\tau_{1,n}, \dots, \tau_{3,n+e_2}$ and 6 points $X_{n_1}, X'_{n_1}, Y_{n_2}, Y'_{n_2}, Z_{n_3}, Z'_{n_3}$:

$$U_{1,n} = \frac{Y_{n_2} - Z'_{n_3}}{Y_{n_2} - Z_{n_3}} \frac{\tau_{2,n}}{\tau_{2,n+e_3}}, \quad W_{1,n} = - \frac{Z_{n_3} - Y'_{n_2}}{Z_{n_3} - Y_{n_2}} \frac{\tau_{3,n+e_2}}{\tau_{3,n}}, \quad \text{etc.}$$

then $\mathcal{R}_n^{(f)}$ takes the form of 3 trilinear Hirota equations: Pakuliak, Sergeev 2002

$$\begin{aligned}
& (X_\alpha - X_\beta)(X'_\beta - X'_\gamma)(X_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}} \\
& \quad + (X_\alpha - X'_\beta)(X_\beta - X_\gamma)(X'_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta} \\
& = (X_\alpha - X_\beta)(X'_\beta - X_\gamma)(X'_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}} \\
& \quad + (X_\alpha - X'_\beta)(X_\beta - X'_\gamma)(X_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta} ,
\end{aligned}$$

$$(\alpha, \beta, \gamma) = (1, 2, 3), (2, 3, 1), (3, 1, 2); \quad X_1 = X_{n_1}, X'_1 = X'_{n_1}, X_2 = Y'_{n_2}, X'_2 = Y_{n_2}, X_3 = Z_{n_3}, X'_3 = Z'_{n_3}.$$

Now well-known since 1978-84 : *any discrete integrable system can be solved by methods of algebraic geometry.* Kirchever, Shiota, Mulase

Trilinear Hirota eqs. identically solved by Double-Fay-Identity.

Practical managable solutions: "rational Θ -functions" H :

$$H^{(g)} \left(\{P'_j, P_j, f_j\}_{k=0}^{g-1} \right) = \frac{\det | P_j^i - f_j P_j'^i |_{i,j=0}^{g-1}}{\prod_{i>j} (P_i - P_j)}$$

$$H(f_0) = 1 - f_0; \quad H(f_0, f_1) = 1 - \frac{P_1 - P'_0}{P_1 - P_0} f_0 - \frac{P_0 - P'_1}{P_1 - P_0} f_1 + \frac{P'_1 - P'_0}{P_1 - P_0} f_0 f_1;$$

$$\begin{aligned}
H(f_0, f_1, f_2) = 1 - \frac{(P'_0 - P_2)(P'_0 - P_1)}{(P_0 - P_1)(P_0 - P_2)} f_0 + \dots + \frac{(P'_1 - P_2)(P'_1 - P_0)(P'_1 - P'_0)}{(P_2 - P_1)(P_2 - P_0)(P_1 - P_0)} f_0 f_1 + \dots \\
+ \dots + \frac{(P'_0 - P'_1)(P'_1 - P'_2)(P'_2 - P_0)}{(P_2 - P_1)(P_2 - P_0)(P_1 - P_0)} f_0 f_1 f_2.
\end{aligned}$$

The solutions for the Weyl centers can be written in terms of the functions:

$$U(\{f_k\}, \mathcal{A}, \mathcal{B}) =: -\varepsilon \frac{A - B'}{A - B} \frac{H(\{P'_j, P_j, \mathbf{f}_j(A)\})}{H(\{P'_j, P_j, \mathbf{f}_j(A) \sigma_k(\mathcal{B})\})}$$

$\mathcal{A} = (A', A), \dots, \mathcal{Y} = (Y', Y),$ etc. stand for pairs of points, and

$$\mathbf{f}_j(Y) = \frac{P_j - Y}{P'_j - Y} f_j; \quad \sigma_j(\mathcal{Y}) = \begin{bmatrix} P'_j & P_j \\ Y' & Y \end{bmatrix}.$$

Then

$$U_{1,\mathbf{n}} = U(\{\mathbf{f}_j I_{j:\mathbf{n}}\}, \mathcal{Y}_{n_2}, \mathcal{Z}_{n_3}); \quad U_{2,\mathbf{n}} = U(\{\mathbf{f}_j I_{j:\mathbf{n}} \sigma_j(\mathcal{Y}_{n_2})\}, \mathcal{Y}_{n_2}, \mathcal{Z}_{n_3});$$

$$W_{1,\mathbf{n}} = -U(\{\mathbf{f}_j I_{j:\mathbf{n}}\}, \mathcal{Z}_{n_3}, \mathcal{Y}_{n_2}); \quad W_{2,\mathbf{n}} = -U(\{\mathbf{f}_j I_{j:\mathbf{n}} \sigma_j(\mathcal{Z}_{n_3})\}, \mathcal{Z}_{n_3}, \mathcal{Y}_{n_2});$$

$$U_{3,\mathbf{n}} = U(\{\mathbf{f}_j I_{j:\mathbf{n}}\}, \mathcal{X}_{n_1}, \mathcal{Y}'_{n_2}); \quad W_{3,\mathbf{n}} = -U(\{\mathbf{f}_j I_{j:\mathbf{n}} \sigma_j(\mathcal{Y}'_{n_2})\}, \mathcal{Y}'_{n_2}, \mathcal{X}_{n_1}).$$

where

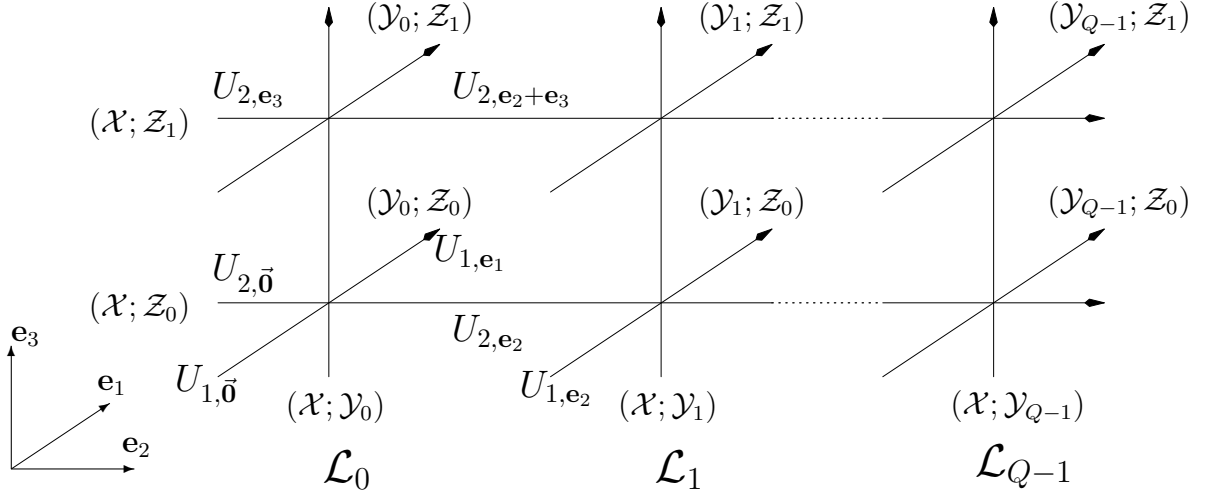
$$I_{j:\mathbf{n}} = \prod_{m_1=0}^{n_1-1} \frac{\sigma_j(\mathcal{X}'_{m_1})}{\sigma_j(\mathcal{X}_{m_1})} \prod_{m_2=0}^{n_2-1} \frac{\sigma_j(\mathcal{Y}_{m_2})}{\sigma_j(\mathcal{Y}'_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{\sigma_j(\mathcal{Z}_{m_3})}{\sigma_j(\mathcal{Z}'_{m_3})}.$$

7 The classical inhomogenous BS-Chain

We consider slice of the 3D-lattice:

Chain in \vec{e}_2 -direction, single layer in \vec{e}_1 , periodic after two steps in \vec{e}_3 :

$$\tilde{U}_{i,\mathbf{n}} \equiv U_{i,\mathbf{n}+\mathbf{e}_3}, \quad \tilde{W}_{i,\mathbf{n}} \equiv W_{i,\mathbf{n}+\mathbf{e}_3}$$



$$\mathcal{L}_n(\Lambda) = \begin{pmatrix} \tilde{W}_{1,ne_2} + \Lambda U_{1,ne_2} \tilde{U}_{1,ne_2} K_{1:n} W_{1,ne_2} & U_{1,ne_2} (\tilde{W}_{1,ne_2} + K_{1:n} W_{1,ne_2}) \\ \Lambda \tilde{U}_{1,ne_2} (W_{1,ne_2} + \tilde{K}_{1:n} \tilde{W}_{1,ne_2}) & W_{1,ne_2} + \Lambda \tilde{K}_{1:n} U_{1,ne_2} \tilde{U}_{1,ne_2} \tilde{W}_{1,ne_2} \end{pmatrix}.$$

Classical monodromy matrix (chain length \mathcal{N})

$$\mathcal{M}(\Lambda) = \left(\prod_{n'=0}^{\mathcal{N}-1} \tilde{W}_{1,n'\mathbf{e}_2}^{-1} \right) \mathcal{L}_0(\Lambda) \mathcal{L}_1(\Lambda) \cdots \mathcal{L}_n(\Lambda) \cdots \mathcal{L}_{\mathcal{N}-1}(\Lambda).$$

Now we derive an *isospectrality transformation* of the transfer matrix $\text{Tr } \mathcal{M}(\Lambda)$ by moving an auxiliary classical operator

$$\mathcal{L}_0^{aux}(\Lambda) = \begin{pmatrix} \tilde{W}_{2,\vec{0}} + \Lambda U_{2,\vec{0}} \tilde{U}_{2,\vec{0}} K_{2:\vec{0}} W_{2,\vec{0}} & U_{2,\vec{0}} (\tilde{W}_{2,\vec{0}} + K_{2:\vec{0}} W_{2,\vec{0}}) \\ \Lambda \tilde{U}_{2,\vec{0}} (W_{2,\vec{0}} + \tilde{K}_{2:\vec{0}} \tilde{W}_{2,\vec{0}}) & W_{2,\vec{0}} + \Lambda \tilde{K}_{2:\vec{0}} U_{2,\vec{0}} \tilde{U}_{2,\vec{0}} \tilde{W}_{2,\vec{0}} \end{pmatrix}$$

through the monodromy

$$\mathcal{L}_0^{aux} \mathcal{L}_0 \mathcal{L}_1 \cdots \mathcal{L}_{\mathcal{N}-1} = \mathcal{L}_0^* \mathcal{L}_1^{aux} \mathcal{L}_1 \cdots \mathcal{L}_{\mathcal{N}-1} = \cdots = \mathcal{L}_0^* \mathcal{L}_1^* \cdots \mathcal{L}_{\mathcal{N}-1}^* \mathcal{L}_{\mathcal{N}}^{aux},$$

requiring $\mathcal{L}_0^{aux} = \mathcal{L}_{\mathcal{N}}^{aux}$ so that $\text{Tr } \mathcal{M}(\Lambda) = \text{Tr } \mathcal{M}^*(\Lambda)$.

We have studied the intertwining of two operators \mathcal{L} :

$$\begin{aligned} & \mathcal{L}(U_2, W_2, \tilde{U}_2, \tilde{W}_2, \Lambda) \cdot \mathcal{L}(U_1, W_1, \tilde{U}_1, \tilde{W}_1, \Lambda) \\ &= \mathcal{L}(U_1^*, W_1^*, \tilde{U}_1^*, \tilde{W}_1^*, \Lambda) \cdot \mathcal{L}(U_2^*, W_2^*, \tilde{U}_2^*, \tilde{W}_2^*, \Lambda) \end{aligned}$$

was described by the mapping

$$S_{12}^{(f)} : U_1, W_1, U_2, W_2, \tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2 \mapsto U_1^*, W_1^*, U_2^*, W_2^*, \tilde{U}_1^*, \tilde{W}_1^*, \tilde{U}_2^*, \tilde{W}_2^* .$$

The mapping $S_{12}^{(f)}$ takes a simple form in terms of the functions U if the periodicity condition $U_3 = U_3^*$, which here is $\sigma_j(\mathcal{Z}_0)\sigma_j(\mathcal{Z}_1) = 1$, is imposed.

Indicating the arguments of the variables U_j only, this reads:

$$\begin{aligned} & \mathcal{L}_0^{aux}(\{f_j\sigma_j(\mathcal{Y}_0)\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1) \cdot \mathcal{L}_0(\{f_j\}, \mathcal{Y}_0, \mathcal{Z}_0, \mathcal{Z}_1) \\ &= \mathcal{L}_0^*(\{f_j\sigma_j(\mathcal{X})\}, \mathcal{Y}_0, \mathcal{Z}_0, \mathcal{Z}_1) \cdot \mathcal{L}_1^{aux}(\{f_j\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1) : \end{aligned}$$

In the U_2, W_2 arguments of operator \mathcal{L}_0^{aux} the intertwining mapping is

$$f_j \rightarrow f_j \sigma^{-1}(\mathcal{Y}_0),$$

while the U_1, W_1 -arguments of the \mathcal{L}_0 are mapped $f_j \rightarrow f_j\sigma_j(\mathcal{X})$.

Finally, periodicity

$$\mathcal{L}_0^{aux}(\{f_j\sigma_j(\mathcal{Y}_0)\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1) = \mathcal{L}_N^{aux}\left(\left\{f_j\sigma_j(\mathcal{Y}_0) \prod_{i=0}^{N-1} \sigma_j^{-1}(\mathcal{Y}_i)\right\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1\right)$$

requires

$$\prod_{i=0}^{N-1} \sigma_j^{-1}(\mathcal{Y}_i) = 1 .$$

If this is satisfied, the transfer matrices $\text{Tr } \mathcal{M}(\Lambda)$ and $\text{Tr } \mathcal{M}^*(\Lambda)$ are *isospectral*. This is non-trivial if some $f_j \neq 0$.

8 Conclusions

- Well-known are the relations between the BS-model and the Chiral Potts model, and between Chiral Potts model and 3D ZBB-model.
- Here we studied in detail the direct relation between BS-model and 3D ZBB-model and its generalizations.
- BS-Lax-operator is obtained from the Sergeev linear problem.
- The BS quantum intertwiner matrix S is obtained as the product of two 3D mapping operators.
- 3d parameterization (Fermat points and Weyl centers) leads to a simple cross-ratio parameterization of BS operators which can be generalized to the inhomogenous case.
- Parameterization of inhomogenous BS model in terms of rational Theta-functions allows derivation of new BS isospectral transformations.

Proposition: Let M be length of the periodic BS-chain. Consider the system

$$\prod_{q=0}^{M-1} \frac{(P' - Y'_q)(P - Y_q)}{(P - Y'_q)(P' - Y_q)} = \prod_{j=0,1} \frac{(P' - Z'_j)(P - Z_j)}{(P - Z'_j)(P' - Z_j)} = 1.$$

where Y_q, Y'_q, Z_0, \dots are given by the CP-variables:

$$kx_q^N = \frac{(Y_q - Z_0)(Z'_1 - Z'_0)}{(Y_q - Z'_0)(Z'_1 - Z_0)}, \quad ky_q^N = \dots; \quad k^2 = \frac{(Z_0 - Z_1)(Z'_0 - Z'_1)}{(Z'_0 - Z_1)(Z_0 - Z'_1)}.$$

This system has $g = M - 1$ solutions $\{(P'_k, P_k)\}$. Define

$$I_k(q) = \prod_{j=0}^{q-1} \frac{\sigma_k(Y_j)}{\sigma_k(Y'_j)} : \quad I_k(0) = I_k(Q) = 1.$$

Then, the spectrum of the BS transfer matrix with

$$\mathbf{X}_q^N = \frac{Y_q - Z'_0}{Y_q - Z_0} \frac{H(\{f_k I_k(q) \sigma_k^{-1}(Y'_q)\})}{H(\{f_k I_k(q) \sigma_k(Z'_0) \sigma_k^{-1}(Y'_q) \sigma_k^{-1}(Z_0)\})},$$

$$\mathbf{Z}_q^N = \frac{Y'_q - Z_0}{Y_q - Z_0} \frac{Y_q - Z'_1}{Y'_q - Z_1} \frac{H(\{f_k I_k(q+1) \sigma_k^{-1}(Z_0)\}) H(\{f_k I_k(q) \sigma_k^{-1}(Z_1)\})}{H(\{f_k I_k(q) \sigma_k^{-1}(Z_0)\}) H(\{f_k I_k(q+1) \sigma_k^{-1}(Z_1)\})}$$

does not depend on the set of $\{f_k\}$.

Theta functions

Concepts: Γ_g : generic algebraic curve of genus g

ω canonical g -dim. vector of the homomorphic differentials.

For $X, Y \in \Gamma_g$ let $\mathbf{I}_Y^X : \Gamma_g^2 \mapsto \text{Jac}(\Gamma_g)$ be $\mathbf{I}_Y^X \stackrel{\text{def}}{=} \int_Y^X \omega$

$E(X, Y) = -E(Y, X)$: prime form on Γ_g^2 ,

$\Theta(\mathbf{v})$: theta-function on $\text{Jac}(\Gamma_g)$:

$$\Theta^{(g)}(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp(i\pi(\mathbf{n}, \Omega \mathbf{n}) + 2i\pi(\mathbf{n}, \mathbf{z}))$$

Fay identity for $A, B, C, D \in \Gamma_g$ and a $\mathbf{v} \in \text{Jac}(\Gamma_g)$:

$$\begin{aligned} \Theta(\mathbf{v}) \Theta(\mathbf{v} + \mathbf{I}_B^A + \mathbf{I}_D^C) &= \Theta(\mathbf{v} + \mathbf{I}_D^A) \Theta(\mathbf{v} + \mathbf{I}_B^C) \frac{E(A, B) E(D, C)}{E(A, C) E(D, B)} \\ &+ \Theta(\mathbf{v} + \mathbf{I}_B^A) \Theta(\mathbf{v} + \mathbf{I}_D^C) \frac{E(A, D) E(C, B)}{E(A, C) E(D, B)}, \end{aligned}$$

In "physical" expressions prime forms only as cross ratios: expressible in thetas:

$$\left[\begin{array}{cc} X & X' \\ Y & Y' \end{array} \right]_E \stackrel{\text{def}}{=} \frac{E(X, Y) E(X', Y')}{E(X, Y') E(X', Y)} = \frac{\Theta_{\epsilon_{\text{odd}}}(\mathbf{I}_X^Y) \Theta_{\epsilon_{\text{odd}}}(\mathbf{I}_{X'}^{Y'})}{\Theta_{\epsilon_{\text{odd}}}(\mathbf{I}_X^{Y'}) \Theta_{\epsilon_{\text{odd}}}(\mathbf{I}_{X'}^Y)}.$$

More useful here: **Rational** limit of Θ :

Sergeev 2000, Pakuliak, Sergeev 2002

$$\Theta^{(g)}(\mathbf{z}) \Rightarrow H^{(g)}\left(\{p_k, q_k, f_k\}_{k=0}^{g-1}\right) = \frac{\det |q_j^i - f_j p_j^i|_{i,j=0}^{g-1}}{\prod_{i>j} (q_i - q_j)}.$$

f_0, f_1, \dots, f_{g-1} remember \mathbf{z} and $p_0, q_0, \dots, p_{g-1}, q_{g-1}$ remember Ω .

$$\left[\begin{array}{cc} X & X' \\ Y & Y' \end{array} \right]_E \Rightarrow \frac{(X - Y)(X' - Y')}{(X - Y')(Y - X')}; \text{ analogous Fay-Identity also for } H^{(g)}.$$

9 Conclusions

The intimate relation of the BS-model

to the 3D ZBB-model is no surprise:

Bazhanov and Stroganov found the relation of the BS-model to the integrable chiral Potts model,

Bazhanov and Baxter found that the \mathbb{Z}_N -generalization of the Zamolodchikov model for two layers leads to the integrable chiral Potts model.

Here, the relation of the Sergeev mapping and the Weyl Linear Problem, to the BS-model is made explicit.

The cross-ratio parametrization simplifies most functional variable changes. It is related to the rational limit of the Theta-function parametrization of the functional mapping.

The introduction of a third pair of variables, which is afterwards eliminated by a periodicity condition, gives a transparent 3D-method to solve the intertwining relations of the BS-model.