

# Enumerative Geometry on Enriques surfaces



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## Outline:

- ⇒ Topological String Theory and Symplectic Invariants
- ⇒ Computability: Noncompact Large N duality, Vertex
- ⇒ Compact case: String duality and the FHSV model
- ⇒ Geometrical Checks
- ⇒ Conclusions

## Topological String Theory:

Map from an oriented Riemann surface  $\Sigma$  to a manifold  $M$

$$X : \Sigma \rightarrow M$$

Characterize  $X$  by

⇨ the **genus**  $g$  of  $\Sigma_g$  ,

⇨ the **cohomology** class of the image  $\beta = [X(\Sigma_g)] \in H_{comp}^2(M, \mathbf{Z})$

Evaluate a string amplitude for the theory on  $M \times \mathbf{R}_{1,3}$ .

Vacuum amplitude  $Z$  :

$$Z = \int \mathcal{D}X \mathcal{D}h e^{iS(h, X, G, \dots)},$$

where metric  $G$  of  $X$  is a background field.

In critical dimension  $\int \mathcal{D}h$  collapses

$$\int \mathcal{D}h \rightarrow \sum_g \int_{\overline{\mathcal{M}}_g} d\mu_g$$

to a sum of finite dimensional integrals over moduli space  $\overline{\mathcal{M}}_g$  of  $\Sigma_g$ .

Topological String Theory is a simplified version of string theory in which  $\int \mathcal{D}X$  collapses in a similar way. E.g. for the  $A$  model, where we assume  $(M, \omega)$  is Kähler

$$\int \mathcal{D}X \mathcal{D}h e^{iS(h, X, G, \dots)} \rightarrow \sum_g \sum_{\beta} \int_{\overline{\mathcal{M}}_g(X, \beta)} c_g^{vir}(X, \beta) \lambda^{2g-2} q^{\beta}.$$

⇒ This is a semi-classical approximation with  $g, \beta$  labelling instanton sectors.

⇒ Dependence on  $G$  is not completely erased:  $q^{\beta} := \exp(-\int_{[\beta]} \omega) = \exp(-\text{Area}(X_{\beta}(\Sigma)))$ .

⇨ Instantons are  $(j, J)$  holomorphic maps

$$\bar{\partial}_{j,J} X = \frac{1}{2}(dX + J \circ dX \circ j) = 0 .$$

Calabi-Yau 3-fold case **very special**

$$\dim_{vir}(\overline{\mathcal{M}}_g(X, \beta)) = \int_{\beta} c_1(TM) + (\mathbf{dim}M - 3)(1 - g) = 0$$

⇨ worldsheet super symmetry provides BRST formalism  
→ exact correlators in cohomological theory.

⇨ Exact terms in the low energy eff. supergravity action.  
Gauge coupling, BPS masses, grav. coupl.  $F_-^{2g-2} R_-^2$

## Symplectic invariants:

$r_{\beta}^g = \int_{\overline{\mathcal{M}}_g(X, \beta)} c_g^{vir}(X, \beta) \in \mathbf{Q}$  Gromov-Witten invariant  
 $\rightarrow$  symplectic invariant of  $M$ .

Gromov–Witten  
invariants



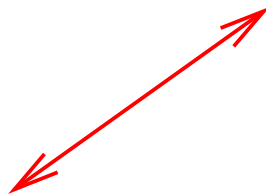
Gopakumar–Vafa  
invariants

(associated to the moduli  
space of D0–D2 BPS  
states)



Donaldson–Thomas  
invariants

(associated to moduli–space  
of free sheaves)



# The relation between DT- GV- and GW invariants

## OMNP 2004

$$Z_{\text{GV}}^{\text{hol}}(M, q_\lambda, q) (q_\lambda)^{\frac{\chi(M)}{2}} = Z_{\text{DT}}^{\text{hol}}(M, -q_\lambda, q) \quad (1)$$

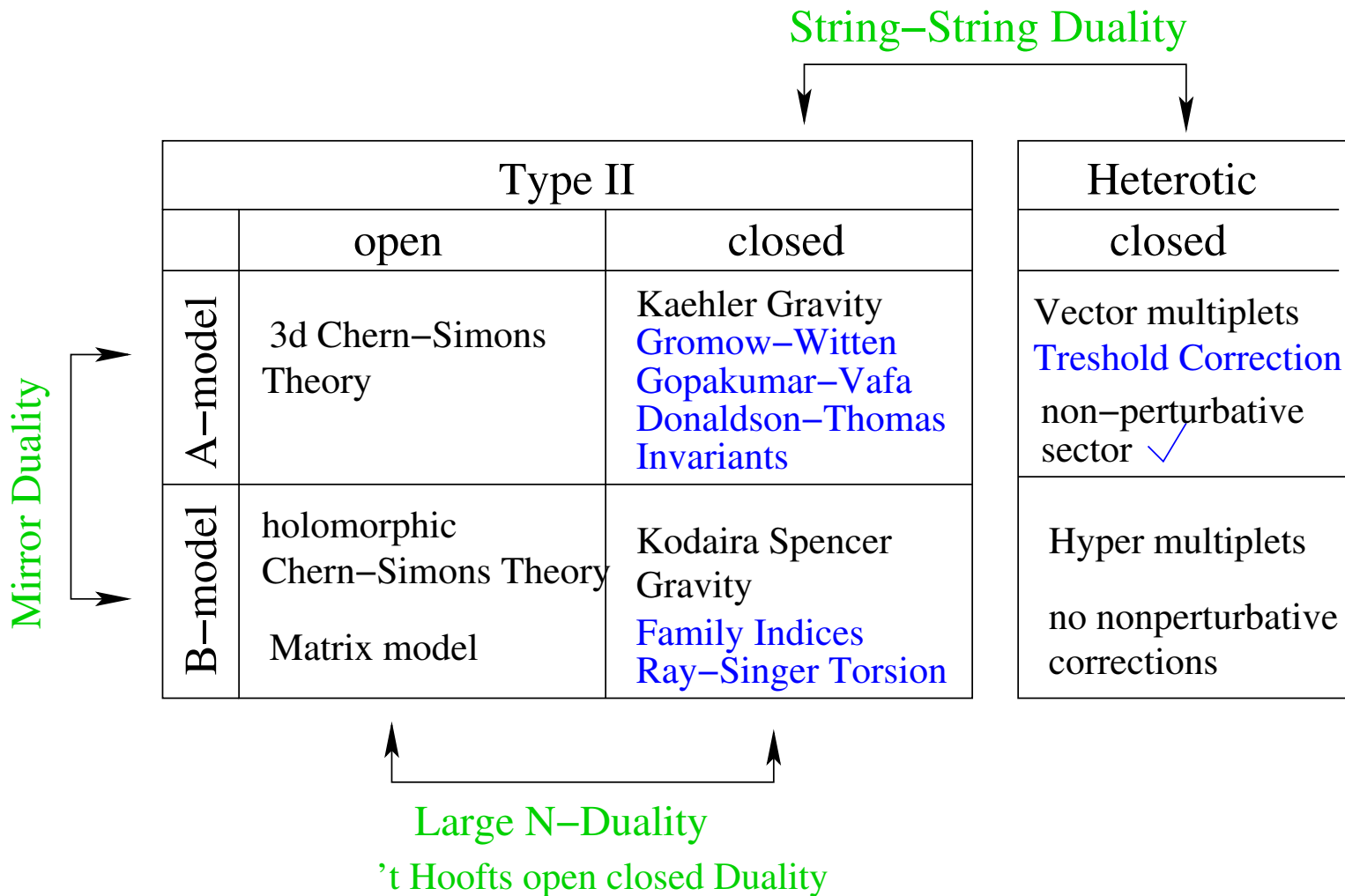
with  $q_\lambda = e^{i\lambda}$ ,  $Z_{\text{DT}}^{\text{hol}}(M, q_\lambda, q) = \sum_{\beta, k \in \mathbf{Z}} n_\beta^{(k)} q_\lambda^k q^\beta$  and

$$Z_{\text{GV}}^{\text{hol}} = e^{\frac{c(t)}{\lambda^2} + l(t)} \exp \left( \sum_{g=0}^{\infty} \sum_{\beta \in H_2(M, \mathbf{Z})} \sum_{m=1}^{\infty} \tilde{n}_\beta^{(g)} \frac{1}{m} \left( 2 \sin \frac{m\lambda}{2} \right)^{2g-2} q^{\beta m} \right) \quad (2)$$

$$Z_{\text{GV}}^{\text{hol}} = \prod_{\beta} \left[ \left( \prod_{r=1}^{\infty} (1 - q_{\lambda}^r q^{\beta})^{r \tilde{n}_{\beta}^{(0)}} \right) \prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} (1 - q_{\lambda}^{g-l-1} q^{\beta})^{(-1)^{g+r} \binom{2g-2}{l} \tilde{n}_{\beta}^{(g)}} \right]$$

where  $n_{\beta}^{(k)} \in \mathbf{Z}!$  and  $\tilde{n}_{\beta}^{(k)} \in \mathbf{Z}!$

# Computability :



Beautiful example of large duality:

$$M = \text{Point}$$

$\overline{\mathcal{M}}_g(X, \beta) \rightarrow \mathcal{M}_{g,n}$  moduli space of  $n$ -punctured Riemann-surface.

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n = \text{ghost number} > 0$$

Need operator insertions.

$$\Psi(x_i) = c_1(T\Sigma_{g,n}|_{x_i}), \quad (1, 1) - \text{form}$$

on  $\overline{\mathcal{M}}_{g,n}$ .  $\tau_n(x_i) := (2n + 1)!! \Psi_i^n$

$Z(t) = e^F$  with  $F := \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$  connected correlators and

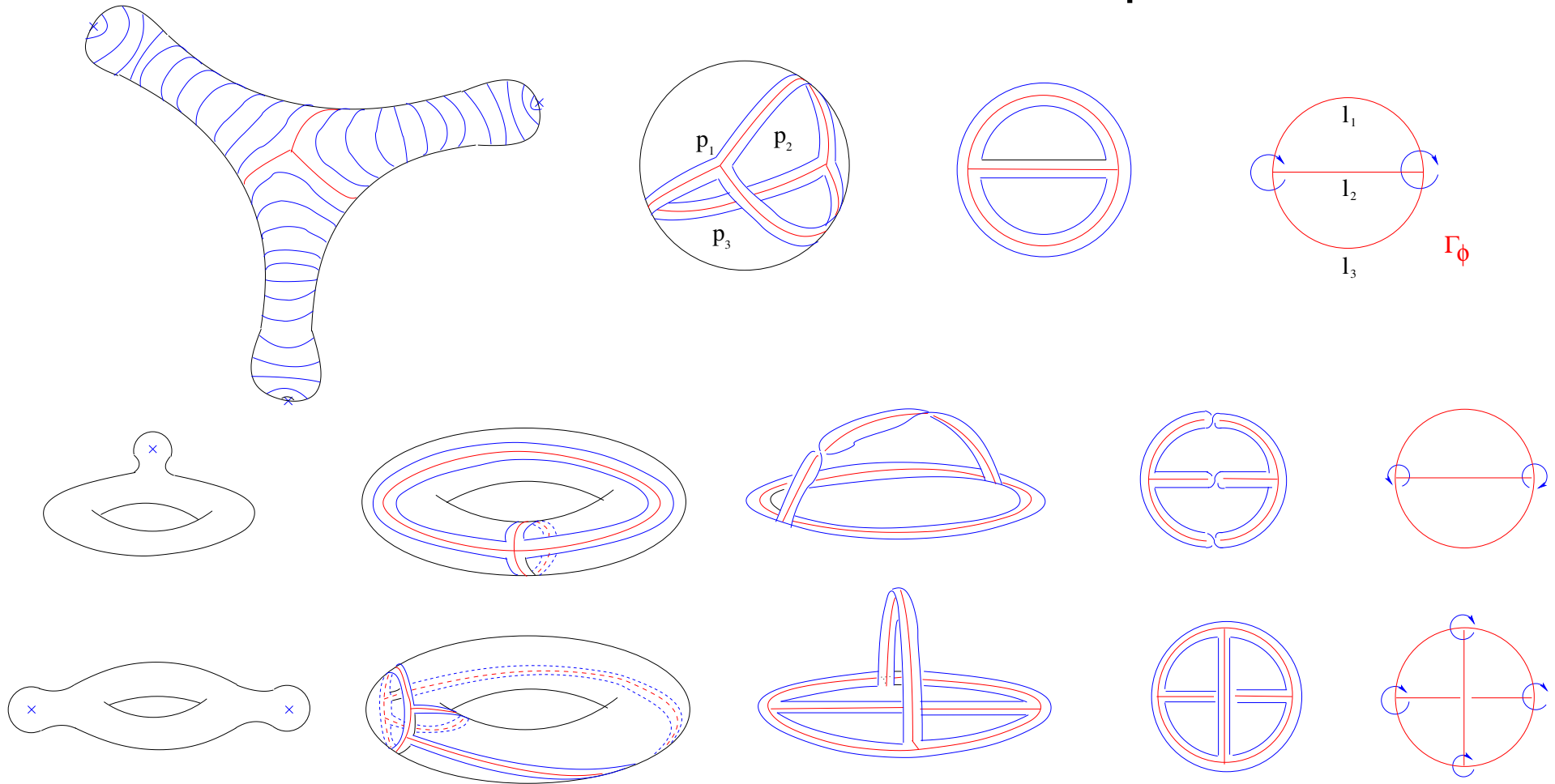
$$F_g(t_0, t_1, \dots) = \sum_{\{d_i\}} \langle \prod \tau_{d_i} \rangle_g \prod_{r>0} \frac{t_r^{n_r}}{n_r!},$$

where  $n_r = \text{Card}(i : d_i = r)$ .

Jenkins-Strebel quadratic differentials  $\phi = \phi(z)dz^2$  defines flat non-degenerate metric  $|\phi(z)||dz|^2$

- horizontal trajectories of  $\phi$  are curve in  $\Sigma_{g,n}$  along which  $\phi(z)$  is real and positive.
- the union of **non-closed trajectories** has measure zero.

# Combinatorial description of the moduli space:



Kontsevich 92:  $C_{\Lambda}^{-1} := \int DX e^{-\frac{1}{2}\text{tr}X^2\Lambda}$

$$Z(\Lambda) = C_{\Lambda} \int DX e^{\frac{i}{6}\text{tr}(X^3) - \frac{1}{2}\text{tr}(X^2\Lambda)}$$

With

$$t_i(\Lambda) = -(2i - 1)!! \text{tr} \Lambda^{-(2i+1)}$$

$$Z(\underline{t}(\Lambda)) = \tau(\underline{t}(\Lambda)) = Z(\Lambda)$$

$\tau(\underline{t})$  **Tau-function** of KDV hierarchy + string- and dilaton equation  $\rightarrow$  **Virasoro constraints:**  $L_n Z = 0 \quad n \geq -1$ .

Exact calculation of finite subsets of correlators at finite  $N$ .

non-compact CY  
(toric)

compact CY  
(compl. Int. in toric A)

A-model	localisation	✓	$g=0$ Kontsevich
	Pandharipande, Graber, Zaslow, Liu, Katz		$g>0?$ Givental, Yau, Lian..
	large N duality	✓	?
	Vertex		
	Aganagic, Klemm, Marino, Vafa		Pandharipande, Okounkov, Gathman
	Relative G-W	✓	in principle
	Pandharipande		$g$ small
<hr/>			
B-model	large N duality	✓	?
	Matrix model		
	Aganagic, Klemm, Marino, Vafa		
	DT	✓	?
	Maulik, Nekrasov, Okounkov, Pandharipande		
	Holomorphic	✓	$g$ small
	anomaly BCOV		BCOV, Klemm, Katz, Vafa
	KS	?	?
	BCOV		

## Stringduality and the FHSV model :

Type II compactification with  $N = 2$  supergravity in 4d  
 $\Leftrightarrow M$  Kähler + 2 covariant constant spinors  $\Rightarrow$   
 $\text{Hol}(M) = \text{SU}(3)$  & exceptional case  $\text{Hol}(M) = \text{SU}(2) \times \mathbf{Z}_2$   
 (Voison & Borcea, FHSV model).

$$M = K3 \times T2/\sigma, \quad \sigma \text{ free } \mathbf{Z}_2 \text{ inv.}$$

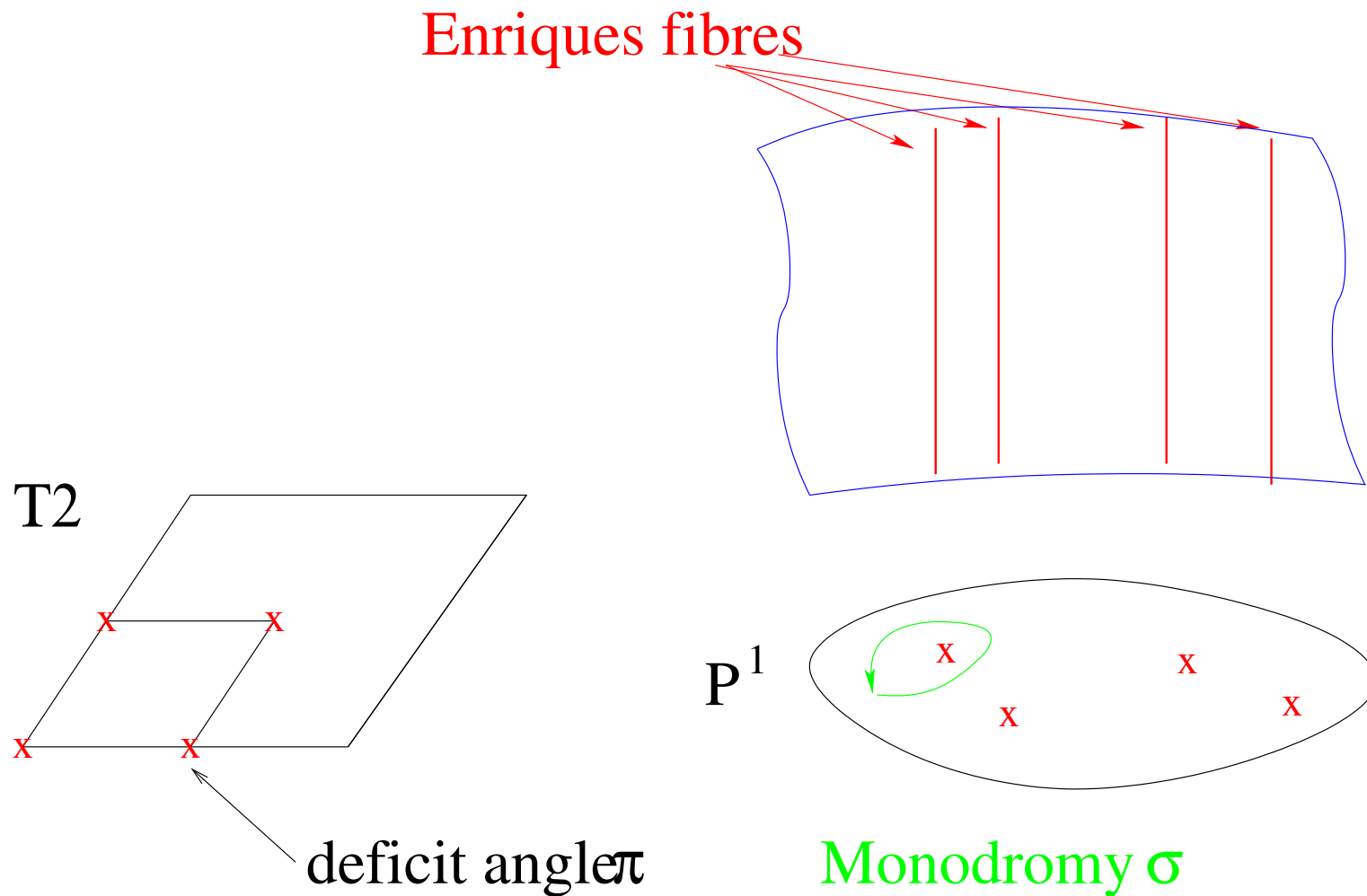
Algebraic realisation:  $K3$  double cover, branched over a  
 bidegree (4,4) hypersurface in  $\mathbf{P}^1 \times \mathbf{P}^1$

$$Y^2 = f_{4:4}(s : t, u : v),$$

$\sigma(Y, s, t, u, v) \mapsto (-Y, -s, t, -u, v)$ , free. (2, 0)-form  
 $\Omega = su \frac{dt \wedge dv}{y}$  **anti-invariant**.

Similarly: Elliptic curve  $y^2 = f_4(w : z)$ ,  
 $\sigma(y, w, z) \mapsto (-y, w, z)$  with four fixpoints  $f_4(w : z) = 0$ .  
 $T\mathbb{2} = \mathbf{C}/\mathbf{Z}^2$ ,  $z \in \mathbf{C}$   $\sigma(z) \mapsto -z$ ,  $dz$  **anti-invariant**.

$M = K3 \times T2/\sigma$  is a  $K3$  fibration over  $\mathbf{P}^1$ .



Project with **Marcos Mariño, CERN** solve topological string on VBFHSV model.

$K3$  fibrations over  $\mathbf{P}^1$  exhibit weak coupling limit of heterotic  $\leftrightarrow$  Type II String duality.

$$vol_C(\mathbf{P}^1) = 4\pi S = \frac{4\pi}{g_{het}^2}$$

Weakly coupled heterotic string is **CFT** based on Narain compactification. **Duality predicts** that a suitable subsector solves topological string on the fibre.

- **What is the map of parameters ?**  $\rightarrow$  tracing duality

through orbifoldization.

- What quantities correspond to symplectic invariants ?  
→ Gopakumar-Vafa BPS states can be identified with CFT states, essentially elliptic genus.

Before the twist: Narain lattice for heterotic compactification on  $T^4 \times T^2$  dual to Type II on  $K3 \times T^2$ .

$$\Gamma_{22,6} = \Gamma^{9,1} \oplus \Gamma^{9,1} \oplus \Gamma_s^{1,1} \oplus \Gamma^{2,2} \oplus \Gamma_g^{1,1},$$

$\Gamma^{9,1}$  decomposes

$$\Gamma^{9,1} = \Gamma_d^{1,1} \oplus E_8(-1). \quad (3)$$

Starting point of Heterotic/TypeII duality:

$\Gamma_{19,3} = \Gamma^{9,1} \oplus \Gamma^{9,1} \oplus \Gamma_g^{1,1}$  is  $H^2(K3, \mathbf{Z})$  lattice  $\sigma$  acts as

follows:

$$|p_1, p_2, p_3, p_4, p_5\rangle \rightarrow e^{2\pi i \delta \cdot p_3} |p_2, p_1, p_3, -p_4, -p_5\rangle$$

where  $\delta \in \Gamma_s^{1,1}$ ,  $\delta^2 = 2$ .  $\sigma$  exchanges summands in  $\Gamma^{9,1} \oplus \Gamma^{9,1}$ , shifts  $\Gamma_s^{1,1}$ ,  $\{H^0(K3), H^4(K3)\}$ , and by  $-1$  on  $\Gamma^{2,2} \oplus \Gamma_g^{1,1}$ .

On  $K3$  lattice we have Enriques involution **Horikawa 70's**

$$\begin{aligned} \Gamma_{10,2}^+ &= E_8(-2) \oplus \Gamma^{1,1}(2) \oplus \Gamma_s^{1,1}(1), & p_1 + p_2 \\ \Gamma_{10,2}^- &= E_8(-2) \oplus \Gamma^{1,1}(2) \oplus \Gamma^{1,1}(1), & p_1 - p_2 . \end{aligned}$$

On T2

$$\begin{aligned}\Gamma_{1,1}^- &= \{dz, d\bar{z}\} \\ \Gamma_{1,1}^+ &= \{1, dz \wedge d\bar{z}\}\end{aligned}$$

$h^{11}(M) = h^{21}(M) = 11$  and the **vector multiplet moduli space** for this compactification is given by

$$\mathrm{SL}(2, \mathbf{Z}) \backslash \mathrm{SL}(2, \mathbf{R}) / \mathrm{SO}(2) \times \mathcal{M}, \quad (4)$$

where

$$\mathcal{M} = O(\Gamma^1) \backslash O(10, 2) / [O(10) \times O(2)], \quad (5)$$

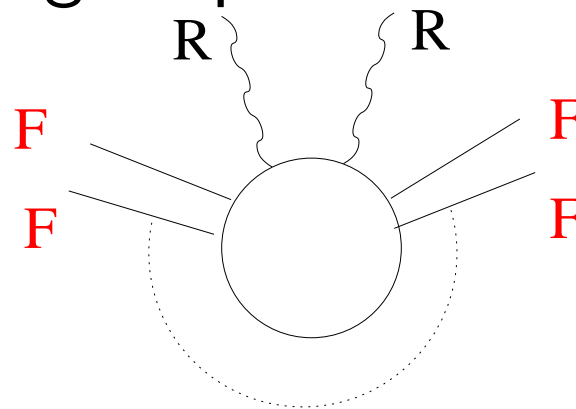
and  $O(\Gamma^1)$  is the group of automorphisms of the lattice

$$\Gamma^1 = \Gamma_s^{1,1} \oplus \Gamma_d^{1,1}(2) \oplus E_8(-2). \quad (6)$$

$$S_{eff} = \sum_{g>0} \int_{\mathbf{R}_{1,3}} F_g(t, \bar{t}) F_-^{2g-2} R_- \wedge R_-$$

$F^{top} = \lim_{\bar{t} \rightarrow \infty} F_g(t, \bar{t})$ . Calculate term in effective action.

Consider the following amplitude:



- One loop exact by N=2 **non-renormalisation theorem**
- Only **BPS-states contribute** in the loop (BPS saturated)

- depends on off-shell  $q \neq SU(2)_L \otimes SU(2)_R$ ,  $SU(2)_R$  does not couple explicitly  $\rightarrow$  only multiplicity  $(2j_R^3 + 1)(-1)^{2j_R^3}$

String loop calculation:

$$F_g = \int_{\mathcal{F}} d^2\tau \tau_2^{2g-1} \frac{1}{|\eta|^4} \sum_{\text{even}} \frac{i}{\pi} \partial_\tau \left( \frac{\vartheta_{[b]}^{[a]}(\tau)}{\eta(\tau)} \right) C_g^{\text{int}} [a, b],$$

where

$$Z_g^{\text{int}} [a, b] = \langle : (\partial X)^{2g} : \rangle$$

evaluated in the internal conformal field theory.  $X$  is the

complex boson, right-moving mode on the  $\mathbf{T}^2$ .

$$F_g = \int_{\mathcal{F}} d^2\tau \tau_2^{2g-1} \sum_{J=1}^3 \mathcal{I}_J^g,$$

with  $\mathcal{I}_J^g = \frac{2\mathcal{P}_{2g}}{Y^{g-1}} \overline{\Theta}_J^g(\tau) f_J(q)$ , the Narain theta function with insertion

$$\Theta_{\Gamma}^g(\tau, \alpha, \beta) = \sum_{p \in \Gamma} p_R^{2g-2} \exp \left\{ \pi i \tau (p + \beta/2)_+^2 + \pi i \bar{\tau} (p + \beta/2)_-^2 + \pi i (p + \beta/2, \alpha) \right\}$$

$$\begin{aligned}
f_1(q) &= \frac{64}{\eta^6(\tau)\vartheta_2^6(\tau)} = \frac{1}{q} \prod_{n=1}^{\infty} (1 - q^{2n})^{-12}, \\
f_2(q) &= \frac{2}{\eta^6(\tau)\vartheta_4^6(\tau)} = 2q^{-\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^n)^{-12} (1 - q^{n-1/2})^{-12}, \\
f_3(q) &= \frac{2}{\eta^6(\tau)\vartheta_3^6(\tau)} = 2q^{-\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^n)^{-12} (1 + q^{n-1/2})^{-12}
\end{aligned}$$

and

$$\left( \frac{2\pi i \eta^3 \lambda}{\vartheta_1(\lambda|\tau)} \right)^2 = \sum_{g=0}^{\infty} (2\pi \lambda)^{2g} \mathcal{P}_{2g}.$$

How to do the Integral ?

Guess the answer: For genus 1 this was attempted, based on expectation that  $F_1$  modular form of weight 4 for  $\Gamma^1$ .

Borcherds 95

$$\Phi_B = e^{2\pi i \rho y} \prod_{r^2 > 0} (1 - e^{2\pi i r \cdot y})^{(-1)^{m+n} d(-r^2/2)}$$

where  $y = R^{9,1} + iC^+$  and

$$\sum_n d(n) q^n = q^{-1} \frac{\prod (1 + q^{2n+1})^8}{\prod (1 - q^{2n})^8}$$

Harvey & Moore 95. Right function, wrong cusp for GW-expansion.

Do the Integral: Dixon, Kaplunovski, Louis, H&M, Borchers Using  $SL(2, Z)$  invariance of integrand and lattice reduction.

Borchers Reduction: Reduction on  $\Gamma_g^{1,1}$ , polylogarithm

$$Li_n(x) = \sum_r \frac{x^r}{r^n}$$

$$F_g = \sum_r c_g(\vec{q}^2 + nm)(-1)^{n+m} \text{Li}_{3-2g}(e^{-r \cdot t}) \quad (7)$$

where  $c_g(n)$  are defined by

$$\sum_n c_g(n) q^n = \frac{128}{\eta^6(\tau) \vartheta_2^6(\tau)} \left\{ 2^{1-g} \mathcal{P}_{2g}(q) - 2^{1+g} \mathcal{P}_{2g}(q^4) \right\},$$

Geometric Reduction: Reduction on  $\Gamma_s^{1,1}$ ,

$$x^{\frac{1}{2}} := \exp(-r \cdot t).$$

The norm squared of  $r$  will be taken in the lattice  $\Gamma^{1,1} \oplus E_8(-1)$ .

$$F_g = \sum_r c_g(r^2) \left\{ 2^{3-2g} \text{Li}_{3-2g}(e^{-r \cdot t}) - \text{Li}_{3-2g}(e^{-2r \cdot t}) \right\},$$

where

$$\sum_n c_g(n) q^n = \frac{128}{\eta^6(\tau) \vartheta_2^6(\tau)} \mathcal{P}_{2g}(G_k).$$

Heterotic string prediction for topological string amplitudes

$g$	$r^2 = 0$	2	4	6	8	10
0	0	0	0	0	0	0
1	8	128	1152	7680	42112	200448
2	0	-16	-288	-2880	-21056	-125280
3	0	0	24	480	5264	41760
4	0	0	0	-32	-704	-8400
5	0	0	0	0	40	960
6	0	0	0	0	0	-48

**Table 1:** BPS invariants  $n_g^{odd}(r) \in \mathbf{Z}$  for the odd classes  $r$  in the fiber direction.

## Geometrical Checks:

$K_E \neq \mathcal{O}_E$  but  $K_E^2 = \mathcal{O}_E$ ,  $h^{2,0}(E) = h^{1,0} = 0$ .  $\forall C$  in class  $r$  with  $[C]^2 = r^2 \exists C + K_E [C]^2 = [C + K_E]^2 = r^2$ . All contribution to  $n_r^g$  come from classes in the four Enriques singular fibres  $\rightarrow n_r^g = 0 \pmod{8}$ , ok with table.

Moduli space of  $D2 - D0$  brane system given by the cohomology of the fibration of  $Jac(C) \rightarrow \mathcal{M}$ , where  $Jac(C)$  is the Jacobian of the curve and  $\mathcal{M}$  is the geometric deformation space. The  $su(2)_L \times su(2)_R$  action is given by the **Lefschetz action** in base and fibre direction.

For smooth curves:

$$n_g^r \sum (-1)^{2j_r^3} (2j_r^3 + 1) = (-1)^{\dim \mathcal{M}} \chi(\mathcal{M})$$

Adjunction formula

$$2g - 2 = [C]^2 + [C][K_E] ,$$

$[C][K_E] = 0$  on Enriques. The moduli space  $\mathcal{M}_C = \mathbf{P}H^0(\mathcal{O}(C))$  with  $h^0(\mathcal{O}(C))$  calculated by Riemann-Roch theorem

$$\chi(\mathcal{O}(C)) = \frac{[C]^2 + [C][K_S]}{2} + \chi(\mathcal{O}(S)) .$$

$\chi(\mathcal{O}(C)) = \sum_{i=1}^n (-1)^i h^i(\mathcal{O}(C))$ . For smooth curves in a  
 Enriques surface  $H^1(\mathcal{O}(C)) = H^2(\mathcal{O}(C)) = 0$  and  
 $\chi(\mathcal{O}(C)) = h^0(\mathcal{O}(C))$ .  $\chi(\mathcal{O}(E)) = \sum_{i=1}^2 h^{i,0}(E) = 1$

$$\mathcal{M}_C = \mathbf{P}^{g-1} .$$

$$n_g(r) = 8 \cdot (-1)^{\frac{r^2}{2}} \chi(\mathbf{P}^{\frac{r^2}{2}}) = 8 \cdot (-1)^{\frac{r^2}{2}} \left( \frac{r^2}{2} + 1 \right)$$

For nodal curves we can do Hilbert scheme calculation  
 similar to [Katz, Klemm, Vafa 99](#) → many further checks.

$g$	$r^2 = 0$	2	4	6	8	10
0	0	0	0	0	0	0
1	0	0	0	0	24	288
2	0	0	0	0	0	0
3	0	0	0	0	0	0

**Table 2:** Differences between the heterotic BPS prediction in Tab. 1 and the geometric BPS calculation.

## Conclusion:

We provided the full all genus expansion for the topological string on  $M$  for 10 out of 11 classes, namely the one in the fibre.

What is with the pure base class direction ? Blow up the  $K3$  in the covering.  $\mathcal{O}(0) \otimes \mathcal{O}(0) \rightarrow T^2$ . This is determined by 2d Yang-Mills theory on the  $T^2$  **Dijkgraaf 96**. Integrable and can be solved by free fermion system. Answer written by **M. Kaneko & D. Zagier 96**

$$\Theta(q_\lambda, q) = \prod_{n>0} (1 - q^n) \prod_{n>0, n \text{ odd}} (1 - q_\lambda^{n^2/8} q^{n/2} y) (1 - q_\lambda^{-n^2/8} q^{n/2} y^{-1})$$

What is with the mixed classes. Know the action of the  $U$  duality group on the periods of  $M$ . However we have not yet found the expected product form of  $Z$ , which describes all classes.

Is there an integrable model governing that .....