Center of small quantum group and affine Springer fibers

(joint with R. Bezrukavnikov, P. Boixeda-Alvarez and P. Shan)

G connected quasi-simple linear group / $\mathbb{C}$

$\check{G}$ dual group, simply connected

$T \subset B \subset G$ Cartan and Borel subgroups

$\Lambda = X^{\vee}(T) \supset Q = \mathbb{Z}\Phi \supset \Phi$ = root system

$\mathfrak{t} = \mathrm{Lie}\, T$, $\mathfrak{g} = \mathrm{Lie}\, G$, $U(\mathfrak{g})$ = enveloping algebra

Define $\check{T}$, $\check{\Lambda}$, $\check{\Phi}$, $\check{\mathfrak{g}}$ similarly

W = Weyl group

$W_{\ell, af} = W \ltimes \ell Q^{\vee} \subset W_{\ell, ex} = W \ltimes \ell \check{\Lambda}$

Ⓐ Positive characteristic :

Assume G defined over $\Bbbk = \overline{\Bbbk}$, $\mathrm{char}(\Bbbk) = p > 0$

$\mathrm{Fr}: G \longrightarrow G$ Frobenius map

$G \supset G_1 = \mathrm{Ker}(\mathrm{Fr})$ = Frobenius kernel = a group scheme

$\Bbbk[G] \twoheadrightarrow \Bbbk[G_1]$ = finite dimensional Hopf algebra

$\mathfrak{g}$ = restricted Lie algebra with p-operation

$$\mathfrak{g} \to \mathfrak{g} \quad , \quad x \mapsto x^{[p]}$$

$u(\mathfrak{g})$ = restricted enveloping algebra

$$= U(\mathfrak{g}) / (x^p - x^{[p]} \; ; \; x \in \mathfrak{g})$$

$= \Bbbk[G_1]^*$ as Hopf algebra (finite dimensional)

$\mathrm{Rep}(G_1)$ = finite dimensional rational G_1 - modules

$= \mathrm{Comod}(\Bbbk[G_1])$

$= \mathrm{Rep}(u(G))$

$\mathrm{Dist}(G) \subset \Bbbk[G]^*$ distribution algebra

$\mathrm{Rep}(G)$ = finite dimensional rational G - modules

$= \mathrm{Rep}(\mathrm{Dist}(G))$

= finite dimensional integrable $\mathrm{Dist}(G)$ - modules

There are algebra homomorphisms

$$U(\mathfrak{g}) \twoheadrightarrow u(G) \hookrightarrow \mathrm{Dist}(G)$$

Steinberg tensor product formula $\Rightarrow \mathrm{Irr}(G_1)$ determins $\mathrm{Irr}(G)$

EX :

(a) $G = GL_n$, $Fr^* \in \mathrm{Aut}_{k\text{-alg}}(k[GL_n])$, $Fr^*(X_{ij}) = X_{ij}^p$

$x^{[p]} = x^p \quad \forall x \in \mathfrak{gl}_n$

(b) $G = \mathbb{G}_a$, $k[\mathbb{G}_a] = k[t]$, $Fr^*(t) = t^p$

$u(\mathbb{G}_a) = k[\mathbb{G}_{a,1}] = k[t]/(t^p)$ self-dual

(B) Quantum groups:

$\zeta \in \mathbb{C}^\times$ root of 1 of order $= \ell$, ℓ is good

(odd, $> h =$ Coxeter number, prime to 3 in type G_2)

We have the following quantum groups attached to $\check{G}$:

(a) $\mathcal{U}_\zeta$ = DeConcini-Kac quantum group (an. of $U(\check{\mathfrak{g}})$)

(b) U_ζ = Lusztig quantum group (an. of $\mathrm{Dist}(\check{G})$)

(c) u_ζ = small quantum group (an. of $u(\check{G})$)

There are algebra homomorphisms

$$\mathcal{U}_\zeta \twoheadrightarrow u_\zeta \hookrightarrow U_\zeta \xrightarrow{\mathrm{Fr}} U_1 = U(\check{\mathfrak{g}})$$

$$\mathrm{Fr}^*: \mathrm{Rep}(\check{G}) = \mathrm{Rep}(U_1) \longrightarrow \mathrm{Rep}(U_\zeta)$$

$\mathrm{Rep}(U_\zeta) = (\check{\Lambda}$ graded, integrable$)$ finite dimensional modules

$\mathrm{Rep}(U_1) = \mathrm{Rep}(\check{G})$

$\mathrm{Rep}(u_\zeta) =$ finite dimensional modules

NB: $A =$ abstract group

$$\begin{array}{ccc} \text{Braided tensor} & \text{de-equivariantization} & \text{Braided tensor} \\ \mathbb{C}\text{-linear categories} & \xrightarrow{\hspace{3cm}} & \mathbb{C}\text{-linear categories} \\ & \xleftarrow[\text{equivariantization}]{\hspace{3cm}} & \\ \text{with } \mathrm{Rep}(A)\text{-action } \mathcal{C}^A & & \text{with } A\text{-action } \mathcal{C} \end{array}$$

$$\mathrm{Obj}(\mathcal{C}^A) = \{ (X, \phi_a) \,;\, X \in \mathrm{Obj}(\mathcal{C}),\ a \in A,\ \phi_a : a(X) \xrightarrow{\sim} X \}$$

$$= \{ A\text{-equivariant objects} \}$$

Idem if A is an affine algebraic group

EX:

(a) G-variety $X \Rightarrow \mathrm{Coh}(X)^G = \mathrm{Coh}_G(X)$

(b) (Arkhipov – Gaitsgory)

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Fr}^*(-)\otimes -}{\circlearrowright} \mathrm{Rep}(U_\zeta) \underset{\text{eq.}}{\overset{\text{de-eq.}}{\rightleftarrows}} \mathrm{Rep}(u_\zeta) \circlearrowleft \check{G}$$

$$\mathrm{Rep}(u_\zeta)^{\check{G}} = \mathrm{Rep}(U_\zeta)$$

$$\mathrm{Rep}(u_\zeta) = \mathrm{Vect} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(U_\zeta)$$

$$\Rightarrow Z(u_\zeta)^{\check{G}} = Z(U_\zeta) \cap u_\zeta$$

Block decomposition:

$$u_\zeta = \bigoplus_{\omega \in \check{\Lambda}/W_{\ell,af}} u_\zeta^\omega$$

$$u_\zeta^0 = \text{principal block}$$

CONJ (Lachowska – Qi) : Type A

(a) $\dim Z(u_\zeta) = \dfrac{1}{(h+1)\ell}\dbinom{(h+1)\ell}{h}$

(b) $\dim Z(u_\zeta^\circ) = (h+1)^{h-1}$

(c) $\check{G}$ acts trivially on $Z(u_\zeta)$

NB: $\{\text{coinvariants}\} = \mathbb{C}[\mathfrak{t}]/(\mathbb{C}[\mathfrak{t}]^W_+)$

$= H^\bullet(G/B)$

$=$ $W \ltimes \mathbb{C}[\mathfrak{t}]$ – module of dim $\#W$

$\{\text{diagonal coinvariants}\} = \mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee]/(\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee]^W_+)$

$=$ $W \ltimes \mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee]$ – module with

quotient to $\mathbb{C}[Q/(h+1)Q]$ as W-module

$W \ltimes \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^\vee] = gr$ (Cherednik's rational algebra DAHA)

In type A we have (Haiman, Gordon)

$$\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^\vee] / (\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^\vee]^W_+) = \mathbb{C}[Q/(h+1)Q]$$

$$= gr(\text{simple DAHA - module})$$

$$\dim(\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^\vee] / (\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^\vee]^W_+)) = (h+1)^{h-1}$$

© Geometrization:

* Affine flag manifolds:

$$\check{\Lambda} \subset \check{\Lambda} \underset{\mathbb{Z}}{\otimes} \mathbb{R} = \mathfrak{t}_{\mathbb{R}}$$

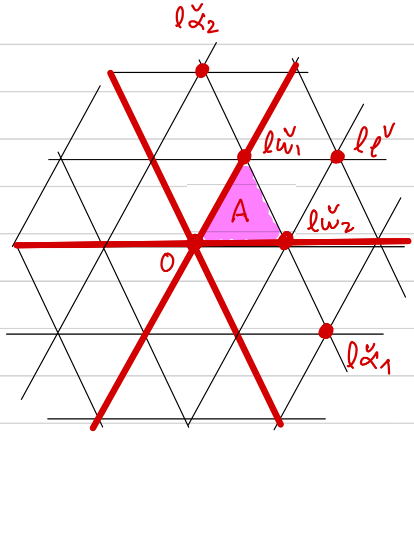
$\mathfrak{t}_{\mathbb{R}} \supset \bar{A} =$ closed fundamental alcove

$$= \{ \lambda \in \mathfrak{t}_{\mathbb{R}} \; ; \; 0 \leq (\mu, \check{\alpha}) \leq \ell \; , \; \forall \check{\alpha} \in \bar{\check{\Phi}} \}$$

$= \text{fundamental domain for } W_{\ell, af} \circlearrowright \mathfrak{t}_{\mathbb{R}}$

$$\check{\Lambda} / W_{\ell, af} \simeq \check{\Lambda} \cap \bar{A}$$

EX: $G = PSL_3$



$K^{(\ell)} = \mathbb{C}((\pi^\ell)) \subset K = \mathbb{C}((\pi)) \supset \mathcal{O} = \mathbb{C}[\![\pi]\!]$

$\omega \in \bar{A}$ labels a parahoric subgroup $P^\omega \subset G(K^{(\ell)})$

$Fl^{\omega,(\ell)} = G(K^{(\ell)})/P^\omega$ partial affine flag manifold $/G$

$Fl^\omega = Fl^{\omega,(1)}$

EX:

(a) $\omega = 0 \Rightarrow Fl^\omega = Gr = G(K)/G(\mathcal{O})$

with $G(\mathcal{O})$ maximal compact in $G(K)$

(b) $\omega \in A \Rightarrow Fl^\omega = Fl = G(K)/I$

with $I \subset G(\mathcal{O})$ Iwahori

$\mathbb{C}^\times \circlearrowright Fl^{\omega}$ by loop rotation, $\zeta \in \mathbb{C}^\times$

$$Gr^{\zeta} = \bigsqcup_{\omega \in \check{\Lambda}/W_{\ell,ex}} Fl^{\omega,(\ell)}$$

as homogeneous spaces over $G(K^{(\ell)})$

* Affine Springer fibers :

Choose $\gamma \in (\mathfrak{g} \otimes K)^{rs}$ compact and regular semi-simple

$Fl^{\omega}_{\gamma} \subset Fl^{\omega}$ affine Springer fiber

$$= \{ Ad(g)(P^{\omega}) \; ; \; g \in G(K), \; Ad(g^{-1})(\gamma) \in rad(Lie\, P^{\omega}) \}$$

$$\subset Ad(G(K))(P^{\omega}) = Fl^{\omega}$$

$\overline{K} = \bigcup_{\ell} K^{(\ell)}$ = algebraic closure of K

γ homogeneous $\overset{def}{\iff}$ $G(\overline{K})$ conjugate to $\mathfrak{g} \otimes t^{d}$ with $d \in \mathbb{Q}$

$\Rightarrow$ Fl^{ω}_{γ} ind-scheme, pure, equidimensional,

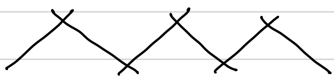
finite dimensional

$\gamma_{\ell} = s \otimes t^{\ell}$, $\gamma = \gamma_1$, $s \in \mathfrak{g}^{rs}$ regular semisimple

LEM:

$$Gr^{\xi}_{\gamma_\ell} = \bigsqcup_{\omega \in \check{\Lambda}/W_{\ell,ex}} Fl^{\omega,(\ell)}_{\gamma_\ell} \simeq \bigsqcup_{\omega} Fl^{\omega}_{\gamma}$$

EX: $G = SL_2$, $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$

$Fl_\gamma = \ldots$  $\ldots$ chain of $\mathbb{P}^1$'s

$Fl_\gamma / \mathbb{Z} =$  $\Rightarrow \dim H^*(Fl_\gamma)^{\mathbb{Z}} = 3$

* **Computation of $H^\bullet(Gr^{\xi}_{\gamma_e})$:**

(a) $T \curvearrowright Fl^{\omega}_{\gamma}$ with $(Fl^{\omega}_{\gamma})^T = (Fl^{\omega})^T = W_{ex}/W_{\omega}$

W_{ω} = stabilizer of $\omega \in \Lambda^{\vee}$ in W_{ex}

(b) Fl^{ω}_{γ} has a paving by affine cells

$\Rightarrow H^\bullet_T(Fl^{\omega}_{\gamma})$ equivariantly formal

(c) There are finitely many 1 dimensional T-orbits

in Fl^{ω}_{γ} between any 2 fixed points

$\Rightarrow$ GKM applies to $H^\bullet_T(Gr^{\xi}_{\gamma_e})$

$$H^\bullet_T(Fl^{\omega}_{\gamma}) = \{ (a_w) \in Fun(W_{ex}/W_{\omega}, H^\bullet_T) ; \quad (*)$$
$$a_w \equiv a_{s_{\alpha,m}\cdot w} \mod \alpha, \ \forall (\alpha,m) \in \Phi \times \mathbb{Z} \}$$

$GL(t) \ni s_{\alpha,m}$ = reflexion / affine hyperplane

LEM + $\check{\Lambda} = \coprod_{\omega \in \check{\Lambda}/W_{ex}} W_{ex}/W_\omega$ + $(*)$ give

$$H_T^\bullet(Gr_{re}^3) = \{ (a_x) \in \mathrm{Fun}(\Lambda^\vee, H_T^\bullet) ;$$
$$a_x \equiv a_{s_{\alpha,m}\cdot x} \bmod \alpha, \ \forall (\alpha,m) \in \Phi \times \mathbb{Z} \}$$

* Symmetries of affine Springer fibers :

$$G(K) \circlearrowright Fl^{\omega} \quad \text{and} \quad T(K) \circlearrowright Fl^{\omega}_{\gamma}$$

Left W_{ex} - action on $H^{\bullet}_{T}(Fl^{\omega}_{\gamma})$:

(a) $W_{ex} \circlearrowright H^{\bullet}_{T}(Fl^{\omega})$

(b) $\check{\Lambda} = X_*(T) = \pi_0(T(K)) \circlearrowright H^{\bullet}_{T}(Fl^{\omega}_{\gamma})$

extends to $W_{ex} \circlearrowright H^{\bullet}_{T}(Fl^{\omega}_{\gamma})$ via GKM

Right action $W_{ex} \circlearrowright H^{\bullet}_{T}(Fl_{\gamma})$ = Springer action

PROP:

(a) $$\dim H^\bullet(\mathcal{F}\ell_\gamma)^{W_{ex}} = (h+1)^{rk}$$

(b) $$\dim H^\bullet(Gr^{\xi}_{\gamma_\ell})^{W_\ell, ex} = \frac{1}{\# W} \prod_{i=1}^{rk\, G} ((h+1)\ell - h + e_i)$$

$\{e_i\} = \{\text{exponents of } W\}$

NB:

(a) In type $\neq A$ part (a) is due to Boixeda Alvarez - Losev [BL]

(b) Proof of (b) uses reduction to $\dim H^\bullet(Gr^{\xi}_{\gamma'_\ell})$ with

γ'_ℓ elliptic (Sommers)

CONJ: We have a commutative diagram

$$\begin{tikzcd} H^\bullet(Gr^{\xi}_{\gamma_\ell})^{W_{\ell,ex}} \arrow[r, "\sim"] \arrow[d, hook] & Z(u_\xi)^{\check{G}} \arrow[d, hook] \\ H^\bullet(Gr^{\xi}_{\gamma_\ell})^{\ell\check{\Lambda}} \arrow[r, "\sim"] & Z(u_\xi)^{\check{T}} \end{tikzcd}$$

with W-invariant lower map.

THM: We have a commutative diagram as above with injective horizontal maps. The lower map is W-invariant

NB: (a) Restricting to principal block we get

$$\begin{tikzcd} H^\bullet(Fl_\gamma)^{W_{ex}} \arrow[r, "A"] \arrow[d, hook] & Z(u_\zeta^\circ)^{\check{G}} \arrow[d, hook] \\ H^\bullet(Fl_\gamma)^{\check{\Lambda}} \arrow[r, "B"] & Z(u_\zeta^\circ)^{\check{T}} \end{tikzcd}$$

(b) In type A, the left map is invertible. Proving B is surjective implies also that $Z(u_\zeta^\circ)^{\check{G}} = Z(u_\zeta^\circ)$ [BL]

(c) Compare with Soergel theorem:

$\mathcal{O}(\mathfrak{g})$ = BGG category $\mathcal{O}$ of $\mathfrak{g}$

= finitely generated B-integrable $U(\mathfrak{g})$-modules

$Z(\mathcal{O}(\mathfrak{g})) = H^\bullet(G/B)$

$= \mathbb{C}[\mathfrak{t}]/(\mathbb{C}[\mathfrak{t}]^W_+)$

(d) DAHA's act on the cohomology of affine Springer fibers

$\Rightarrow$ The conjecture relates $Z(u_3^\circ)$ with DAHA's

(e) To construct B we use GKM to define an

isomorphism $H_T^\bullet(Fl_\gamma) \xrightarrow{\sim} Z(T \ltimes u_3^\circ)$. The

equivariant formality of L.H.S. gives a map

$H^\bullet(Fl_\gamma) \hookrightarrow Z(u_3^\circ)$ which restricts to B

Ⓓ Definition of the map $\dot{A}$:

* Mixed geometry approach:

$D^b_{m,I^u}(Gr)$ = I^u-equt derived cat^y of mixed complexes on Gr

$D^b_{m,IW}(Fl)$ = Iwahori-Whittaker derived category of mixed complexes on Fl

[BY13] $\Rightarrow$ $\exists$ (Koszul) equivalence $D^b_{m,I^u}(Gr) \longrightarrow D^b_{m,IW}(Fl)$

$$\begin{tikzcd}
D^b_{m,I^u}(Gr) \arrow[r] \arrow[loop below, "(-1/2)"] & D^b_{m,IW}(Fl) \arrow[loop below, "{[1](1/2)}"]
\end{tikzcd}$$

$$\begin{tikzcd}
H^\bullet(Fl) \arrow[d, "\text{purity}"'] \arrow[ddd, bend left, "(*)"] \\
\mathrm{Hom}\left(\mathrm{id}_{D^b_{m,IW}(Fl)}, \mathrm{id}_{D^b_{m,IW}(Fl)}[-\bullet](-\tfrac{\bullet}{2})\right) \arrow[d, "{[BY13]}"'] \\
\mathrm{Hom}\left(\mathrm{id}_{D^b_{m,I^u}(Gr)}, \mathrm{id}_{D^b_{m,I^u}(Gr)}(\tfrac{\bullet}{2})\right) \arrow[d, "\text{degrading functor}"'] \\
Z(D^b_{I^u}(Gr)) \arrow[r, equal, "{[ABG04]}"] & Z(Rep(U_\zeta)^\circ)
\end{tikzcd}$$

$$Z(\mathfrak{u}_\zeta^\circ)^{\check{G}} = Z(U_\zeta) \cap \mathfrak{u}_\zeta^\circ \longrightarrow Z(\mathrm{Rep}(U_\zeta)^\circ)$$

CLAIM: $(*)$ factors through $H^\bullet(Fl) \longrightarrow Z(\mathfrak{u}_\zeta^\circ)^{G^\vee}$

$*$ Harish-Chandra approach:

$Y = \mathrm{Spec}(A/I) \subset X = \mathrm{Spec}(A)$

$\tilde{N}_Y(X)$ = deformation to normal cone

$$= \mathrm{Spec}(A[\hbar, \hbar^{-1} I])$$

$N_Y(X)$ = normal cone of Y into X

$$= \mathrm{Spec}\Big(\bigoplus_{n \geq 0} I^n / I^{n+1}\Big)$$

$$= \tilde{N}_Y(X) \times_{\mathbb{A}^1} \{0\}$$

** Apply Borel's construction to $H^\bullet(Gr^3)$:

$\mathcal{L}(\lambda) \in \mathrm{Pic}(Fl)$ for all $\lambda \in \Lambda \times \mathbb{Z} = X^*(T \times \mathbb{C}^*)$

T-equivariance + cup product by $c_1(\mathcal{L}(\lambda))$'s give a map

$$\begin{array}{ccc} H_T^\bullet \otimes H_T^\bullet & \longrightarrow & H_T^\bullet(Fl) \\ \| & & \| \\ \mathbb{C}[t \times t] & \longrightarrow & \mathbb{C}[N_\Delta(t \times t)]^{\oplus \pi_1(G)} \end{array}$$

with $\Delta = t \times_{t/W} t \subset t \times t$

Similarly, let $\Omega = \{1\} \times_{T/W} (T/W)$ relative to the map

$$T/W \to T/W, \quad W.t \mapsto W.t^\ell$$

$\Rightarrow H^\bullet(Gr^3) = \mathbb{C}[N_\Omega(T/W)]$ by equivariant formality

** Harish Chandra isomorphism for quantum groups (Rosso):

U_ζ = Lusztig quantum group at $q=\zeta$ $= U_q/(q-\zeta)$

U_q = a $\mathbb{C}[q,q^{-1}]$-algebra

$U_{\hat{\zeta}}$ = completion of U_q at $q=\zeta$

= a $\mathbb{C}[\![\hbar]\!]$-algebra with $\hbar = q-\zeta$

$HC : \mathbb{C}[\![\hbar]\!][T/W] \xrightarrow{\sim} Z(U_{\hat{\zeta}})$ Harish Chandra isomorphism

Specialize $\hbar=0 \Rightarrow$ HC factorizes through

$$\mathbb{C}[\tilde{N}_\Omega(T/W)]^{\wedge}_{\hbar=0} \longrightarrow Z(U_{\hat{\zeta}})$$

(Proof uses De Concini-Kac's Theorem on $Z(\mathcal{U}_\zeta)$)

$\Rightarrow$ The fiber at $\hbar=0$ gives a map

$$H^\bullet(Gr^{\mathfrak{Z}}) = \mathbb{C}[N_\Omega(T/W)] \longrightarrow Z(U_{\mathfrak{Z}})$$

PROP: This map factors through

$$\begin{tikzcd} H^\bullet(Gr^{\mathfrak{Z}}) \arrow[rr] \arrow[dr, "res"'] & & Z(u_{\mathfrak{Z}})^{\check{G}} \subset Z(U_{\mathfrak{Z}}) \\ & H^\bullet(Gr^{\mathfrak{Z}}_{\gamma_\ell})^{W_{\ell,ex}} \arrow[ur] & \end{tikzcd}$$

(Proof uses GKM description of $H^\bullet(Gr^{\mathfrak{Z}}_{\gamma_\ell})$)

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