Hyperlogarithm functions for complex curves

B. Enriquez (coll. F. Zerbini)

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Abstract

• Hyperlogarithm (HL) functions: a class of functions on the punctured complex plane, motivated by monodromy computations (Poincaré, Lappo-Danilevskii).

• recent applications: (a) identification of multiple zeta values with certain classes of periods (Goncharov-Manin conj.)(Brown) // (b) Feynman integral computations in QFT (Brown, Panzer)

• Elliptic analogues of the HL functions were also introduced and applied to QFT computations (Brown-Levin, Broedel-Duhr-Dulat-Tancredi).

• goals of lecture : (a) introduce and study analogues of the algebra of HL functions for an arbitrary affine complex curve // (b) explain relation with the construction of an alternative analogue of the algebra of HL functions (d'Hoker-Hidding-Schlotterer)

• joint work w. F. Zerbini.

Bibliography

HL in genus 0 Poincaré 1884 // Lappo-Danilevskii 1953 // F. Brown, Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}$, Ann. Sci. Ec. Norm. Sup. 2009 // E. Panzer, Feynman integrals and HLs, PhD Thesis, arXiv:1506.07243v1

HL in genus 1 J. Broedel, C. Duhr, F. Dulat, L. Tancredi, Elliptic polylogarithms and iterated integrals on elliptic curves I: general formalism. J. High Energy Phys. 2018 // F. Brown, A. Levin, Multiple elliptic polylogarithms, arXiv:1110.6917 // E, F. Zerbini, Elliptic HLs, arXiv:2307.01833

Differential filtrations K.-T. Chen, Extension of C^∞ functions by integrals and Malcev completion of π_1, Adv. in Math. 1977

HL in arb. genus E. D'Hoker, M. Hidding and O. Schlotterer. Constructing polylogarithms on higher-genus Riemann surfaces, arXiv:2306.08644 // E, F. Zerbini, Analogue of HL functions on affine complex curves, arXiv:2212.03119

Plan (21 pp)

- A. Hyperlogarithm (HL) functions on $\mathbb{P}^1$ (3pp)
- B. Minimal stable algebra of multivalued functions on C and algebraic Maurer-Cartan (MC) elements (6pp)
- C. Filtrations of the algebra of multivalued functions (3pp)
- D. Ideas of the proofs (6pp)
- E. Relation w. d'Hoker-Hidding-Schlotterer (DHS) approach to HL functions on C (3pp)

A. HL functions on $\mathbb{P}^1$

A1. HL functions is genus 0: basics

- $S \subset \mathbb{C}$ is a finite subset, $\mathbb{C}_S := \mathbb{C} \smallsetminus S$.
- $\underline{S}$ is S viewed as abstract set, $\underline{S}^* := \sqcup_{n\geq 0} \underline{S}^n$ is the set of words in $\underline{S}$, so $\underline{s}_1 \cdots \underline{s}_n \in \underline{S}^*$
- map $\underline{S}^* \to \mathcal{O}_{hol}(\tilde{\mathbb{C}}_S)$, $w \mapsto L_w$ defined by $L_\emptyset := 1$, $L_{w\underline{s}}(z) := \int^z L_w(t) d\ln(t-s)$.
- generating series : $\mathbf{L}(z) := \sum_w L_w(z) w^*$ in $\mathcal{O}_{hol}(\tilde{\mathbb{C}}_S)\langle\langle \underline{S} \rangle\rangle$ satisfies $d\mathbf{L} = \mathbf{L} \cdot J$ where $J := \sum_s \underline{s} \cdot d(z-s)$ and $\mathbf{L}(z) \sim z^{\sum_s \underline{s}}$ as $z \to \infty$.
- the functions $(L_w)_w$ are the hyperlogarithm (HL) functions (Poincaré 1884, Lappo-Danilesvkii 1953) (also called "Goncharov polylogarithms" by physicists)

A2. HL functions in genus 0: motivation

• original motivation: monodromy computations/Riemann-Hilbert problem.

• HL functions are applied to identification of set of periods arising from moduli space of marked stable genus-zero curves with set of MZVs (solution of Goncharov-Manin conjecture, Brown 2009)

• a large class of Feynman integrals computing scattering amplitudes in QFT can be expressed in terms of HLs (Brown 2009, Panzer 2015)

• genus 1 analogues were introduced and applied to scattering amplitude computations (Broedel, Duhr, Dulat, Tancredi 2018)

• this lecture: construct analogues of HL functions for an arbitrary complex curve

A3. Properties of HL functions in genus 0 (Brown 09)

- set $\mathcal{H} := \mathrm{Span}_{\mathbb{C}}(L_w, w \in \underline{S}^*) \subset \mathcal{O}_{hol}(\tilde{\mathbb{C}}_S)$.
- then $\mathcal{H}$ is a subalgebra, and $\mathrm{Sh}(\mathbb{C}\underline{S}) \to \mathcal{H}$, $w \mapsto L_w$ is an algebra iso ($\mathrm{Sh}(V)$: shuffle algebra over a vector space V)
- $\mathcal{O}(\mathbb{C}_S) := \mathbb{C}[z, 1/(z-s)|s \in S]$
- the family $(L_w)_w$ is $\mathcal{O}(\mathbb{C}_S)$-free, i.e. the algebra morphism $\mathcal{O}(\mathbb{C}_S) \otimes \mathcal{H} \to \mathcal{O}_{hol}(\tilde{\mathbb{C}}_S)$ is injective
- let $A_{\mathbb{C}_S}$ be its image, then $A_{\mathbb{C}_S} \subset \mathcal{O}_{hol}(\tilde{\mathbb{C}}_S)$ is a subalgebra, stable under all the endos $int_{\omega,z_0} : f \mapsto [z \mapsto \int_{z_0}^z f\omega]$ for $\omega \in \mathcal{O}(\mathbb{C}_S)dz = \Omega(\mathbb{C}_S)$ (regular differentials on $\mathbb{C}_S$) and $z_0 \in \tilde{\mathbb{C}}_S$
- $A_{\mathbb{C}_S}$ is the minimal subalgebra of $\mathcal{O}_{hol}(\tilde{\mathbb{C}}_S)$ with this property

B. Minimal stable algebra of multivalued funs on C and alg. MC elts

B1. The minimal stable algebra A_C of C

- C: affine complex curve
- $p : \tilde{C} \to C$ a universal cover // $\Gamma_C := \mathrm{Aut}(\tilde{C}/C)$
- $\mathcal{O}(C)$: the algebra of regular functions on C // $\mathcal{O}_{hol}(\tilde{C})$: the algebra of holomorphic functions on $\tilde{C}$
- $\Omega(C)$: the $\mathcal{O}(C)$-module of regular differentials on C
- there exists a minimal subalgebra $A_C \subset \mathcal{O}_{hol}(\tilde{C})$, stable under all the endos $int_{\omega,z_0} : f \mapsto [z \mapsto \int_{z_0}^{z} f\omega]$ for $\omega \in \Omega(C)$ and $z_0 \in \tilde{C}$

B2. Maurer-Cartan elements for C

- $\mathrm{H}_C := \Omega(C)/d\mathcal{O}(C)$ ($= \mathrm{H}_1^{dR}(C)$ as C is affine)
- $\mathfrak{g} := \mathbb{L}(\mathrm{H}_C^*)$ (free Lie alg. gen. by H_C^*), $\hat{\mathfrak{g}}$:=degree completion

Definition

(a) an alg. Maurer-Cartan (MC) element for C: an elt $J \in \Omega(C)\hat{\otimes}\hat{\mathfrak{g}}$
(b) J is non-degenerate iff $\mathrm{im}(J \in \Omega(C)\hat{\otimes}\hat{\mathfrak{g}} \to \mathrm{H}_C \otimes \mathrm{H}_C^*) = id$.
(c) $\mathrm{MC}_{nd}(C) := \{$non-deg. MC elts for $C\}$

Definition

(a) Σ_C is the set of sections $\sigma : \mathrm{H}_C \to \Omega(C)$ of can. projection.
(b) maps $\Sigma_C \leftrightarrows \mathrm{MC}_{nd}(C)$, $\sigma \mapsto J_\sigma := \sum_i \sigma(h_i) \otimes h^i$ with $(h_i)_i$, $(h^i)_i$ dual bases of H_C and H_C^* and $J \mapsto \sigma_J$ such that $J \equiv J_{\sigma_J}$ mod $\Omega(C)\hat{\otimes}\hat{\mathfrak{g}}_{\geq 2}$.

Then $\sigma_{J_\sigma} = \sigma$.

B3. Algebra morphisms attached to a MC element

to (J, z_0) with J=MC elt and $z_0 \in \tilde{C}$, attach:

• the function $g_{J,z_0} : \tilde{C} \to \exp(\hat{\mathfrak{g}}) = \mathcal{G}((U\mathfrak{g})^\wedge)$ ($\mathcal{G}$: group-like elements) such that $dg = gJ$ and $g(z_0) = 1$; it is holomorphic;

• the map $f_{J,z_0} : \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{hol}(\tilde{C})$, $\xi \mapsto [z \mapsto \xi(g_{J,z_0}(z))]$ based on $\mathrm{Sh}(\mathrm{H}_C) = \oplus_{n\geq 0}(U\mathfrak{g})[n]^* \to (\prod_{n\geq 0} U\mathfrak{g}[n])^* = ((U\mathfrak{g})^\wedge)^*$.

Lemma

(a) The map $f_{J,z_0} : \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{hol}(\tilde{C})$ is a morphism of algebras.
(b) If $\sigma \in \Sigma_C$, $f_{J_\sigma,z_0}([h_1|\dots|h_k]) = (z \mapsto \int_{z_0}^z \sigma(h_1) \circ \cdots \sigma(h_k))$ where $\int_{z_0}^z \sigma(h_1) \circ \cdots \sigma(h_k)$=iterated integral.

B4. Algebra isos attached to a MC element

Theorem A

Let J non-degenerate MC element and $z_0 \in \tilde{C}$.
(a) $\mathrm{im}(\tilde{f}_{J,z_0} : \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{hol}(\tilde{C}))$ *is independent of z_0, denoted $\mathcal{H}_C(J)$.*
(b) The map $f_{J,z_0} : \mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{hol}(\tilde{C})$, $f \otimes a \mapsto p^*(f) \cdot \tilde{f}_{J,z_0}(a)$ is an injective alg. morphism.
(c) $\mathrm{im}(f_{J,z_0}) = A_C$ *(independent of (J, z_0))*
(d) hence alg. iso. $f_{J,z_0} : \mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C) \to A_C$.

- If $C = \mathbb{C}_S$, then $\mathrm{H}_C \simeq \mathbb{C}\underline{S}$, and $\Sigma_C \ni \sigma_0 := [\mathbb{C}\underline{S} \ni \underline{s} \mapsto d\ln(z-s) \in \Omega(\mathbb{C}_S)]$. Then $\mathcal{H} = \mathcal{H}_{\mathbb{C}_S}(J_{\sigma_0})$. Hence $\mathcal{H}_C(J)$=analogue of alg. of HL functions

- whereas $\mathcal{H}_C(J)$ varies with J, the product $\mathcal{O}(C) \cdot \mathcal{H}_C(J)$ does not and $= A_C$.

B5. Group aspects of alg. isos attached to a MC element

Group actions on algebras

• to $\mathfrak{n}$ nilpotent Lie algebra, attach 1-connected complex alg. group $\exp(\mathfrak{n}) = \mathcal{G}((U\mathfrak{n})^\wedge)$. Then alg. of reg. funs on $\exp(\mathfrak{n})$ given by $\mathcal{O}(\exp(\mathfrak{n})) = (U\mathfrak{n})' = \cup_{n\geq 0}((U\mathfrak{n})_+^n)^\perp \subset (U\mathfrak{n})^*$. Right regular action of $\exp(\mathfrak{n})$ on $\mathcal{O}(\exp(\mathfrak{n}))$.

• right regular action of $\exp(\hat{\mathfrak{g}}) = \mathcal{G}((U\mathfrak{g})^\wedge) = \mathcal{G}(\hat{T}(\mathrm{H}_C^*))$ on $\mathcal{O}(\exp(\hat{\mathfrak{g}})) = T(\mathrm{H}_C^*)' = \mathrm{Sh}(\mathrm{H}_C)$, hence on $\mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C)$.

• right regular action of Γ_C on $\mathcal{O}_{hol}(\tilde{C})$ by $(f_{|\gamma})(z) := f(\gamma z)$, hence of $\mathbb{C}\Gamma_C$; restricts to action on $A_C \subset \mathcal{O}_{hol}(\tilde{C})$.

• action of $\mathbb{C}\Gamma_C$ on A_C extends to action of $(\mathbb{C}\Gamma_C)^\wedge := \varprojlim_n (\mathbb{C}\Gamma_C)/(\mathbb{C}\Gamma_C)_+^n$, which restricts to action of prounipotent completion $\Gamma_C(\mathbb{C}) = \mathcal{G}((\mathbb{C}\Gamma_C)^\wedge)$ on A_C.

B6. Group aspects of alg. isos (cont'd)

Theorem B

(a) For J non-deg. MC elt and $z_0 \in \tilde{C}$, the map

$$\mathbb{C}\Gamma_C \otimes \mathrm{Sh}(H_C) \to \mathbb{C}, \quad \gamma \otimes \xi \mapsto \xi(g_{J,z_0}(\gamma z_0))$$

is a Hopf algebra pairing. It induces an iso $i_{J,z_0} : \Gamma_C(\mathbb{C}) \to \exp(\hat{\mathfrak{g}})$ of prounipotent groups.
(b) The alg. iso. $f_{J,z_0} : \mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C) \to A_C$ is compatible with i_{J,z_0} and the action of its source and target on the target and source of f_{J,z_0}.

C. Filtrations of the algebra of multivalued funs on C

C1. Group-action induced filtrations

Definition

$\mathcal{O}_{mod}(\tilde{C}) \subset \mathcal{O}_{hol}(\tilde{C})$ is the subalgebra of functions with moderate growth at the cusps of C.

The action of Γ_C on $\mathcal{O}_{hol}(\tilde{C})$ restricts to an action on $\mathcal{O}_{mod}(\tilde{C})$.

Definition

For $n \geq 0$, $F_n^{gp}\mathcal{O}_{mod}(\tilde{C}) := \{f \in \mathcal{O}_{mod}(\tilde{C}) | f_{|(\mathbb{C}\Gamma_C)_+^{n+1}} = 0\}$.

Lemma

$F_\bullet^{gp}\mathcal{O}_{mod}(\tilde{C})$ is an increasing algebra filtration of $\mathcal{O}_{hol}(\tilde{C})$ with $F_0^{gp} = \mathcal{O}(C) \subset F_1^{gp} \subset \cdots$, stable under action of Γ_C.

C2. Differential filtrations

Definition

(a) $F_0^\delta \mathcal{O}_{hol}(\tilde{C}) := \mathbb{C}$
(b) for $n \geq 0$, $F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C}) := \{f \in \mathcal{O}_{hol}(\tilde{C}) | d(f) \in \Omega(C) \cdot F_n^\delta \mathcal{O}_{hol}(\tilde{C})\}$
(c) for $n \geq 0$, $F_n^\mu \mathcal{O}_{hol}(\tilde{C}) := \mathcal{O}(C) \cdot F_n^\delta \mathcal{O}_{hol}(\tilde{C})$

Definitions inspired by (Chen 1977).

Lemma

(a) $F_\bullet^\delta$ *and* $F_\bullet^\mu$ *are increasing algebra filtrations of* $\mathcal{O}_{hol}(\tilde{C})$.
(b) $F_0^\delta \subset F_0^\mu \subset F_1^\delta \subset F_1^\mu \subset \cdots$

C3. Filtrations induced by iterated integration

Lemma

For $z_0 \in \tilde{C}$, the map $I_{z_0} : \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{hol}(\tilde{C})$ given by $[\omega_1|\cdots|\omega_n] \mapsto [z \mapsto \int_{z_0}^{z} \omega_1 \circ \cdots \circ \omega_n]$ is an algebra morphism.

Definition

For $n \geq 0$, set $F_n\mathrm{Sh}(V) := \oplus_{k\leq n}\mathrm{Sh}_n(V)$.

Then $F_\bullet\mathrm{Sh}(V)$ is an algebra filtration of $\mathrm{Sh}(V)$ and $I_{z_0}(F_\bullet\mathrm{Sh}(\Omega(C)))$ is an algebra filtration of $\mathcal{O}_{hol}(\tilde{C})$, which is independent of z_0.

C4. Comparison of filtrations

Theorem C

(a) For any $(J, z_0) \in \mathrm{MC}_{nd}(C) \times \tilde{C}$*, equalities*

$$F_\bullet^{gp}\mathcal{O}_{mod}(\tilde{C}) = F_\bullet^{\mu}\mathcal{O}(\tilde{C}) = f_{J,x_0}(\mathcal{O}(C) \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C))$$

and

$$I_{z_0}(F_\bullet \mathrm{Sh}(\Omega(C))) = F_\bullet^{\delta}\mathcal{O}(\tilde{C}) = f_{J,x_0}(\mathbb{C} \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C) + \mathcal{O}(C) \otimes F_{\bullet-1}\mathrm{Sh}(\mathrm{H}_C))$$

of algebra filtrations of $\mathcal{O}_{hol}(\tilde{C})$*.*
(b) Equality

$$F_\infty^{gp}\mathcal{O}_{mod}(\tilde{C}) = F_\infty^{\mu}\mathcal{O}(\tilde{C}) = f_{J,x_0}(\mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C)) = I_{z_0}(\mathrm{Sh}(\Omega(C))) \\ = F_\infty^{\delta}\mathcal{O}(\tilde{C}) = A_C$$

of subalgebras of $\mathcal{O}(\tilde{C})$*.*

D. Ideas of the proofs

D1. Easy statements

Thm. A: Let J non-degenerate MC element and $z_0 \in \tilde{C}$.
(a) $\mathrm{im}(\tilde{f}_{J,z_0} : \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{hol}(\tilde{C}))$ is independent of z_0, denoted $\mathcal{H}_C(J)$.
(b) The map $f_{J,z_0} : \mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{hol}(\tilde{C})$, $f \otimes a \mapsto p^*(f) \cdot \tilde{f}_{J,z_0}(a)$ is an alg. morphism.
Thm. B: (a) For J non-deg. MC elt and $z_0 \in \tilde{C}$, the map

$$\mathbb{C}\Gamma_C \otimes \mathrm{Sh}(H_C) \to \mathbb{C}, \quad \gamma \otimes \xi \mapsto \xi(g_{J,z_0}(\gamma z_0))$$

is a Hopf algebra pairing. It induces an iso $i_{J,z_0} : \Gamma_C(\mathbb{C}) \to \exp(\hat{\mathfrak{g}})$ of prounipotent groups [based on freeness of Γ_C].
(b) The alg. morphism $f_{J,z_0} : \mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C) \to A_C$ is compatible with i_{J,z_0} and the action of its source and target on the target and source of f_{J,z_0}.

D2. Statements based on iterated integrals

Thm. C: For any $(J, z_0) \in \mathrm{MC}_{nd}(C) \times \tilde{C}$, equalities

$$I_{z_0}(F_\bullet \mathrm{Sh}(\Omega(C))) = F_\bullet^\delta \mathcal{O}(\tilde{C}) = f_{J,x_0}(\mathbb{C} \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C) + \mathcal{O}(C) \otimes F_{\bullet-1} \mathrm{Sh}(\mathrm{H}_C))$$

of filtrations of $\mathcal{O}_{hol}(\tilde{C})$ and

$$I_{z_0}(\mathrm{Sh}(\Omega(C))) = A_C$$

of subalgebras of $\mathcal{O}_{hol}(\tilde{C})$ based on study of iterated integrals. Therefore

$$F_\bullet^\mu \mathcal{O}(\tilde{C}) = f_{J,x_0}(\mathcal{O}(C) \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C))$$

(multiplying by $\mathcal{O}(C)$) and

$$A_C = F_\infty^\delta \mathcal{O}(\tilde{C}) = F_\infty^\mu \mathcal{O}(\tilde{C}) = f_{J,z_0}(\mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C))$$

(2nd eq. due to relations $F_\bullet^\mu / F_\bullet^\delta$, 3rd eq. to relations between $\mathbb{C} \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C) + \mathcal{O}(C) \otimes F_{\bullet-1} \mathrm{Sh}(\mathrm{H}_C)$ and $\mathcal{O}(C) \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C)$).

D3. Remaining statements

- Remaining statements:

Thm. C: $\boxed{F_\bullet^{gp}\mathcal{O}_{mod}(\tilde{C}) = f_{J,x_0}(\mathcal{O}(C)\otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C))}$

Thm. A: $\boxed{f_{J,z_0} \text{ is an algebra iso } \mathcal{O}(C)\otimes \mathrm{Sh}(\mathrm{H}_C) \to A_C.}$

- Since $A_C = f_{J,x_0}(\mathcal{O}(C)\otimes \mathrm{Sh}(\mathrm{H}_C))$, both are consequences of:

$\boxed{f_{J,z_0} : \mathcal{O}(C)\otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C) \to F_\bullet^{gp}\mathcal{O}_{mod}(\tilde{C}) \text{ is an iso of filtrations.}}$

D4. Hopf algebras w. a comodule algebra (HACAs)

- A Hopf algebra w. comodule algebra (HACA) is a pair (O, A) where O is a Hopf algebra and A is an algebra, equipped with a left coaction of O.
- For O Hopf algebra, set $F_nO := \mathrm{Ker}(O \to O^{\otimes n} \to (O/\mathbb{C})^{\otimes n})$.
- $F_\bullet O$ is an increasing Hopf algebra filtration ($F_a \cdot F_b \subset F_{a+b}$, $\Delta(F_a) \subset \sum_{a'+a''=a} F_{a'} \otimes F_{a''}$). Then $\mathrm{gr}(O)$ a graded Hopf algebra.
- For (O, A) a HACA, set F_nA :=preimage of $F_nO \otimes A$ under $\Delta_A : A \to O \otimes A$. One has $F_0O = \mathbb{C}$, $F_0A = A^O$.
- then $F_\bullet A$ is an algebra filtration, and $\Delta_A : F_\bullet A \to F_\bullet(O \otimes A)$
- get an associated graded HACA $(\mathrm{gr}O, \mathrm{gr}A)$, hence $\Delta_{\mathrm{gr}A} : \mathrm{gr}A \to \mathrm{gr}O \otimes \mathrm{gr}A$.
- the map $\mathrm{gr}A \to \mathrm{gr}O \otimes \mathrm{gr}_0 A = \mathrm{gr}O \otimes A^O$ is injective.

D5. Examples of HACAs

- if A is equipped w. right action of a group Γ, set $F_nA = \{a \in A | a_{|(\mathbb{C}\Gamma)_+^{n+1}} = 0\}$, then $F_\bullet A$ is an algebra filtration of A
- if Γ is fin. gen., then $(\mathbb{C}\Gamma)' = \cup_{n\geq 0}((\mathbb{C}\Gamma)_+^{n+1})^\perp \subset (\mathbb{C}\Gamma)^*$ is a Hopf algebra, and $(F_\infty A, (\mathbb{C}\Gamma)')$ is a HACA (*)
- associated HACA filtration : $F_\bullet A$ and $F_n(\mathbb{C}\Gamma)' = ((\mathbb{C}\Gamma)_+^{n+1})^\perp$
- the map $F_nA \otimes (\mathbb{C}\Gamma)_+^n \to A^\Gamma$, $a \otimes x \mapsto a_{|x}$ induces left nondeg. pairing $\mathrm{gr}_n A \otimes (\mathbb{C}\Gamma)_+^n/(\mathbb{C}\Gamma)_+^{n+1} \to A^\Gamma$, hence injection $\mathrm{gr}_n A \hookrightarrow (\ldots)^* \otimes A^\Gamma$.
- example of construction (*): $A = \mathcal{O}_{mod}(\tilde{C})$, $\Gamma = \Gamma_C$, then get a HACA $(F_\infty \mathcal{O}_{mod}(\tilde{C}), (\mathbb{C}\Gamma_C)')$.
- another example of HACA: $\mathrm{Sh}(\mathrm{H}_C)$ is a HA, hence $(\mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C))$ is a HACA
- $F_n^{hpf}\mathrm{Sh}(\mathrm{H}_C) = F_n\mathrm{Sh}(\mathrm{H}_C)$, hence associated filtration given by $\mathcal{O}(C) \otimes F_\bullet\mathrm{Sh}(\mathrm{H}_C)$ and $F_\bullet\mathrm{Sh}(\mathrm{H}_C)$.

D6. Proof of remaining statement: a HACA morphism and its associated graded morphism

• $i_{J,z_0} : \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'$ is a Hopf algebra iso (already seen) therefore induces iso of graded Hopf algebras $\mathrm{gr}(i_{J,z_0}) : \mathrm{Sh}(\mathrm{H}_C)) \to \mathrm{gr}(\mathbb{C}\Gamma_C)'$

• $(f_{J,z_0}, i_{J,z_0}) : (\mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C)) \to (F_\infty^{gp}\mathcal{O}_{mod}(\tilde{C}), (\mathbb{C}\Gamma_C)')$ is a HACA morphism.

• induces sequence of graded algebra morphisms

$$\mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}_C) \overset{\mathrm{gr} f_{J,z_0}}{\to} \mathrm{gr}^{gp}\mathcal{O}_{mod}(\tilde{C}) \hookrightarrow \mathcal{O}(C) \otimes \mathrm{gr}(\mathbb{C}\Gamma_C)'$$

• which coincides with $id \otimes \mathrm{gr}(i_{J,z_0})$ therefore is an iso of graded algebras

• therefore $f_{J,z_0} : \mathcal{O}(C) \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C) \to F_\bullet^{gp}\mathcal{O}_{mod}(\tilde{C})$ is an iso of filtrations.

E. Relation w. the DHS approach

E1. The DHS construction

- let Σ be a Riemann surface, $p \in \Sigma$, let $h :=$genus(Σ).
- $H_1(\Sigma) := H_1(\Sigma, \mathbb{C})$ is a symplectic $2h$-dimensional vector space, $\omega \in \Lambda^2(H_1(\Sigma))$ the elt induced by sympl. form
- fix a decomposition $H_1(\Sigma) = L_a \oplus L_b$ as a sum of Lagrangian subspaces
- $\Gamma(\Sigma, \Omega^{0,1})$ is a h-dimensional vector space
- integration is a perfect pairing $\Gamma_{hol}(\Sigma, \Omega^{0,1}) \otimes L_b \to \mathbb{C}$, hence dual element $\mathcal{J}^{0,1}_{\mathrm{DHS}} := \sum_i b_i \otimes \overline{\omega^i} \in L_b \otimes \Gamma_{hol}(\Sigma, \Omega^{0,1})$
- $\Gamma(\Sigma, \Omega^{1,0}(p))$ is the space of sections of $\Omega^{1,0}$, smooth outside p, with local expansion $a \cdot dz/z+$bounded at p; then a is called the residue at p
- the Lie algebra $\mathfrak{g} := \hat{\mathbb{L}}(H_1(\Sigma))$ is graded: $\deg(L_b) = 0$, $\deg(L_a) = 1$

E2. The DHS construction (cont'd)

- there exists unique $\mathcal{J}^{1,0}_{\mathrm{DHS}} \in \mathfrak{g}[1]\hat{\otimes}\Gamma(\Sigma, \Omega^{1,0}(p))$, such that $\overline{\partial}\mathcal{J}^{1,0}_{\mathrm{DHS}} = [\mathcal{J}^{0,1}_{\mathrm{DHS}}, \mathcal{J}^{1,0}_{\mathrm{DHS}}]$ (equality in $\mathfrak{g}[1]\hat{\otimes}\Gamma(\Sigma, \Omega^{1,1}(p)))$ and $\mathrm{res}_p(\mathcal{J}^{1,0}_{\mathrm{DHS}}) = \omega \in \Lambda^2(H_1(\Sigma)) \subset \mathfrak{g}$
- set $\mathcal{J}_{\mathrm{DHS}} := \mathcal{J}^{1,0}_{\mathrm{DHS}} + \mathcal{J}^{0,1}_{\mathrm{DHS}}$, then $\mathcal{J}_{\mathrm{DHS}}$ is a MC elt, so $d - \mathcal{J}_{\mathrm{DHS}}$ is a flat connection
- $\mathcal{J}_{\mathrm{DHS}}$ can be expressed explicitly in terms of the Arakelov Green function (element of $C^\infty(\Sigma \times \Sigma - \Sigma_{diag})/\mathbb{C}$, independent on choice of L_a, L_b).
- the flat connection $d - \mathcal{J}_{\mathrm{DHS}}$ gives rise to an alg. morphism $\mathrm{Sh}(H_1(\Sigma)^*) \to C^\infty(\tilde{\Sigma}_p)$, where $\Sigma_p := \Sigma \smallsetminus p$.

E3. HL functions attached to the DHS element

- define $\mathcal{H}_C(\mathcal{J}_{\mathrm{DHS}}) := \mathrm{im}(\mathrm{Sh}(H_1(\Sigma)^*) \to C^\infty(\tilde{\Sigma}_p))$.
- for $J \in \mathrm{MC}_{nd}(\Sigma_p)$, there exists $\alpha : \hat{\mathbb{L}}(\mathrm{H}^*_{\Sigma_p}) \to \hat{\mathbb{L}}(H_1(\Sigma))$ and $g \in C^\infty(\Sigma_p, \exp(\hat{\mathbb{L}}(\mathrm{H}^*_C)))$ such that

$$d - \mathcal{J}_{\mathrm{DHS}} = \alpha_*(g(d-J)g^{-1}).$$

- therefore $\boxed{\mathcal{H}_C(\mathcal{J}_{\mathrm{DHS}}) \cdot C^\infty(\Sigma_p) = \mathcal{H}_C(J) \cdot C^\infty(\Sigma_p)}$
- therefore also equal to $A_C \cdot C^\infty(\Sigma_p)$
- for $J, J' \in \mathrm{MC}_{nd}(\Sigma_p)$, there exists $\tilde{\alpha}$ aut. of $\hat{\mathbb{L}}(\mathrm{H}^*_{\Sigma_p})$ and $g \in C^\infty(\Sigma_p, \exp(\hat{\mathbb{L}}(\mathrm{H}^*_C)))$ with $d - J' = \tilde{\alpha}_*(\tilde{g}(d-J)\tilde{g}^{-1})$.
- therefore $\boxed{A_C \cdot C^\infty(\Sigma_p) = \overline{A_C} \cdot C^\infty(\Sigma_p)}$

Thanks for your attention!


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