Homotopy moment maps and differential characters

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Homotopy moment maps

Let X be a manifold, with de Rham complex $\Omega^*(X)$. Let n be a natural number.

Definition

An n-multisymplectic form on X is a closed differential form $\omega \in \Omega^{n+1}(X)$ such that the morphism

$$\xi \in \mathrm{Vec}(X) \mapsto \iota(\xi)\omega \in \Omega^n(X)$$

is injective.

The case $n = 1$ of symplectic forms is special, since if X is finite-dimensional, the morphism $\xi \mapsto \iota(\xi)\omega$ is not just injective, but bijective.

Let $\mathfrak{h}$ be a Lie algebra with differential action on X:

$$\rho : \mathfrak{h} \longrightarrow \mathsf{Vec}(X).$$

Let $C^*(\mathfrak{h})$ be the Chevalley–Eilenberg complex of $\mathfrak{h}$, with differential δ_0. Denote the tensor product of $C^*(\mathfrak{h})$ and $\Omega^*(X)$ by

$$C^*(\mathfrak{h}) \otimes \Omega^*(X).$$

Choose a basis $\{c_i\}$ of $\mathfrak{h}$, and dual basis $\{c^i\}$ of $\mathfrak{h}^*$; denote the structure coefficients of $\mathfrak{h}$ by A_{ij}^k. Form the operator

$$\iota = \sum_i c^i \otimes \iota(\rho_i) : C^p(\mathfrak{h}) \otimes \Omega^q(X) \longrightarrow C^{p+1}(\mathfrak{h}) \otimes \Omega^{q-1}(X),$$

of total degree 0.

Definition (Callies, Frégier, Rogers, Zambon)

A homotopy moment map for the action ρ of $\mathfrak{h}$ on an n-multisymplectic manifold (X, ω) is an element

$$\mu \in \bigoplus_{i=1}^{n-1} C^i(\mathfrak{h}) \otimes \Omega^{n-i-1}(X)$$

such that

$$(\delta_0 + d)\mu = e^\iota \omega - \omega.$$

The case of symplectic manifolds: $n = 1$

If $n = 1$, we have $\iota\omega = \sum_i c^i \iota(\rho_i)\omega \in \mathfrak{h}^* \otimes \Omega^1(X)$. Thus, a homotopy moment map is an element $\mu = \sum_i c^i \mu_i \in \mathfrak{h}^* \otimes \Omega^0(X)$ such that

$$d\mu_i = -\iota(\rho_i)\omega$$

and

$$\sum_k A_{ij}^k \mu_k = \iota(\rho_i)\iota(\rho_j)\omega = -\mathcal{L}(\rho_i)\mu_j = \{\mu_i, \mu_j\}.$$

In other words, μ is a moment map for the action ρ of $\mathfrak{h}$ on the symplectic manifold (X, ω).

I am not aware of an analogue of symplectic reduction for higher moment maps.

Reformulation for exact n-multisymplectic manifolds

Definition

An n-multisymplectic manifold (X, ω) is exact if $\omega = d\alpha$, for $\alpha \in \Omega^n(X)$.

The Lie derivative $c_i \mapsto \mathcal{L}(\rho_i)$ makes the de Rham complex $\Omega^*(X)$ into a differential graded $\mathfrak{h}$-module. This motivates replacing the complex $C^*(\mathfrak{h}) \otimes \Omega^*(X)$ by the complex $C^*(\mathfrak{h}, \Omega^*(X))$, with differential $\delta + d$, where

$$\delta = \delta_0 + \sum_i c^i \mathcal{L}(\rho_i).$$

Lemma

$$\delta + d = e^{\iota} \circ (\delta_0 + d) \circ e^{-\iota}$$

The equation for a homotopy moment map becomes

$$(\delta_0 + d)(\alpha + \mu) = e^{\iota}\omega.$$

Define

$$\nu = e^{\iota}(\alpha + \mu) \in C^*(\mathfrak{h}, \Omega^*(X)),$$

or equivalently,

$$\mu = e^{-\iota}\nu - \alpha.$$

For exact n-multisymplectic manifolds, a homotopy moment map is an equivariant extension of α:

$$(\delta + d)\nu = \omega.$$

The variational bicomplex

Let $p : E \to M$ be a fiber bundle, with jet-space $J_\infty(p)$. An adapted coordinate system at a point $e \in E$ is a coordinate system $(t^1, \ldots, t^n; u^1, \ldots, u^N)$ such that the coordinates $(t^1, \ldots, t^n)$ are pulled back from a coordinate system around $p(e) \in M$.

The coordinates t^μ are the independent coordinates, and M is the worldsheet; the coordinates u^a are the coordinates, identified with the fields of the theory. Denote $\partial/\partial t^\mu$ by ∂_μ.

An adapted coordinate system gives rise to coordinates

$$(t^1, \dots, t^n; \partial^I u^1, \dots, \partial^I u^N)$$

on the jet-space $J_\infty(p)$, where I ranges over multi-indices $(i_1, \dots, i_n) \in \mathbb{N}^n$, and

$$\partial^I = \partial_1^{i_1} \dots \partial_n^{I_n}.$$

Denote partial differentiation $\partial/\partial(\partial^I u^a)$ with respect to a jet coordinate by $\partial_{a,I}$.

Let $\mathcal{O}_\infty$ be sheaf of algebras over M whose sections are smooth functions in the coordinates $(t^1, \dots, t^n; \partial^I u^1, \dots, \partial^I u^N)$.

The variation de Rham complex

The variational de Rham complex Ω_∞^* is the de Rham complex associated to $\mathcal{O}_\infty$.

It is bigraded: the differentials dt^μ have bidegree $(1,0)$, and the differentials

$$\theta_I^a = d(\partial^I u^a) - \sum_\mu \partial_\mu \partial^I u^a \, dt^\mu$$

have bidegree $(0,1)$.

The differential d on Ω_∞ breaks into horizontal and vertical parts

$$d^{1,0} = \sum_\mu dt^\mu D_\mu : \Omega_\infty^{p,q} \to \Omega_\infty^{p+1,q}, \quad d^{0,1} = \sum_{a,I} \theta_I^a \, \partial_{a,I} : \Omega_\infty^{p,q} \to \Omega_\infty^{p,q+1}.$$

where

$$D_\mu = \partial_\mu + \sum_{a,I} \partial_\mu \big(\partial^I u^a\big) \, \partial_{a,I}.$$

The variational n-multisymplectic form of a classical field theory

A first-order Lagrangian density L determines a classical field theory. In an adapted coordinate system,

$$L = L(t^i, u^a, \partial_\mu u^a)\, dt^1 \wedge \ldots \wedge dt^n \in \Omega_\infty^{n,0}.$$

From L, we construct a Lepage form

$$\alpha = L - \partial_{a,\mu} L\, \theta^a \wedge \iota(\partial_\mu)\, dt^1 \wedge \ldots \wedge dt^n \in \Omega_\infty^{n,0} \oplus \Omega_\infty^{n-1,1} \subset \Omega_\infty^n.$$

The differential $\omega = d\alpha \in \Omega_\infty^{n+1}$ extends Noether's symplectic form off-shell:

$$\omega = \frac{\delta L}{\delta u_i}\, du_i \wedge dt^1 \wedge \ldots \wedge dt^n + \cdots \in \Omega_\infty^{n,1} + \Omega_\infty^{n-1,2}.$$

Modulo a mild nondegeneracy condition on L, ω is a variational analogue of a n-multisymplectic form.

Variational Cartan calculus

Derivations $\xi \in \mathsf{Der}(\mathcal{O}_\infty)$ of $\mathcal{O}_\infty$ are the variational vector fields. There is a variational Cartan calculus: to a variational vector field ξ, we associate the contraction

$$\iota(\xi) : \Omega^*_\infty \to \Omega^{*-1}_\infty$$

and the Lie derivative

$$\mathcal{L}(\xi) = d \circ \iota(\xi) + \iota(\xi) \circ d : \Omega^*_\infty \to \Omega^*_\infty.$$

Fix a fiber bundle $p : E \longrightarrow B$ and a (first-order) Lagrangian density $L \in \Omega_\infty^{n,0}$ with associated Lepage form $\alpha \in \Omega_\infty^n$ and variational n-multisymplectic form $\omega \in \Omega_\infty^{n+1}$.

Consider a Lie algebra $\mathfrak{h}$ and a variational action of $\mathfrak{h}$

$$\rho : \mathfrak{h} \longrightarrow \mathsf{Der}(\mathcal{O}_\infty)$$

such that $\mathcal{L}(\rho)\omega = 0$.

Definition

A variational homotopy moment map for ρ is an element $\nu \in C^*(\mathfrak{h}, \Omega_\infty^*)$ of total degree $n - 1$ such that

$$(\delta + d)\nu = \omega.$$

The Chern–Simons classical field theory

Let M be a closed oriented 3-manifold. Let $\mathfrak{g}$ be a reductive Lie algebra, with invariant inner product $(-,-)$. Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$.

Let P be a G-principal bundle over M. We consider a field theory over M, where the fields (dependent coordinates) are the components of a connection over P.

Over a chart $U \subset M$ where P is trivialized, the Lepage form of the Chern–Simons Lagrangian theory equals

$$\alpha = \tfrac{1}{2}(A, dA) + \tfrac{1}{6}(A, [A, A]).$$

If G is simply connected, then P is a trivial bundle, and the Lepage form is globally defined. We discuss the general case later.

The 3-multisymplectic form of Chern–Simons field theory

The 3-multisymplectic form equals

$$\omega = \tfrac{1}{2}(\mathbb{F},\mathbb{F}) = (F, d^{0,1}A) + \tfrac{1}{2}(d^{0,1}A, d^{0,1}A),$$

where $F = d^{1,0}A + \tfrac{1}{2}[A,A]$ is the curvature, and

$$\mathbb{F} = dA + \tfrac{1}{2}[A,A] = d^{0,1}A + F$$

is its analogue in the variational bicomplex.

This 3-form is nondegenerate on generalized vector fields: those of the form

$$\xi^\mu D_\mu + \mathrm{pr}\left\langle X, \frac{\partial}{\partial A}\right\rangle = \xi^\mu D_\mu + \sum_I \left\langle \partial^I X, \frac{\partial}{\partial(\partial^I A)}\right\rangle,$$

where $\xi \in \mathcal{O}_\infty \otimes_{\mathcal{O}} \mathrm{Vec}(M)$ and $X \in \Omega^{1,0}_\infty \otimes \mathfrak{g}$.

The Atiyah Lie algebra

The Atiyah Lie algebra of a principal bundle P is the space of first-order differential operators $\partial_{\xi,\eta} = \xi^\mu \partial_\mu + \eta$, where $\xi \in \mathsf{Vec}(M)$ is a vector field on M, and $\eta \in \Gamma(M, P \times_G \mathfrak{g})$ is a gauge transformation.

The generalized vector fields

$$\rho(\partial_{\xi,\eta}) = \xi^\mu D_\mu + \mathsf{pr}\left\langle -\mathcal{L}(\xi)A + \nabla^A \eta, \frac{\partial}{\partial A} \right\rangle$$

yield an action of the Atiyah Lie algebra on the variational de Rham complex.

The (local) homotopy moment map

Since $\delta A = \nabla^A \eta$ and $\delta\eta = \frac{1}{2}[\eta, \eta]$, we see that

$$(\delta + d)(A - \eta) + \tfrac{1}{2}[A - \eta, A - \eta] = \mathbb{F}.$$

This allows us to prove the following theorem.

Theorem

$$\nu = \tfrac{1}{2}(A - \eta, (\delta + d)(A - \eta)) + \tfrac{1}{6}(A - \eta, [A - \eta, A - \eta])$$

is a homotopy moment map for the action of the Atiyah Lie algebra in Chern–Simons theory.

Note that ν is independent of ξ, even though $\rho(\partial_{\xi,\eta})$ is not. Of course, the homotopy moment map $\mu = e^{-\iota}\nu - \alpha$ does depend on ξ.

Differential characters

The main point of this talk is that when P is not a trivial bundle, which can only happen when the connected compact Lie group G is not simply connected, the multisymplectic form ω should be interpreted as a variational analogue of a differential character.

We follow Brylinski's discussion in his book Loop Spaces, Characteristic Classes and Geometric Quantization. He developed a differential geometric version of Deligne cohomology, building on pioneering work of Gawędzki.

Let $A \subset \mathbb{C}$ be a subgroup, and let $A(d)$ be complex of sheaves

$$0 \longrightarrow (2\pi i)^d A \longrightarrow \Omega^0 \longrightarrow \cdots \longrightarrow \Omega^d \longrightarrow 0$$

in degrees $[0, d+1]$. A differential character of degree $d+1$ on a manifold is a Čech cohomology class $\alpha \in \check{H}^{d+1}(M, A(d))$.

We write $A(d)$ for brevity: it is usually written $A_D(d)$, and $A(d)$ is the constant sheaf $(2\pi i)^d A = \sigma^{\leq 0} A_D(d)$.

Given a cover $\mathcal{U} = \{U_a\}$ of M, a differential character is given by data $\alpha_{a_0 \dots a_k} \in \Omega^{d-k+1}(U_{a_0 \dots a_k})$, $0 \le k \le d$, forming a cocycle

$$d\alpha_{a_0 \dots a_k} + \sum_{i=0}^{k} (-1)^{k-i} \alpha_{a_0 \dots \widehat{a}_i \dots a_k} = 0, \qquad 0 < k \le d,$$

and such that

$$\sum_{i=0}^{d+1} (-1)^{d+1-i} \alpha_{a_0 \dots \widehat{a}_i \dots a_{d+1}} \in \Omega^0(U_{a_0 \dots a_{d+1}})$$

is a locally constant function with values in $(2\pi i)^d A$. This is a Čech $(d+1)$-cocycle with values in $(2\pi i)^d A$, whose cohomology class in $\check{H}^{d+1}(M, (2\pi i)^d A)$ is the characteristic class $c(\alpha)$ of α.

The curvature $\omega(\alpha)$ of a differential character α is the global closed differential $(d+1)$-form defined over $U_a \subset M$ by the differential form $d\alpha_a$. This differential form is a de Rham representative of the characteristic class $c(\alpha)$, thought of as an element of $\check{H}^{d+1}(M, \mathbb{C})$.

Examples

In the case of $\mathbb{Z}(1)$, such a cocycle is precisely the data for a complex line bundle L on M, with cocycle $g_{a_0a_1} = e^{\alpha_{a_0a_1}}$, and a connection with connection one-forms $\alpha_a \in \Omega^1(U_a)$. The characteristic class is the Chern class $c_1(L) \in \check{H}^2(M, 2\pi i\mathbb{Z})$. The surjectivity of the curvature map

$$\omega : \check{H}^2(M, \mathbb{Z}(1)) \to \{\text{closed } 2\text{-forms on } M\}$$

is the Kostant–Souriau theorem, associating a prequantization line bundle to a symplectic form with periods in $2\pi i\mathbb{Z}$.

The case of $\mathbb{Z}(2)$ is also familiar: this a gerbe with curving, the subject of Brylinski's book.

Homotopy moment map in terms of differential characters

We may now generalize the definition of a homotopy moment maps to the setting of differential characters.

Definition

A homotopy moment map is a lift of the differential character

$$\alpha \in \check{C}^*(\mathcal{U}, A(d))$$

to the Chevalley–Eilenberg complex, that is, an element

$$\nu \in \check{C}^*(\mathcal{U}, C^*(\mathfrak{g}, A(d)))$$

of total degree $d+1$ satisfying the equation

$$(\delta + \check{\delta} + d)\nu = \omega,$$

where $\omega = (\check{\delta} + d)\alpha$.

The Lepage form of Chern–Simons theory

If G is a compact Lie group, let $\mathcal{G}$ be the sheaf of gauge transformations over M: this is the sheaf of all smooth functions from an open subset of M to G.

Let $\mathcal{U}$ be a cover of M trivializing the principal bundle P, with cocycle $g_{a_0a_1} \in \Gamma(U_{a_0a_1}, \mathcal{G}) = C^\infty(U_{a_0a_1}, G)$. A Chern–Simons field is a Čech 0-cochain

$$A_a \in \Omega^1(U_a, \mathfrak{g})$$

satisfying

$$A_{a_0} = \mathsf{ad}(g_{a_0a_1})A_{a_1} + g^*_{a_0a_1}\theta_R,$$

where $\theta_R \in \Omega^1(G, \mathfrak{g})$ is the right-invariant Maurer–Cartan form on G.

We will also need the left-invariant Maurer–Cartan form θ_L. For a matrix group, $g^*\theta_R$ equals $-dgg^{-1}$ and $g^*\theta_L$ equals $g^{-1}dg$.

The Lepage form

$$\alpha_a = \tfrac{1}{2}(A_a, dA_a) + \tfrac{1}{6}(A_a, [A_a, A_a]) \in \check{C}^0(\mathcal{U}, \Omega^3_\infty)$$

has coboundary

$$\alpha_{a_0} - \alpha_{a_1} = \tfrac{1}{12} g^*_{a_0a_1}(\theta_L, [\theta_L, \theta_L]) - d\Big(\tfrac{1}{2}(A, g^*_{a_0a_1}\theta_L)\Big).$$

We must complete the definition of the Lepage form for Chern–Simons by adding the Čech 1-cochain

$$\alpha_{a_0a_1} = -\tfrac{1}{2}(A_{a_1}, g^*_{a_0a_1}\theta_L).$$

We obtain a Čech cochain of total degree 3 for the complex $A_\infty(3)$ of sheaves

$$0 \longrightarrow (2\pi i)^3 A \longrightarrow \Omega^0_\infty \longrightarrow \Omega^1_\infty \longrightarrow \Omega^2_\infty \longrightarrow \Omega^3_\infty \longrightarrow 0$$

This Lepage form is a differential character twisted by Gawędzki's Chern–Simons gerbe: its Čech coboundary is the sum of the multisymplectic form

$$\omega = \tfrac{1}{2}(\mathbb{F},\mathbb{F}) = (F, d^{0,1}A) + \tfrac{1}{2}(d^{0,1}A, d^{0,1}A),$$

which is independent of the gauge used in its definition, and the pullback of a differential character of degree 3 on M, with vanishing $\omega_a \in \check{C}^0(\mathcal{U},\Omega^4)$,

$$\omega_{a_0a_1} = \tfrac{1}{12}g^*_{a_0a_1}(\theta_L,[\theta_L,\theta_L]) \in \check{C}^1(\mathcal{U},\Omega^3),$$

and

$$\omega_{a_0a_1a_2} = \tfrac{1}{2}\big(g^*_{a_0a_1}\theta_L, g^*_{a_1a_2}\theta_R\big) \in \check{C}^2(\mathcal{U},\Omega^2).$$

Since this cochain is pulled back from M, it has no influence on the variational calculus, but it must be taken into account when quantizing the beginframe

These formulas globalize the Polyakov–Wiegman formula. In modern language, we recognize the 2-shifted symplectic form on BG, originally constructed by Shulman in his 1972 Berkeley thesis.

For Chern–Simons theory, the homotopy moment ν is obtained from α by replacing A by $A-\eta$, exactly as in the local case.

Joyeux anniversaire Michèle !