The Belkale-Kumar cohomology of complete flag manifolds

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In honour of Michèle Vergne 80^{th} birthday

Cohomology of G/P

$$H^*(G/P,\mathbb{Z}) = \bigoplus_{w\in W^P} \mathbb{Z}[X_w], \text{ and}$$

$$[X_u].[X_v] = \sum_{w\in W^P} c_{uv}^w [X_w].$$

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It is graded by the degree:
Setting $T_w := T_{P/P} w^{-1} X_w$, if $c_{uv}^w \neq 0$ then

$$\dim(T_u) + \dim(T_v) = \dim(G/P) + \dim(T_w),$$

BK-produc on G/P

Consider the decomposition under the action of L:

$$T_{P/P}G/P = V_1 \oplus \cdots \oplus V_s.$$

Easily, if $T_w^i = V_i \cap T_w$, we have

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Hence, if $c_{uv}^w \neq 0$ then

$$\sum_{i=1}^{s} \Big(\dim(T_u^i) + \dim(T_v^i) \Big) = \sum_{i=1}^{s} \Big(\dim(V_i) + \dim(T_w^i) \Big).$$

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To get the BK-product, reinforce this condition replacing the sum by a $\forall i = 1, \ldots, s$.

Theorem (Belkale-Kumar 2006)

Replacing c_{uv}^w by 0 if

$$\forall 1 \leq i \leq s \quad \dim(T_u^i) + \dim(T_v^i) = \dim(V_i) + \dim(T_w^i),$$

does NOT hold and kepping the other ones unchanged, one gets a product $\odot_0$ on $H^*(G/P, \mathbb{Z})$ that is still associative and satisfies Poincarré duality.

Eigencone

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Set

$$\Gamma(G) = \{(\lambda_1, \lambda_2, \lambda_3) \in (\mathfrak{h}_+)^3 \ :: \mathcal{O}_{\lambda_1} + \mathcal{O}_{\lambda_2} + \mathcal{O}_{\lambda_3} \ni 0\}.$$

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Theorem (Belkale-Kumar, R.)

The regular faces of $\Gamma(G)$ correspond bijectively with the structure coefficients of $\odot_0$ equal to one, for the various G/P.

Our results

Theorem (Francone-R. 2023)

For G/B, the coefficients for $\odot_0$ are 0 or 1.

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Theorem (Francone-R. 2023)

Let Φ_1, Φ_2 and Φ_3 be three biconvex subsets of Φ^+ such that $\Phi_3 = \Phi_1 \sqcup \Phi_2$. Let β and γ be two positive roots such that

1. $\beta \in \Phi_1$;
2. $\gamma \notin \Phi_3$;
3. $\gamma + \beta \in \Phi_3$.

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Proof : reduction to the rank 7, and a very long checking.

About the proof

Step 1 : Sufficient to prove that some map η is birational.
Start with u, v and w in W such that $\Phi(w) = \Phi(u) \sqcup \Phi(v)$, where $\Phi(w) := \Phi^+ \cap w^{-1}\Phi^-$.
In general

$$c_{uv}^w = \sharp g_u X_u \cap g_v X_v \cap g_w X_{w^\vee}.$$

We have an incidence variety $\eta : \mathcal{X} \longrightarrow X := (G/B)^3$ and want to prove that η is birational.

About the proof

Step 2 : Using G/B simply connected.
Let $R \subset \mathcal{X}$ be the (Weyl) ramification of η. Since X is simply connected, it is sufficient to prove that the codimenssion of $\eta(R)$ is at least 2.

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Step 2 : Using G/B simply connected.
Let $R \subset \mathcal{X}$ be the (Weyl) ramification of η. Since X is simply connected, it is sufficient to prove that the codimenssion of $\eta(R)$ is at least 2.
Step 3 : Construct an explicit open subset Ω where η is unramified.

About the proof

Step 4: Fix an irreducible component D of $\mathcal{X} - \Omega$.
Because of the form of Ω, D comes from a Schubert divisor in X_u, X_v or $X_{w^\vee}$. So, D comes from a Schubert covering relation, say of of u.

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Step 5: If it is the weak Bruhat order.
Prove by explicit computation of the tangent map that D is not a component of R.

About the proof

Step 6: D comes from a covering relation of the stroong Bruhat order.
To be proved: for any $x \in D$ the tangent map of $\eta_{|D}$ has a Kernel in $T_x D$. Let M be the matrix of the linerar map.

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The trick consists in proving that $\det(M') = \det(N')$, for some similar matrix N occuring when one applies the process in the case of Poincaré duality.
In this last case η is biratiional. Hence $\det(N') = 0$. The proof is ended.

Applications

1- Let $d(w)$ denote the number of descents of w. A descent is a simple rooot α such that $\ell(s_\alpha w) = \ell(w) - 1$. Then,

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2- The regular faces of the eigencone of minimal dimension are simplicial cones.
3- A component of $V(\lambda_1) \otimes V(\lambda_2)$ is said to be cohomological if comes from the surjectivity of some cup product map:

$$\mathrm{H}^{\ell(w_1)}(G/B, \mathcal{L}(w_1 \cdot \lambda_1)) \otimes \mathrm{H}^{\ell(w_2)}(G/B, \mathcal{L}(w_2 \cdot \lambda_2)) \longrightarrow \mathrm{H}^{\ell(w_3^\vee)}(G/B, \mathcal{L}(w_3^\vee$$

Our result implies that a component is cohomological if and only if it relongs to somme regular face of minimal dimension.

A conjecture

Come back to the situation of G/P. Given $u, v \in W^P$, set

$$\Sigma_u^v := \overline{u^{-1}X_u^\circ \cap w_{0,P}v^{-1}X_v^\circ}$$

The conjecture is

$$[\Sigma_u^v]_{\odot_0} = [X_u]\odot_0[X_v].$$

Thank you

HAPPY BIRTHDAY Michèle !