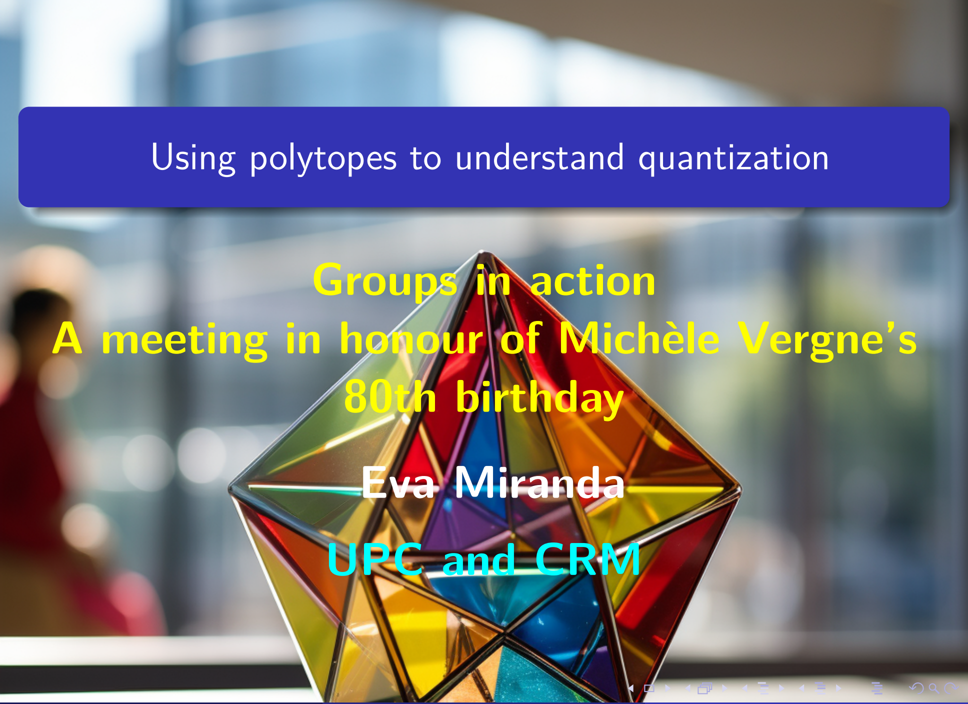


Using polytopes to understand quantization

Groups in action
A meeting in honour of Michèle Vergne's 80th birthday

Eva Miranda

UPC and CRM

Michèle in action

Some of Michèle's playgrounds

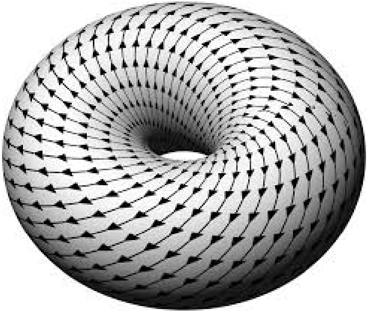
- Cohomologies (in its different reincarnations).
- Toric manifolds.
- Counting integral points on polytopes.
- Quantization.

Extending de Rham cohomology

Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



b-forms

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

Key point: A b-form of degree k decomposes as:

$$\omega = \alpha \wedge \frac{dz}{z} + \beta, \quad \alpha \in \Omega^{k-1}(M),\ \beta \in \Omega^k(M) \quad d\omega := d\alpha \wedge \frac{dz}{z} + d\beta.$$

- This defines the b-cohomology groups. (Mazzeo-Melrose)

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2. It is also a Poisson structure!.
- This dual point of view, allows us to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path methods.

Symplectic manifolds with boundary

- Consider formal deformation quantization of manifolds with boundary à la Fedosov.
- These symplectic manifolds with boundary have local normal form of type (b-symplectic):

$$\omega = \frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i \geq 2} dx_i \wedge dy_i$$

Theorem (Nest-Tsygan)

Equivalence classes of star products on a b-symplectic manifold are in one-to-one correspondence with elements in

$${}^bH^2(M, \mathbb{C}[\hbar]) \simeq H^2(M, \mathbb{C}[\hbar]) \oplus H^1(\partial M, \mathbb{C}[\hbar]).$$

Deformation quantization of E-manifolds

- The b-tangent bundle can be replaced by other algebroids (E-symplectic) known to Nest and Tsygan.
- An important class is that of b^m-tangent bundle defined as the bundle whose sections are given by vector which are tangent to an hypersurface to order m.
- Deformation quantization of E-manifolds:

Theorem (Nest-Tsygan)

The set of isomorphism classes of E-deformations is in bijective correspondence with the space

$$\frac{1}{i\hbar}\omega + \ {}^{E}H^2(M, \mathcal{C}[[\hbar]])$$

Example 1: Geodesics on pseudo-Riemannian manifolds

- Given a pseudo-Riemannian manifold (M, g), let $\mathcal{L}$ be the space of all oriented non-parametrized geodesics.
- $\mathcal{L}$ splits as $\mathcal{L}_{\pm}$, the space of space-like $(g(\dot{\gamma}, \dot{\gamma}) > 0)$- and time-like $(g(\dot{\gamma}, \dot{\gamma}) < 0)$ geodesics, and $\mathcal{L}_0$, the space of light-like geodesics ($g(\dot{\gamma}, \dot{\gamma}) = 0$).
- $\mathcal{L}_{\pm}$ is even dimensional and symplectic.
- $\mathcal{L}_0$ can be seen as the common boundary of $\mathcal{L}_{\pm}$ and has an induced contact structure.
- In dimension 2, this structure is indeed b-symplectic.

Example 2: The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

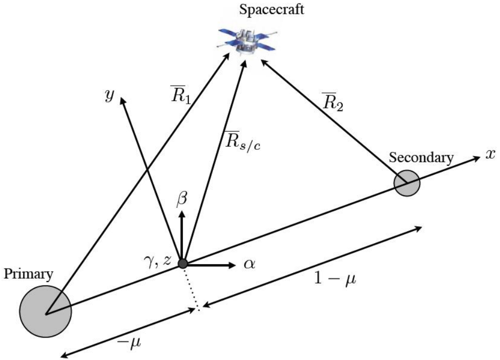


Figure: Circular 3-body problem

Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$, with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbb{R}^+$, can be then extended to infinity $(x = 0)$.
- The symplectic structure becomes a singular object $\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG$. for $x > 0$

Why singular?

1. The 2-dimensional space of geodesics on a pseudoriemannian surface is singular.
2. Some non-compact symplectic manifolds can be compactified as singular symplectic manifolds.
3. Singular forms appear after regularization transforms in celestial mechanics and sigma coordinates in Painlevé equations.
4. They model certain manifolds with boundary.
5. Why not?

Singular symplectic manifolds as Poisson manifolds

The local model

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

does not define a smooth form but its dual defines a smooth Poisson structure!

$$\mathbf{\Pi} = \mathbf{x_1^m} \frac{\partial}{\partial \mathbf{x_1}} \wedge \frac{\partial}{\partial \mathbf{y_1}} + \sum_{\mathbf{i \geq 2}}^{\mathbf{n}} \frac{\partial}{\partial \mathbf{x_i}} \wedge \frac{\partial}{\partial \mathbf{y_i}}$$

The structure Π is a bivector field which satisfies the integrability equation $[\Pi, \Pi] = 0$. The Poisson bracket associated to Π is given by the equation

$$\{f, g\} := \Pi(df, dg)$$

The local Poisson case. Splitting Theorem.

The local structure for Poisson manifolds is given by the following:

Theorem (Weinstein)

Let (M^n, Π) *be a smooth Poisson manifold and let* p *be a point of* M *of rank* $2k$*, then there is a smooth local coordinate system* $(x_1, y_1, \dots, x_{2k}, y_{2k}, z_1, \dots, z_{n-2k})$ *near* p*, in which the Poisson structure* Π *can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where f_{ij} *vanish at the origin.*

b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a b-**Poisson structure** on (M, Z).

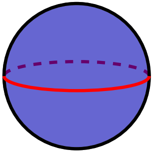
Theorem

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \ldots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and
$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$

b-Poisson structures are dual to b-symplectic forms

Examples

- A Radko surface.

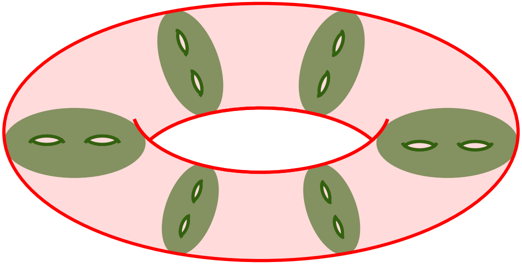
- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial \theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

Poisson Geometry of the critical hypersurface

This last example is semilocally the *canonical* picture of a b-Poisson structure .

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field v transverse to the symplectic foliation induced on Z (modular vector field).
3. (Guillemin-M. Pires) Z is a mapping torus with glueing diffeomorphism the flow of v.



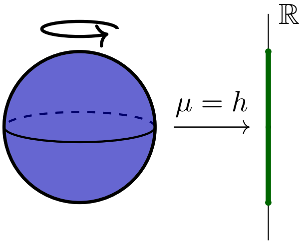
Toric manifolds

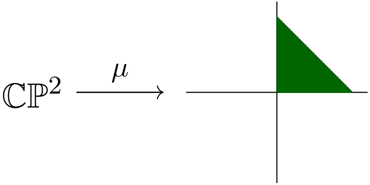
Toric manifolds

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes and the bijective correspondence is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$



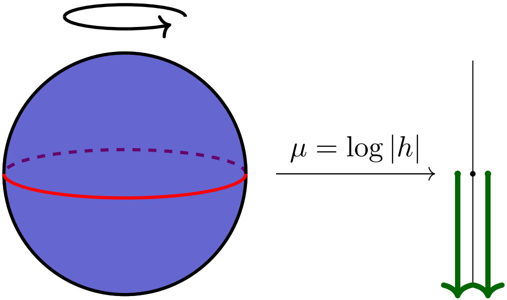


$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

Radko surfaces and their symmetries

$$(S^2, \tfrac{1}{h} dh \wedge d\theta) \leftrightsquigarrow (S^2, h\tfrac{\partial}{\partial h} \wedge \tfrac{\partial}{\partial \theta}).$$

We want to study generalizations of rotations on a sphere.



b-Hamiltonian actions

- Denote by ${}^bC^\infty(M)$ the space of functions which are C^∞ on $M \setminus Z$ and near each Z_i can be written as a sum,

$$c_i \log|f| + g \tag{1}$$

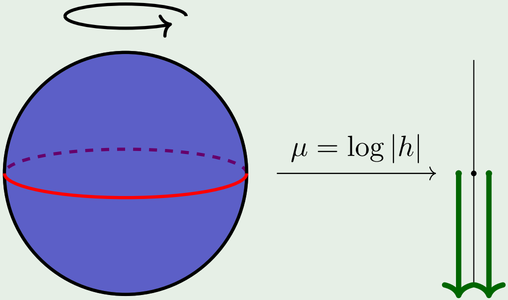
  with $c_i \in \mathbb{R}$ and $g \in C^\infty(M)$.
- let T be a torus and $T \times M \to M$ an action of T on M. We will say that this action is b-*Hamiltonian* if the elements, $X \in \mathfrak{t}$ of the Lie algebra of T satisfy

$$\iota(X^M)\,\omega = d\phi, \phi \in {}^bC(M), \tag{2}$$

The S^1-b-sphere

Example

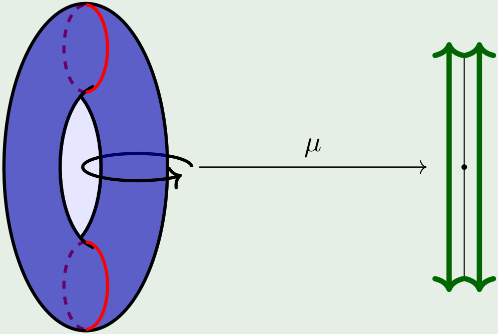
$(\mathbb{S}^2, \omega = \frac{dh}{h} \wedge d\theta)$, with coordinates $h \in [-1, 1]$ and $\theta \in [0, 2\pi]$. The critical hypersurface Z is the equator, given by $h = 0$. For the $\mathbb{S}^1$-action by rotations, the moment map is $\mu(h, \theta) = \log |h|$.



The S^1-b-torus

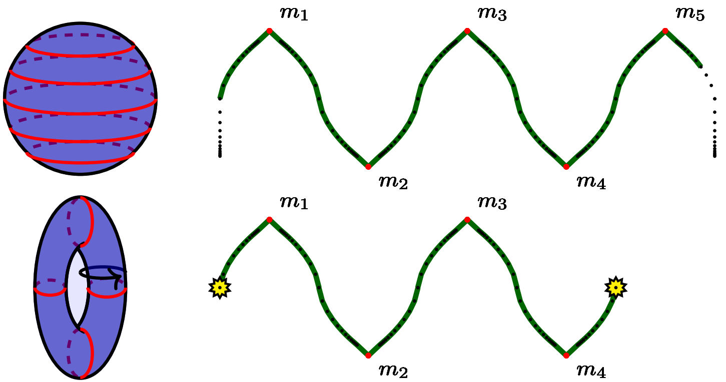
Example

On $(\mathbb{T}^2, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$, with coordinates: $\theta_1, \theta_2 \in [0, 2\pi]$. The critical hypersurface Z is the union of two disjoint circles, given by $\theta_1 = 0$ and $\theta_1 = \pi$. Consider rotations in θ_2 the moment map is $\mu : \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ is given by $\mu(\theta_1, \theta_2) = \log\left|\frac{1+\cos(\theta_1)}{\sin(\theta_1)}\right|$.



b-surfaces and their moment map

A toric b-surface is defined by a smooth map $f : S \longrightarrow {}^b\mathbb{R}$ or $f : S \longrightarrow {}^b\mathbb{S}^1$ (a posteriori the moment map).



Classification of toric b-surfaces

Theorem (Guillemin, M., Pires, Scott)

A toric b-symplectic surface is equivariantly b-symplectomorphic to either $(\mathbb{S}^2, Z)$ or $(\mathbb{T}^2, Z)$, where Z is a collection of latitude circles.

The action is the standard rotation, and the b-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.

The weights $w(a)$ of the codomain of the moment map are given by the modular periods of the connected components of the critical hypersurface.

The semilocal model

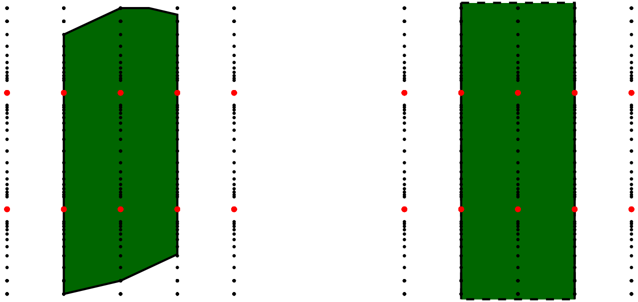
Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$.
For any Delzant polytope $\Delta \subseteq \mathfrak{t}_Z^*$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, the **semilocal model** of the b-symplectic manifold as

$$M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{\mathrm{lm}} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{\mathrm{lm}}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$.

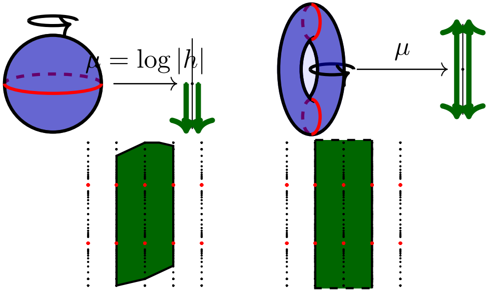
From local to global....

We can reconstruct the b-Delzant polytope from the Delzant polytope on a mapping torus via *symplectic cutting* in a neighbourhood of the critical hypersurface.



This information can be recovered by doing reduction in stages: Hamiltonian reduction of an action of $\mathbb{T}_Z^{n-1}$ and the classification of toric b-surfaces.

We can reconstruct the b-Delzant polytope from the Delzant polytope on a mapping torus via *symplectic cutting* close to the critical hypersurface.

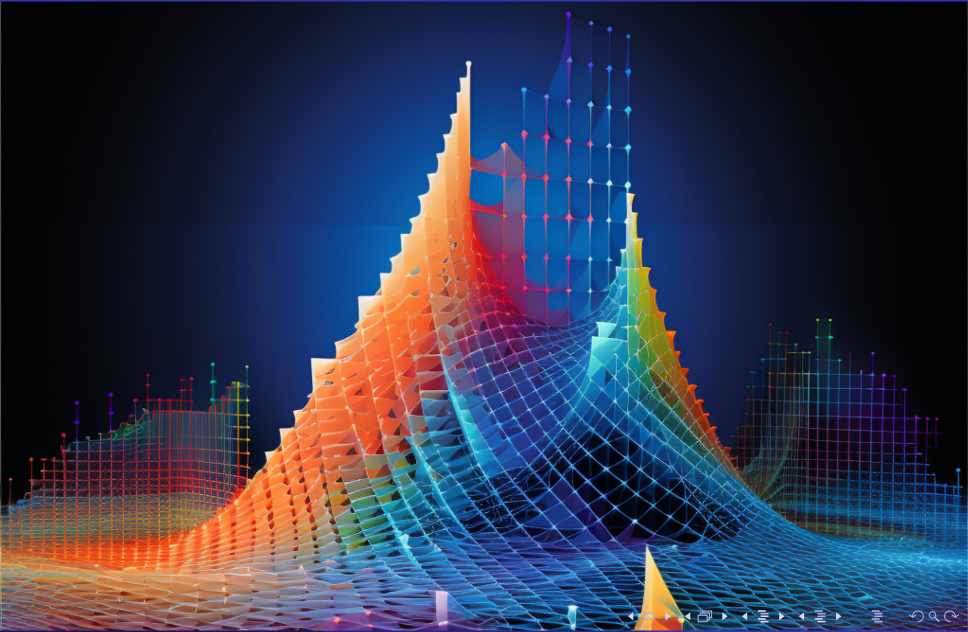


Guillemin-M.-Pires-Scott

There is a one-to-one correspondence between b-toric manifolds and b-Delzant polytopes and toric b-manifolds are either:

- ${}^b\mathbb{T}^2 \times X$ (X a toric symplectic manifold of dimension $(2n-2)$).
- obtained from ${}^b\mathbb{S}^2 \times X$ via symplectic cutting.

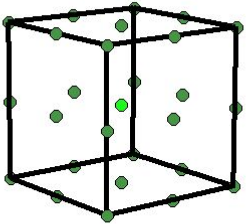
Counting integral points on polytopes and quantization



Geometric quantization of toric manifolds

In symplectic geometry we can quantize counting Bohr-Sommerfeld leaves (Bohr-Sommerfeld quantization).

BS leaves of the polarization correspond to the **integer points in the interior of the Delzant polytope (Guillemin-Sternberg)**.



Bohr-Sommerfeld leaves are the integer points in the Delzant polytope.

What about Geometric quantization of Poisson manifolds?

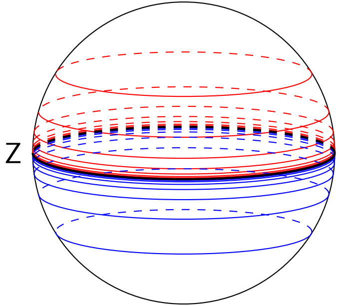
What is a polarization?

Geometric quantization of b-manifolds?

- By extending the admissible Hamiltonian functions with b-functions, we can consider toric actions on b-symplectic manifolds with n-dimensional orbits (also along the critical set).
- Their orbits would be an example of *"Lagrangian"* submanifold $\rightsquigarrow$ polarization.
- There is a analogue of Delzant theorem for b-toric actions.
- Lagrangian orbits (*polarization*) can be read as points on the image polytope.

We can use the polytopes to quantize in the b-case too!

$$({}^bS^2, Z = \{h = 0\}, \omega = \frac{1}{h} dh \wedge d\theta, \mu = -\log |h|)$$



The b-sphere contains as many Bohr-Sommerfeld leaves on the northern hemisphere (in red) as on the southern hemisphere (in blue).

Formal quantization (Meinrenken, Paradan, Vergne, Weitsman)

1. (M,ω) compact symplectic manifold and $(\mathbb{L},\nabla)$ line bundle with connection of curvature ω.
2. By twisting the spin-$\mathbb{C}$ Dirac operator on M by $\mathbb{L}$ we obtain an elliptic operator $\bar{\partial}_{\mathbb{L}}$.

Since M is compact, $\bar{\partial}_{\mathbb{L}}$ is Fredholm, and we define the geometric quantization $Q(M)$ by

$$Q(M) = \mathrm{ind}(\bar{\partial}_{\mathbb{L}})$$

as a virtual vector space.

Formal Quantization

What if M is non-compact?

- Describe a method for quantizing non-compact prequantizable Hamiltonian T-manifolds based upon the "quantization commutes with reduction" principle.
- Important assumption: The moment map ϕ is proper.
- Apply this method to b-symplectic manifolds.

Assume M is non-compact but ϕ proper:

- Let $\mathbb{Z}_T \in \mathfrak{t}^*$ be the weight lattice of T and α a regular value of the moment map.
- Let V be an infinite-dimensional virtual T-module with finite multiplicities. We say $V = Q(M)$, formal quantization if for any compact Hamiltonian T-space N with integral symplectic form, we have

$$(V \otimes Q(N))^T = Q((M \times N)//_0 T). \tag{3}$$

In other words, denote by $M_\alpha = \phi^{-1}(\alpha)/T$ the reduced space, $[Q, R] = 0$ implies $Q(M)_\alpha = Q(M_\alpha)$ where $Q(M)_\alpha$ is the α-weight space of $Q(M) \rightsquigarrow Q(M) = \bigoplus_\alpha Q(M_\alpha)\alpha$

Theorem (Braverman-Paradan)

$Q(M) = ind(\overline{\partial})$

Formal quantization of b-symplectic manifolds

A b-symplectic manifold is prequantizable if:

- $M \setminus Z$ is prequantizable
- The cohomology classes given under the Mazzeo-Melrose isomorphism applied to $[\omega]$ are integral.

Theorem (Guillemin-M.-Weitsman)

- $Q(M)$ *exists.*
- $Q(M)$ *is finite-dimensional.*

Idea of proof
$Q(M) = Q(M_+) \bigoplus Q(M_-)$
and an ϵ-neighborhood of Z does not contribute to quantization.

Braverman, Loizides, and Song

$Q(M) = ind(D_{APS})$
with D_{APS} the Dolbeault-Dirac operator endowed with the Atiyah-Patodi-Singer boundary conditions.

Bohr-Sommerfeld leaves and formal geometric quantization

Assume that there is a T-action with non-vanishing modular weight,

$$Q(M) = \oplus_\alpha \epsilon_\alpha Q(M//_\alpha T)\alpha,$$

For the toric case, the quotient $M//_\alpha T$ is a point. This proves,

Theorem (Mir, M., Weitsman)

*Let (M^{2n}, Z, ω, μ) be a b-symplectic toric manifold Then, the formal geometric quantization of M coincides with the count of its Bohr-Sommerfeld leaves **with sign** (Bohr-Sommerfeld quantization).*

Geometric quantization of symplectic toric manifolds

↓

Count of the integer points in the image of the moment map

↑

Formal geometric quantization of symplectic toric manifolds

Geometric quantization of b-symplectic toric manifolds

↓

Count **with sign** of the integer points in the image of the moment map

↑

Formal geometric quantization of b-symplectic toric manifolds

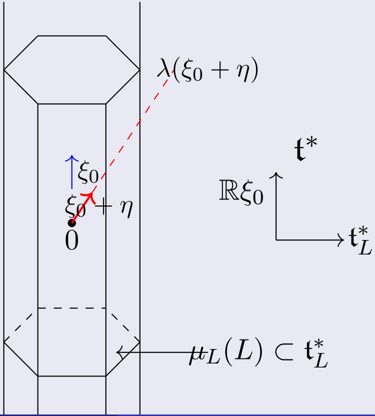
What about quantization of b^m-symplectic manifolds?

Theorem (Guillemin, M., Weitsman)

1. *If m is odd, $Q(M)$ is a finite dimensional virtual T-module.*
2. *If m is even, there exists a weight $\xi \in \mathfrak{t}^*$, integers $c_\pm$, and $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, and $\eta \in \mathfrak{t}^*$ is a weight of T,*

$$\dim Q(M)^{\lambda\eta} = \begin{cases} 0 & if \quad \eta \neq \pm\xi \\ c_\pm & if \quad \eta = \pm\xi \end{cases}$$

(In fact, $c_\pm = \epsilon_\pm \dim Q(M)^{\pm\lambda\xi}$, where $\epsilon_\pm \in \{\pm 1\}$, for any λ sufficiently large.)



(Singular) symplectic manifolds

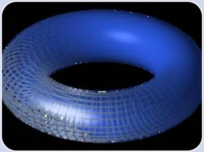
b^m -Symplectic

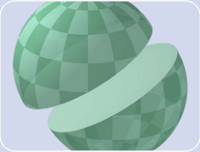
Symplectic

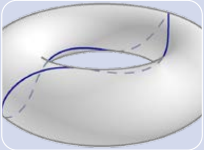
Folded symplectic

(Singular) symplectic manifolds

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k + 1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

In particular:

- Any b^{2k}-symplectic manifold admits a symplectic structure.
- Any b^{2k+1}-symplectic manifold admits a folded symplectic structure.
- The converse is not true: S^4 admits a folded symplectic structure but no b-symplectic structure.

Theorem (M.-Weitsman)

For any b^m-manifold endowed with a T-action with non-vanishing modular weight,

$$Q(M) = ind(D).$$

Outside the b-box

To b or not to b...

Work in progress, M-Nest

Semisimple linearizable Poisson structures have associated E-symplectic manifolds.
This association is done via a *desingularization process* gives a hierarchy of E-symplectic manifolds.

What comes next?

- Understand Poisson Geometry through E-glasses.
- Compare Deformation quantization of E-symplectic manifolds and Poisson manifolds through the desingularization scheme.
- Dream: Understand Geometric Quantization of Poisson manifolds.

Stratification by coadjoint orbits

- $M = \mathfrak{g}^*$ - Poisson manifold, $\mathfrak{g}$-real Lie algebra of compact type, with Lie group G.
  M has a stratification of the form

  $$M = \cup O_i$$

  where O_i is the union of orbits of codimension i. Moreover,

  $$\partial \overline{O_i} = \cup_{k>i} O_k.$$

- For each connected component $U \subset O_i$, the diffeomorphism class of G-orbits in U is constant, say G/G_λ, $\lambda \in U$ fixed.
- Let T be the maximal torus in G, $\mathfrak{t}$ its Lie algebra. Denote by W the Weyl group $N(T)/T$.

A Hironaka desingularization for $\mathfrak{g}^*$

- (Bott) Each orbit of the coadjoint action of G intersects $\mathfrak{t}^*$ precisely in an orbit of W.
- The restriction map

$$M = \mathfrak{g}^* \to \mathfrak{t}^* = V$$

  intertwines the G-action on M and W-action on V.
- Let $\lambda \in V$ be a generic element (in the top stratum) and $O_\lambda = G/G_\lambda$ the associated symplectic orbit.

Theorem

A resolution of the action has the form of a Poisson map

$$(V \times O_\lambda)^W \to M.$$

A simple example $SU(2)$

- For $SU(2)$ the generic orbit is: $\mathcal{O} = \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \simeq \mathbb{C}P^1$.
- And the desingularization

$$(\mathbb{R} \times O_\lambda)^W \to \mathfrak{su}(2)^*$$

- In this case the desingularization yields the regular symplectic foliation $\mathbb{R} \times S^2$ and the desingularization transformation is given by *spherical coordinates*. The symplectic form ω on the regular symplectic foliation projects to $\frac{\omega}{r}$ on each generic sphere (coadjoint orbit).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