Local Zeta functions for a class of p-adic symmetric spaces

Pascale Harinck (CNRS - CMLS- Ecole polytechnique - France)

with Hubert Rubenthaler (IRMA-Strasbourg-France)
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Introduction.

Goal: To generalize the Tate functional equation for local zeta functions to a class of p-adic reductive symmetric spaces.
Tate's functional equation (Thesis 1950) on a p-adic field F of caract. 0, q=cardinal of the residual field of F. For $\varphi \in \mathcal{S}(F)$ (smooth, compactly supported), χ= character of F^*, $\mathrm{Re}(s) > s_0$.

$$\text{Local zeta function} \quad Z(\varphi, \chi, s) = \int_{F^*} \varphi(t)\chi(t)|t|^s d^*t,$$

***L*-function**: $\exists\, L(\chi, s) = P_\chi(q^{-s})^{-1}$ unique, $P_\chi \in \mathbb{C}[X]$ such that $P_\chi(0) = 1$ and

$$\{Z(\varphi, \chi, s); \varphi \in \mathcal{S}(F)\} = L(\chi, s)\mathbb{C}[q^{-s}, q^s].$$

$$\text{Functional Equation} \quad Z(\mathcal{F}\varphi, \chi^{-1}, 1-s) = \gamma(\chi, s, \psi) Z(\varphi, \chi, s).$$

where $\mathcal{F}\varphi(y) = \int_F \varphi(x)\psi(xy)dx$ = Fourier transform ($\psi \in \widehat{F}$)

$\gamma(\chi, s, \psi) = \dfrac{L(1-s, \chi^{-1})}{L(s, \chi)} \epsilon(\chi, s, \psi)$ with $\epsilon(\chi, s, \psi) = Cq^{-ms}$,
$C \in \mathbb{C}$, $m \in \mathbb{Z}$. (Inverse of the ρ function of Tate).

Generalizations of Tate's functional equation.

• **H.Jacquet-R.Langlands for $GL(2, F)$ (1970), R.Godement-H.Jacquet for $G = GL(n, D)$ (1972)**, where D a central division algebra over F.
(π, W)=smooth irreducible representation of G,

$$Z(\varphi, s, c_{w,\tilde{w}}) = \int_G \varphi(g) c_{w,\tilde{w}}(g) |\nu(g)|^s dg,$$

where $c_{w,\tilde{w}}(g) = \langle \pi(g)w, \tilde{w}\rangle$ and ν= reduced norm on $M_n(D)$.

$$Z(\mathcal{F}\varphi, \check{c}_{w,\tilde{w}}, n-s) = \gamma(\pi, s) Z(\varphi, c_{w,\tilde{w}}, s), \quad \check{c}_{w,\tilde{w}}(g) = c_{w,\tilde{w}}(g^{-1})$$

Existence of L and ϵ-functions.

• **M. Sato, F. Sato (1989) and I. Muller (arXiv 2008)** on a class of prehomogeneous vector space (M, V) (ie. $\exists M(\bar{F})$ open orbit in $V \otimes \overline{F}$), with relative invariant polynomials $P_0, \dots, P_m$

$$Z(\Phi, s, \omega) = \int_V \Phi(X) \prod_{j=0}^{m} \omega_j(P_j(X))\, |P_j(X)|^{s_j} dX,\ \omega_j \in \widehat{F^*},\ s \in \Omega \subset \mathbb{C}^m$$

$\rightarrow$ abstract functional equations and explicit ones on examples.

Generalizations of Tate's functional equation.

- **N.Bopp - H.Rubenthaler (2005)**: classification of commutative regular PV on $\mathbb{R}$ and explicit functional equations for zeta functions associated to minimal spherical principal series.
- **W. W. Li (2018-2021)**: abstract functional equations for zeta functions associated to smooth irreducible rep. on PV whose open orbit is a wavefront spherical variety (which includes symmetric space) (under some assumptions).

Our results are the analog in p-adic case of those of N.Bopp-H.Rubenthaler+existence of L functions.

Perspectives. For a class of commutative irred. regular PV with a unique open orbit, we hope to obtain explicit formulation of results of W.W. Li (and, if it is possible, with existence of L and ϵ-functions).

Structure of 3-graded algebras.

We fix F a p-adic field of charac. 0 and $\overline{F}$ an algebraic closure.

A commutative prehomogeneous vector space over F (called a PV) is constructed from a 3-**graded reductive algebra** $\tilde{\mathfrak{g}}$ defined over F.

$$\begin{array}{rcccc} \tilde{\mathfrak{g}} = & V^- \oplus & \mathfrak{g} & \oplus & V^+ \\ \text{graded by } H_0 : & -2 & 0 & & 2 \end{array}$$

We always suppose that:
1) The representation $(\mathfrak{g}, V^+)$ is **absolutely irreducible** (i.e. $(\mathfrak{g} \otimes \overline{F}, V^+ \otimes \overline{F})$ is irreducible.
2) $\exists X \in V^+, Y \in V^-$ such that $\{Y, H_0, X\}$ is an $\mathfrak{sl}_2$-triple **(Regularity condition)**.

We fix $\mathfrak{a}$ = maximal split abelian subalgebra of $\mathfrak{g}$ containing H_0 (it is also maximal split abelian in $\tilde{\mathfrak{g}}$)

$$\tilde{\Sigma} = \text{ roots of } (\tilde{\mathfrak{g}}, \mathfrak{a}), \quad \Sigma = \text{ roots of } (\mathfrak{g}, \mathfrak{a})$$

Structure of 3-graded algebras.

Theorem

There exists a basis of simple roots $\widetilde{\Pi} \subset \widetilde{\Sigma}$ such that

1. There is a unique root $\lambda_0 \in \widetilde{\Pi}$ with $\lambda_0(H_0) = 2$
2. for $\nu \in \widetilde{\Pi}$, $\nu \neq \lambda_0$ then $\nu(H_0) = 0$. $(\Pi = \widetilde{\Pi} \setminus \lambda_0 = \text{basis of } \Sigma)$.
3. Inductively, $(\tilde{\mathfrak{g}}_1 = Z_{\tilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)$, with $\widetilde{\mathfrak{l}}_0 = \widetilde{\mathfrak{g}}^{-\lambda_0} \oplus [\widetilde{\mathfrak{g}}^{-\lambda_0}, \widetilde{\mathfrak{g}}^{\lambda_0}] \oplus \widetilde{\mathfrak{g}}^{\lambda_0})$, $\exists$ a sequence of strongly orthogonal roots $\lambda_0, \lambda_1, \ldots, \lambda_k$, where k is uniquely determined by the condition

$$V^+ \cap Z_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0 \oplus \ldots \oplus \widetilde{\mathfrak{l}}_k) = \{0\}.$$

   and we have $H_0 = H_{\lambda_0} + \ldots + H_{\lambda_k}$ (H_{λ_j} is the coroot of λ_j).
   $k+1$ is called **the rank** of $\tilde{\mathfrak{g}}$.

Classification of simple regular irreducible 3-graded algebras.

Let $\mathfrak{j}$ be a Cartan subalgebra of $\widetilde{\mathfrak{g}}$ such that $\mathfrak{a} \subset \mathfrak{j} \subset \mathfrak{g} \subset \widetilde{\mathfrak{g}}$.

Let $\widetilde{\Psi}$ be a basis of $\widetilde{\mathcal{R}}=$ roots of $(\widetilde{\mathfrak{g}} \otimes \overline{F}, \mathfrak{j} \otimes \overline{F})$ such that

$\rho(\widetilde{\Psi}) = \widetilde{\Pi} \cup \{0\}$, where $\rho : \overline{\mathfrak{j}}^* \longrightarrow \mathfrak{a}^*$ is the restriction morphism.

Proposition

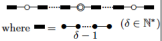
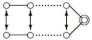
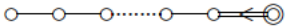
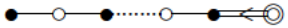
There is a unique simple root $\alpha_0 \in \widetilde{\Psi}$ such that $\rho(\alpha_0) = \lambda_0$.

Consequence

1. In the Satake -Tits diagram of $\tilde{\mathfrak{g}}$, the root α_0 is “white”.
2. The underlying Dynkin diagram, where the root α_0 is distinguished, correspond to a regular 3-graded algebra over $\overline{F}$. (which are classified as on $\mathbb{C}$ (I.Muller- H.Rubenthaler- G.Schiffmann))

TABLE

Table 1 Simple Regular Graded Lie Algebras over a p-adic field

	$\widetilde{\mathfrak{g}}$	$\mathfrak{g}'$	V^+	$\widetilde{\mathcal{R}}$	$\widetilde{\Sigma}$	Satake-Tits diagram	$rank(=k+1)$	ℓ	d	e	Type	1-type
(1)	$\mathfrak{sl}(2(k+1),D)$	$\mathfrak{sl}(k+1,D)$ $\oplus$ $\mathfrak{sl}(k+1,D)$	$\mathrm{M}(k+1,D)$	A_{2n-1} $_{n=(k+1)\delta}$	A_{2k+1}		$k+1$	δ^2	$2\delta^2$	0	I	(A,δ)
(2)	$\mathfrak{u}(2n,E,H_n)$	$\mathfrak{sl}(n,E)$	$\mathrm{Herm}_\sigma(n,E)$	A_{2n-1} $_{n\geqslant 1}$	C_n		n	1	2	2	II	$(A,1)$
(3)	$\mathfrak{o}(q_{(n+1,n)})$	$\mathfrak{o}(q_{(n,n-1)})$	F^{2n-1}	B_n $_{n\geq 3}$	B_n		2	1	$2n-3$	1	II	$(A,1)$
(4)	$\mathfrak{o}(q_{(n+2,n-1)})$	$\mathfrak{o}(q_{(n+1,n-2)})$	F^{2n-1}	B_n $_{n\geqslant 3}$	B_{n-1}		2	1	$2n-3$	3	II	$(A,1)$
(5)	$\mathfrak{o}(q_{(4,1)})$	$\mathfrak{o}(3)$	F^3	B_2	$B_1=A_1$		1	3	$--$	$--$	III	B
(6)	$\mathfrak{sp}(2n,F)$	$\mathfrak{sl}(n,F)$	$\mathrm{Sym}(n,F)$	C_n $_{n\geqslant 2}$	C_n		n	1	1	1	II	$(A,1)$
(7)	$\mathfrak{u}(2n,\mathbb{H},H_{2n})$	$\mathfrak{sl}(n,\mathbb{H})$	$\mathrm{SkewHerm}(\mathrm{n},\mathbb{H})$	C_{2n}	C_n		n	3	4	4	III	B

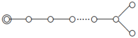
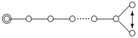
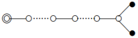
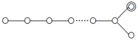
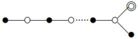
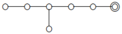
Basic invariants: • $\ell=\dim\widetilde{\mathfrak{g}}^{\lambda_i}$, • $e=\dim\widetilde{\mathfrak{g}}^{\frac{(\lambda_i+\lambda_j)}{2}}\geq 0, \quad i\neq j$

• $d=\dim E_{i,j}(\pm 1,\pm 1) \quad i\neq j$, where $X\in E_{i,j}(p,q)$

$\Longleftrightarrow [H_{\lambda_i},X]=pX,\ [H_{\lambda_j},X]=qX,\ [H_{\lambda_s},X]=0, s\neq i,j,$

TABLE

Table 1 (continued)   Simple Regular Graded Lie Algebras over a p-adic field

	$\widetilde{\mathfrak{g}}$	$\mathfrak{g}'$	V^+	$\widetilde{\mathcal{R}}$	$\widetilde{\Sigma}$	Satake diagram	$rank(=k+1)$	ℓ	d	e	Type	1-type
(8)	$\mathfrak{o}(q_{(m,m)})$	$\mathfrak{o}(q_{(m-1,m-1)})$	F^{2m-2}	D_m $m\geqslant 4$	D_m		2	1	$2m-4$	0	I	$(A,1)$
(9)	$\mathfrak{o}(q_{(m+1,m-1)})$	$\mathfrak{o}(q_{(m,m-2)})$	F^{2m-2}	D_m $m\geqslant 4$	B_{m-1}		2	1	$2m-4$	2	II	$(A,1)$
(10)	$\mathfrak{o}(q_{(m+2,m-2)})$	$\mathfrak{o}(q_{(m+1,m-3)})$	F^{2m-2}	D_m $m\geqslant 4$	B_{m-2}		2	1	$2m-4$	4	I	$(A,1)$
(11)	$\mathfrak{o}(q_{(2n,2n)})$	$\mathfrak{sl}(2n,F)$	$\mathrm{Skew}(2n,F)$	D_{2n} $n\geqslant 3$	D_{2n}		n	1	4	0	I	$(A,1)$
(12)	$\mathfrak{u}(2n,\mathbb{H},K_{2n})$	$\mathfrak{sl}(n,\mathbb{H})$	$\mathrm{Herm}(n,\mathbb{H})$	D_{2n} $n\geqslant 3$	C_n		n	1	4	4	I	$(A,1)$
(13)	split E_7	split E_6	$\mathrm{Herm}(3,\mathbb{O}_s)$	E_7	E_7		3	1	8	0	I	(A,1)

We remark that $\ell = 3$ or $\ell = \delta^2$, $\delta \in \mathbb{N}^*$ and $0 \leq e \leq 4$.

G-open orbits

Let $\mathrm{Aut}_0(\tilde{\mathfrak{g}}) = \mathrm{Aut}(\tilde{\mathfrak{g}}) \cap \mathrm{Aut}_e(\tilde{\mathfrak{g}} \otimes \overline{F})$ where $\mathrm{Aut}_e(\tilde{\mathfrak{g}} \otimes \overline{F}) = \langle e^{\mathrm{ad}(x)}, x \text{ nilpotent in } \tilde{\mathfrak{g}} \otimes \overline{F}\}$ (group of **elementary automorphisms**). We introduce **the group** G

$$G = Z_{\mathrm{Aut}_0(\tilde{\mathfrak{g}})}(H_0)$$

$G = \mathbf{G}(F)$, $\mathbf{G}$ algebraic reductive group defined over F, $Lie(G) = \mathfrak{g}$, G stabilizes $V^{\pm}$ and $\mathfrak{g}$.

Theorem

1. (G, V^+) *is a regular irreducible prehomogeneous vector space. We denote by* Δ_0 *its relative invariant polynomial.*
2. *If* $\ell = \delta^2$, $\delta \in \mathbb{N}^*$ *and* $e = 0$ *or* 4, *the group* G *has a unique open orbit* $\Omega^+ = \{X \in V^+; \Delta_0(X) \neq 0\}$,
3. *In other cases, the number* r *of open* G*-orbits in* V^+ *depends on the parity of* k *and* $r \in \{2, 3, 4, 5\}$ *except when* $e = 2$ *and* k *is even for which* $r = 1$.
4. *Same results for* (G, V^-) *with relative inv. polynomial* ∇_0.

Symmetric spaces and minimal σ_s- parabolic subgroup P

Let $\Omega_s^{\pm} = G.\mathcal{I}_s^{\pm}$, for $s = 1 \ldots, r$, be the open G-orbits in $V^{\pm}$ with

$$\mathcal{I}_s^{\pm} \in \oplus_{j=0}^{k} \tilde{\mathfrak{g}}^{\pm\lambda_j}, \text{ such that } \quad \{\mathcal{I}_s^{-}, H_0, \mathcal{I}_s^{+}\} = \mathfrak{s}l_2 - \text{triple}$$

Let $w_s = e^{\mathrm{ad}\, \mathcal{I}_s^{+}} e^{\mathrm{ad}\, \mathcal{I}_s^{-}} e^{\mathrm{ad}\, \mathcal{I}_s^{+}}$ be the non trivial element of the corresponding Weyl group.

We set $\sigma_s(X) = w_s.X, \; X \in \tilde{\mathfrak{g}}, \quad \sigma_s(g) = w_s g w_s^{-1}, \; g \in \mathrm{Aut}_0(\tilde{\mathfrak{g}})$.

Proposition

1. σ_s is an involution of $\tilde{\mathfrak{g}}$, $\mathfrak{g}$ and G.
2. $H_s = Z_G(\mathcal{I}_s^{+}) = Z_G(\mathcal{I}_s^{-})$ is a subgroup of G^{σ_s} with Lie algebra $\mathfrak{g}^{\sigma_s}$ so that $\Omega_s^{\pm} \simeq G/H_s$ are symmetric spaces.

Let $\mathfrak{a}^0 = \oplus_{j=0}^{k} F H_{\lambda_j}$ and $\mathfrak{n} = \oplus_{0 \leq i < j \leq k} E_{i,j}(1, -1) \subset \mathfrak{g}$ and

$$P = LN \text{ where } L = Z_G(\mathfrak{a}^0) \text{ and } N = \exp^{\mathrm{ad}\, \mathfrak{n}}$$

Lemma

For all $\Omega_s^{\pm}$, $\sigma_s(P)$ is the opposite parabolic subgroup of P (ie. P is a σ_s-parabolic subgroup), and P is minimal for this property.

Prehomogeneous vector spaces $(P, V^{\pm})$

Theorem

Let Δ_j ($j = 0, \ldots, k$) be the relative invariant polynomial of (G_j, V_j^+) corresponding to $\tilde{\mathfrak{g}}_j = Z_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{l}}_0 \oplus \ldots \oplus \tilde{\mathfrak{l}}_j)$. Then

1. (P, V^+) is prehomogeneous, and the Δ_j's are the relative invariants.
2. (P, V^-) is prehomogeneous, with a family of relative invariants ∇_j.
3. If $\ell = \delta^2$ and $e = 0$ or 4, P has a unique open orbit in $V^{\pm}$.
4. In other cases, the number N of open P-orbits depends on e, ℓ and k ($N = 3^{k+1}$ if $\ell = 3$, and for $\ell = 1$: $N = 4^k$ if $e = 1$ or 3 and $N = 2^k$ if $e = 2$).

H-distinguished representation and Zeta functions.

From now, we assume that $\ell = \delta^2$ and $e = 0$ or 4. Hence, G and P have a unique open orbit in $V^{\pm}$. $(\Omega^{\pm} = G.\mathcal{I}^{\pm} \simeq G/H)$.

H-distinguished representation

A smooth admissible representation (π, W) of G is H-distinguished if $(W^*)^H \neq \{0\}$ (space of H-invariant linear forms).

Zeta functions (formally defined).

For $\xi \in (W^*)^H, w \in W$ and $\Phi \in \mathcal{S}(V^+), \Psi \in \mathcal{S}(V^-)$, and $z \in \mathbb{C}$, we set:

$$\mathcal{Z}^+(\Phi, z, \xi, w) = \int_{G/H} \Phi(g.\mathcal{I}^+) < \pi^*(g)\xi, w > |\Delta_0(g.\mathcal{I}^+)|^z dg,$$

$$\mathcal{Z}^-(\Psi, z, \xi, w) = \int_{G/H} \Psi(g.\mathcal{I}^-) < \pi^*(g)\xi, w > |\nabla_0(g.\mathcal{I}^-)|^z dg.$$

Zeta functions.

If $\ell = \delta^2$, $d = 2\ell$ and $e = 0$, then by the classification

$V^+ \simeq \mathfrak{g}l(n, D), \quad G/H \simeq GL(n, D) \times GL(n, D)/diag \simeq GL(n, D)$,
where D is a central division algebra of degree δ.
$\Longrightarrow$ our zeta functions coincide with those of Godement-Jacquet.

Theorem (W.W. Li)

If (π, W) is irreducible, then for Re $z >> 0$, the integrals $\mathcal{Z}^{\pm}(\Phi, z, \xi, w)$ are convergent for $\Phi \in \mathcal{S}(V^{\pm})$, $(\xi, w) \in (W^*)^H \times W$, and extend to rational functions in q^{-z}.

Proof.

W.W. Li proves this result when G is split, using results of Sakellaridis and Venkatesh (on neighborhood at infinity and boundary degenerations). Arguments are valid in general case by P. Delorme's results. □

Functional equation

Fourier transform.
$\mathcal{F}(\Phi)(Y) = \int_{V^+} \Phi(X)\psi(b(X,Y))dX,\ \ Y \in V^-,\ \Phi \in \mathcal{S}(V^+)$,
where $\psi \in \hat{F}$, and b is a suitable normalization of the Killing form.

Theorem (Li, H-Rubenthaler)

Let (π, W) be a H-distinguished smooth irreducible representation of G ($\dim(W^*)^H < \infty$ by Sakellaridis - Venkatesh)).
Then, there exists an endomorphism $\gamma_\psi(\pi, z)$ of W^{*H}, rational in q^{-z}, such that

$$\mathcal{Z}^-(\mathcal{F}(\Phi), m-z, \xi, w) = \mathcal{Z}^+(\Phi, z, \gamma_\psi(\pi, z)\xi, w),\ \ m = \frac{\dim V^+}{k+1}.$$

Proof.

W.W. Li proves this results under assumptions on $V^+ - \Omega^+$. We give a simpler proof in our case using Bruhat's results. □

Functional equation for H- distinguished minimal principal series

Here, we assume that $\ell = 1$ and $e = 0$ or 4.
Recall that $P = LN$ with $L = Z_G(\oplus_j FH_{\lambda_j})$ and $\sigma(P)$ is opposite to P. As $\ell = 1$ the group L acts by a scalar $x_j(\cdot)$ on $\tilde{\mathfrak{g}}^{\lambda_j}$ and

$$L/L\cap H \simeq (F^*)^{k+1} \text{ by the map } l \mapsto (x_0(l), \ldots, x_k(l)).$$

For $\delta = (\delta_0, \ldots, \delta_k) \in \widehat{F^*}^{k+1}$ a unitary character and $\mu \in \mathbb{C}^{k+1}$, we define a character δ_μ of L, which is trivial on $L \cap H$, by

$$\delta_\mu(l) = \prod_{j=0}^{k} \delta_j(x_j(l))\, |x_j(l)|^{\mu_j}.$$

By P.Blanc-P.Delorme, for almost μ, the representation $(Ind_P^G \delta_\mu, I_{\delta_\mu})$ is H-distinguished and $\dim(I^*_{\delta_\mu})^H = 1$.

Functional equation for H- distinguished minimal principal series ($\ell = 1$, $e = 0$ or 4)

Theorem

Recall that $m = \frac{\dim V^+}{k+1}$. Let $\xi \in (I^*_{\delta_\mu})^H$ and $w \in I_{\delta_\mu}$. Then

1. $\mathcal{Z}^\pm(\Phi, z, \xi, w)$ are convergent for Re $z >> 0$ and extend to rational functions in q^{-z},
2. **Existence of L-function**: $\exists! L^\pm(z, \delta_\mu) = P(q^{-z})^{-1}$ such that $P \in \mathbb{C}[X]$, $P(0) = 1$ and $\{\mathcal{Z}^\pm(\Phi, z, \xi, w); \Phi, \xi, w \text{ as usual}\} = L^\pm(z, \delta_\mu)\mathbb{C}[q^{-z}, q^z]$
3. **Explicit functional equation**

$$\mathcal{Z}^-(\mathcal{F}\Phi, \frac{(m+1)}{2} - z, \xi, w) = d(\delta, \mu, z) \;\; \mathcal{Z}^+(\Phi, z + \frac{(m-1)}{2}, \xi, w)$$

with $d(\delta, \mu, z) = C_{\tilde{\mathfrak{g}}} \prod_{j=0}^{k} \gamma(\delta_j, z - \mu_j, \psi)$

where $\gamma(\cdot, \cdot, \psi)$ is the inverse of the ρ function of Tate, $C_{\tilde{\mathfrak{g}}} \in \mathbb{C}^*$.

Moreover $d(\delta, \mu, z) = C' \dfrac{L^-(1 - z, \delta_\mu)}{L^+(z, \delta_\mu)} q^{-sn}$, $C' \in \mathbb{C}, n \in \mathbb{Z}$.

Comments

• **On the proof**:- Relation between $\mathcal{Z}^+(\Phi, z, \xi, w)$ and the zeta functions $Z(\Phi, \omega, s)$ of Sato for (P, V^+) when z and μ are in some convex cones

$$Z(\Phi, \omega, s) = \int_{V^+} \Phi(X) \prod \omega_j(\Delta_j(X))|\Delta_j(X)|^{s_j} dX.$$

- $(P, V^\pm)$ satisfies hypothesis of Theorem k_p of F. Sato (on P singular orbits on $V^+ - \Omega^+$)$\longrightarrow$ abstract functional equation for the zeta functions $Z(\Phi, \omega, s)$ of Sato.

• The last point is an easy consequence of existence of L-functions and functional equation.

• **Open problem:** Explicit expression of L-functions.

Corollary

If (π, W) is an irreducible H-distinguished subrepresentation of I_{δ_μ} then the same results hold for zeta functions associated to (π, W).

Perspectives ($\ell = 1$, $e = 0$ or 4).

Generalize our result to any smooth irreducible H-distinguished representation (π, W): Subrepresentation Theorem for p-adic symmetric spaces of Kato -Takano:
(π, W) **is a subrepresentation of** $Ind_Q^G \tau$, where

- Q **is σ-parabolic subgroup** (ie. $\sigma(Q)$ and Q are opposite)
- τ **is a relatively cuspidal representation** of $M = Q \cap \sigma(Q)$

(ie. $\tau=$ smooth irred., $M \cap H$-distinguished and $\langle \tau^*(m)\xi, w\rangle$ is compactly supported modulo $Z_M(M \cap H)$ for $\xi \in (V_\tau^*)^{H \cap M}$ and $w \in V_\tau$).

Proposition

If (τ, V_τ) is relatively cuspidal representation, then
- there exists L-functions for $\mathcal{Z}^{\pm}(\Phi, \tau, \xi, v)$.
- Moreover, if $\dim(V_\tau^*)^{M \cap H} = 1$, then the factor $\gamma(\tau, z, \psi)$ is scalar and satisfies $\gamma(\tau, z, \psi) = C\frac{L^-(1-z,\tau)}{L^+(z,\tau)} q^{-sn}$ for some $C \in \mathbb{C}^*$, $n \in \mathbb{Z}$.

THANK YOU FOR YOUR ATTENTION