Topological K-theory for discrete groups and index theory

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Group in Action
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Outline

This talk is about index theory in the context of noncommutative geometry.

- Higher index theory
- Topological K-theory of a group and index formulas

Reference:

- Coauthors: Paulo Carrillo-Rouse (Toulouse)
  Bai-Ling Wang (Australian National University)
- *Topological K-theory for discrete groups and index theory*
  Bull. Sci. math. 2023.

1. Higher index theory

Index theory in different contexts

- An elliptic operator D on a closed manifold is Fredholm, and

$$\mathrm{ind} D := \dim \ker D - \dim \mathrm{coker} D \in \mathbb{Z};$$

- If a compact group G acts on the manifold and D is G-invariant, then it has the equivariant index

$$\mathrm{ind}_G D := [\ker D] - [\mathrm{coker} D] \in R(G);$$

- If D is a Dirac type operator twisted by connections from the moduli space T of flat $U(1)$-connections, then there is a family index

$$\mathrm{ind}\{D_\nabla\}_T := \text{“}[\{\ker D_\nabla\}] - [\{\mathrm{coker} D_\nabla\}]\text{”} \in K^0(T).$$

Here $T = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.

Higher index

Let

- M be a complete Riemannian manifold and
- Γ a group acting on M properly with M/Γ compact.

Then a Γ-invariant elliptic operator D on M has a higher index

$$\mathrm{ind}_G D \in K_*(C_r^*(\Gamma)),$$

so that when

- Γ is trivial,

$$\mathrm{ind}_\Gamma D \in \mathbb{Z} \text{ is the Fredholm index;}$$

- Γ is compact,

$$\mathrm{ind}_\Gamma D \in R(\Gamma) \text{ is the equivariant index;}$$

- $\Gamma = \mathbb{Z}^n$,

$$\mathrm{ind}_\Gamma D \in K^0(\hat{\Gamma}) \text{ is the family index.}$$

Group C^*-algebras and K-theory

Let Γ be a locally compact group. The reduced group C^*-algebra $C_r^*(\Gamma)$ is the norm closure of the image for

$$C_c(\Gamma) \to \mathcal{B}(L^2(\Gamma)) \qquad f \mapsto [g \to f * g].$$

Its K-theory are groups made out of equivalence classes of projections or unitaries of the matrix algebra over $C_r^*(\Gamma)$. If

- Γ is trivial, then

$$C_r^*(\{e\}) = \mathbb{C} \quad \& \quad K_0(\mathbb{C}) \simeq \mathbb{Z};$$

- Γ is compact, then

$$C_r^*(\Gamma) \cong \bigoplus_{\pi \in \hat{\Gamma}} M_{n_\pi}(\mathbb{C}) \quad \& \quad K_0(C_r^*(\Gamma)) \simeq \bigoplus_{\pi \in \hat{\Gamma}} \mathbb{Z} \cong R(\Gamma);$$

- $\Gamma = \mathbb{Z}^n$, then

$$C_r^*(\Gamma) \cong C(\hat{\Gamma}) \text{ by Fourier transform} \quad \& \quad K_0(C_r^*(\Gamma)) \simeq K^0(\widehat{\Gamma}).$$

Why higher index?

1. Dirac operator approach in the obstruction of positive scalar curvature metric.

Theorem (Schoen-Yau, Gromov-Lawson)

The n torus does not admit a metric with positive scalar curvature.

Suppose T^n has metric with positive scalar curvature. Then by the Lichnerowich theorem $D^2 = \Delta + \frac{\kappa}{4}$,

$$D \text{ is invertible, and then } \mathrm{ind}_{\mathbb{Z}^n} \widetilde{D} = 0.$$

However, it can be verified that

$$\mathrm{ind}_{\mathbb{Z}^n} \widetilde{D} \neq 0.$$

2. Representation theory of connected reductive groups and their geometric construction through Connes-Kasparov isomorphism, following Borel-Weil-Bott, Parthasarathy, Atiyah-Schmid, ...

Baum-Connes conjecture

Question:

- Does every element of $K_*(C_r^*(\Gamma))$ come from the index of some Γ-invariant elliptic operator?

In 1982, Baum-Connes proposed an algorithm of computing the K-theory, by introducing the topological group of Γ:

$$K_{top}^*(\Gamma)$$

and formulating the assembly map, or the higher index map

$$\mu : K_{top}^*(\Gamma) \to K_*(C_r^*(\Gamma)).$$

The Baum-Connes conjecture claims that μ is an isomorphism.
The conjecture is important in noncommutative geometry because

- $K_*(C_r^*(\Gamma))$ is hard to compute in general;
- Applications in geometry, topology and representation theory.

Up-to-date description of $K^*_{top}(\Gamma)$

Baum-Connes-Higson reformulated the Baum-Connes assembly map using "analytic K-homology":

$$\mu : K^\Gamma_*(\underline{E}\Gamma) \to K_*(C^*_r(\Gamma)).$$

where $\underline{E}\Gamma$ is universal space of proper actions by Γ in the sense that for any Γ-proper cocompact manifold X, there exists a Γ-equivariant continuous map

$$f : X \to \underline{E}\Gamma.$$

About K-homology:

- Atiyah formulated analytic K-homology for a closed manifold M as dual theory of K-theory using elliptic operators;
- K-homology was generalized to Kasparov's KK-theory using abstract elliptic operators.

2. Topological K-theory for a discrete group

Overview

The topological K-theory for a discrete group Γ

- is defined following pushforward maps between Γ proper spaces;
- admits a Chern character map as a result of Riemann-Roch theorem for Γ proper spaces;
- contains and assembles all index pairing information for Γ proper actions.

The group $K^*_{top}(\Gamma)$ represents the "computable part" of the K-theory of $C^*_r(\Gamma)$.

Pushforward map / wrong way functoriality

If $f : X \to Y$ is a continuous function between compact topological spaces, then there is a functorial map on K-theory:

$$f^* : K^0(Y) \to K^0(X) \quad [E] \mapsto [f^* E].$$

There is a pushforward map, or wrong way functorial map

$$f_! : K^*(X) \to K^*(Y),$$

which is specialized to

- Thom isomorphism $i_! : K^*(X) \to K^*(E)$ associated to $i : X \to E$ where $E \to X$ is a complex (spin^c in general) vector bundle.
- Index map $f_! : K^0(X) \to K^0(pt)$ associated to $f : X \to pt$ where X admits a spin^c structure.

Pushforward map

Let Γ be a discrete group. The pushforward map can also be defined to nice Γ-equivariant maps $f : M \to N$:

$$f_! : K_\Gamma^*(M) \to K_\Gamma^*(N)$$

where

- M, N are Γ-proper cocompact manifolds,
- $x \in K_\Gamma^*(M) := K_*(C_r^*(M \rtimes \Gamma))$.

Key construction: Deformation to the normal cone

Let $f : M \to N$ be a Γ-equivariant map.

- $TM \oplus f^*TN$ is identified with the normal bundle of the inclusion

$$M \xrightarrow{\Delta \times f} M \times M \times N \qquad x \mapsto (x, x, f(x)).$$

- There is a deformation groupoid

$$D_f : TM \oplus f^*TN \bigsqcup (M \times M \times N) \times (0,1] \rightrightarrows f^*TN \bigsqcup (M \times N) \times (0,1]$$

Example

Connes' tangent groupoid

$$\mathcal{T} = TM \times \{0\} \bigsqcup M \times M \times (0,1]$$

associated to the map $M \to \{pt\}$.

Pushforward map

Denote

$$T_f := TM \oplus f^*TN \to M$$

with dim r_f and assume T_f^* has a spin^c-structure, i.e., f is K-oriented.

$C^*(D_f)$ has evaluations at end points

$$\begin{aligned} ev_0 &: C^*(D_f) \to C^*(T_f) \\ ev_1 &: C^*(D_f) \to C^*(M \times M \times N). \end{aligned}$$

Since $\ker ev_0$ is contractible, ev_0 induces an isomorphism on K-theory.

The pushforward map $f_! : K_\Gamma^{*-r_f}(M) \to K_\Gamma^*(N)$ is defined by the compositions:

$$\begin{aligned} K_\Gamma^{*-r_f}(M) \xrightarrow{Th} K_\Gamma^*(T_f^*) \xrightarrow{F} K_\Gamma^*(T_f) \xrightarrow{(e_{0,*})^{-1}} K_\Gamma^*(D_f) \\ \xrightarrow{e_{1,*}} K_\Gamma^*(M \times M \times N) \xrightarrow{M} K_\Gamma^*(N). \end{aligned}$$

Example: pushforward as the analytic index

Let $f : M \to pt$ where M is spin^c, the the pushforward map

$$f_! : K^{*-r}(M) \to K^*(pt)$$

is given by

$$K^{*-r}(M) \to K^*(T^*M) \to K_*(C_0(T^*M)) \to$$
$$K_*(C_r^*(TM)) \xrightarrow{(e_{0,*})^{-1}} K_*(C_r^*(\mathcal{T})) \xrightarrow{e_{1,*}} K_*(C_r^*(M \times M)) \to K^*(pt) \simeq \mathbb{Z}.$$

This recovers the analytic index of Atiyah-Singer:

$$\begin{aligned} K^0(T^*M) &\to \mathbb{Z} \\ [\sigma_D] &\mapsto \operatorname{ind} D. \end{aligned}$$

Topological K-theory for Γ

The topological K-theory of Γ $K^*_{top}(\Gamma)$ consists of cycles (M, x) where

- M is a Γ-proper cocompact spin^c manifold;
- $x \in K^*_\Gamma(M) := K_*(C^*_r(M \rtimes \Gamma))$

subject to relations

$$(M, x) \sim (N, f_! x).$$

where $f_! : K^*_\Gamma(M) \to K^*_\Gamma(N)$ is induced from a Γ-equivariant map $f : M \to N$ which is K-oriented, i.e., $TM \oplus f^* TN \to M$ is spin^c. In other words,

$$K^*_{top}(\Gamma) := \varinjlim_{f_!} K^*_\Gamma(M).$$

Assembled Chern character

Suppose a discrete group Γ acts properly and cocompactly on a manifold M.

- For $f : M \to N$ we have a cohomological pushforward on delocalized cohomology

$$f_! : H^{*-r_f}_{\Gamma,deloc}(M) \to H^*_{\Gamma,deloc}(N).$$

  and the delocalized cohomology for discrete groups

$$H^*_{top}(\Gamma) := \varinjlim_{f_!} H^*_{\Gamma,deloc}(M).$$

- For a discrete group Γ, our next result is the formulation of Chern character morphism

$$ch^{top} : K^*_{top}(\Gamma) \longrightarrow H^*_{top}(\Gamma)$$

  due to a Grothendieck-Riemann-Roch theorem.

Delicalized cohomology (Tu-Xu)

Let M be a proper Γ-manifold. Consider the periodic delocalized cohomology groups for $* = 0, 1 \; mod \, 2$:

$$H^{*}_{\Gamma,deloc}(M) := \bigoplus_{g \in \langle \Gamma \rangle^{fin}} \prod_{k = *, \, mod \, 2} H^{k}_{c}(M_g \rtimes \Gamma_g),$$

where

- $g \in \Gamma$ is a fixed, finite order element
- $\langle \Gamma \rangle^{fin}$ stands for the set of conjugacy classes of finite order elements of Γ
- $M_g = \{x \in M : x \cdot g = x\}$
- $\Gamma_g = \{h \in \Gamma : hg = gh\}$
- $M_g \rtimes \Gamma_g$ is the action groupoid

Remark

Equivalent formulations: Brendon (co)homology, Chen-Ruan orbifold (co)homology, cyclic cohomology of crossed products.

Grothendieck-Riemann-Roch

Let Γ be a discrete group.

- Let M, N be proper cocompact Γ-manifolds;
- Let $f : M \to N$ be a Γ-equivariant K-oriented map, i.e., $TM \oplus f^*TN \to M$ is spin^c. Let dim $TM \oplus f^*TN = r$.

Theorem (CR-W-W)

The diagram commutes:

$$\begin{array}{ccc} K_\Gamma^{*-r}(M) & \xrightarrow{f_!} & K_\Gamma^*(N) \\ \Big\downarrow {\scriptstyle ch_{Td_M^\Gamma}} & & \Big\downarrow {\scriptstyle ch_{Td_N^\Gamma}} \\ H_{\Gamma,deloc}^{*-r}(M) & \xrightarrow[f_!]{} & H_{\Gamma,deloc}^*(N). \end{array}$$

where for $E \to M$ a Γ-proper spin^c bundle,

$$\mathrm{ch}_{Td_E^\Gamma} : K_\Gamma^*(M) \to H_{\Gamma,deloc}^*(M) \qquad x \mapsto \mathrm{ch}_M^\Gamma(x) \wedge \mathrm{Td}^\Gamma(E)$$

is the "twisted delocalized Chern character."

Example

The commutative diagram

$$
\begin{array}{ccc}
K_\Gamma^{*-r}(M) & \xrightarrow{f_!} & K_\Gamma^{*}(N) \\
\Big\downarrow{\scriptstyle ch_{Td_M^\Gamma}} & & \Big\downarrow{\scriptstyle ch_{Td_N^\Gamma}} \\
H_{\Gamma,deloc}^{*-r}(M) & \xrightarrow[f_!]{} & H_{\Gamma,deloc}^{*}(N).
\end{array}
$$

reduces to the Riemann-Roch theorem when Γ is trivial:

$$ch(f_!(E)) \wedge \mathrm{Td}(N) = f_!(\mathrm{ch}(E) \wedge \mathrm{Td}(M)).$$

If N is in addition a point, this is

$$f_!(E) = f_!(\mathrm{ch}(E) \wedge \mathrm{Td}(M)) = \int_M \mathrm{ch}(E) \wedge \mathrm{Td}(M) = \mathrm{ind}\, D_E.$$

An example of index formula

Let M be a closed complex (or spin^c) manifold, $E \to M$ be a holomorphic vector bundle, and $\mathcal{O}(E)$ be holomorphic sections. Consider the Dolbeault complex of E:

$$\overline{\partial} : 0 \to \Omega^{0,0}(E) \to \Omega^{0,1}(E) \to \cdots \to \Omega^{0,n}(E) \to 0.$$

Let

$$\chi(M, E) := \sum_q (-1)^q \dim H^q(M, \mathcal{O}(E)).$$

Theorem (Riemann-Roch, Atiyah-Singer)
$D := \overline{\partial} + \overline{\partial}^* : \Omega^{0,even}(E) \to \Omega^{0,odd}(E)$ *is Fredholm and*

$$\operatorname{ind} D = \chi(M, E) = \int_M \mathrm{ch}(E)\mathrm{Td}(M).$$

Idea of proof: N is a point and Γ is trivial

For $f : M \to pt$ K-oriented, $f_! : K^*(M) \to K^*(pt)$ is given by the analytic index of D_M.

$$\begin{array}{ccc}
K^0(M) & \xrightarrow{ch(-)\wedge Td(M)} & H^{ev}(M) \\
\simeq \downarrow Thom & & \simeq \downarrow Thom \\
K^0(T^*M) & \xrightarrow{ch} & H^{ev}(T^*M) \\
\downarrow \text{ind} & & \downarrow \\
K_0(C^*(M \times M)) & \xrightarrow{ch} & HP_0(C^\infty(M \times M)) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathbb{Z} & \xrightarrow[\subset]{} & \mathbb{C}.
\end{array}$$

When M is closed spin^c and D is Dirac, commutativity of the diagram implies the Riemann-Roch theorem

$$\text{ind}\, D_E = \int_M \mathrm{ch}(E) \wedge \mathrm{Td}(M).$$

Assembled Chern character

For a discrete group Γ and for $* = 0, 1 \ mod \ 2$ we define the delocalized cohomology for discrete groups

$$H^*_{top}(\Gamma) := \varinjlim_{f_!} H^*_{\Gamma, deloc}(M).$$

Theorem (CR-W-W)

For a discrete group Γ, there is a well-defined Chern character morphism

$$ch^{top} : K^*_{top}(\Gamma) \longrightarrow H^*_{top}(\Gamma)$$
$$ch^{top}([M, x]) = [M, ch^{\Gamma}_M(x) \wedge Td^{\Gamma}_M].$$

Furthermore, it is an isomorphism once tensoring with $\mathbb{C}$.

Remark

Chern character on the LHS of the assembly map were previously formulated by Lück, Matthey, Voigt.

Pairing on the LHS of the assembly map

Recall that there is an assembly map $\mu : K^*_{top}(\Gamma) \to K_*(C^*_r(\Gamma))$. The LHS can be explicitly calculated and assembles all index pairing information for proper cocompact actions.

Theorem (CR-W-W)

One has a cohomological assembly map

$$H^*_{top}(\Gamma) \to HP_*(\mathbb{C}\Gamma)$$

where a cohomological formula is obtained from pairing with $\tau \in HP^*(\mathbb{C}\Gamma)$.

According to Burghelea:

$$HP_*(\mathbb{C}\Gamma) \simeq (\prod_{\langle\Gamma\rangle^{fin}} \bigoplus_k H_k(\Gamma_g, \mathbb{C})) \bigoplus T_*(\Gamma)$$

Index formula for proper actions

Recall that by Burghelea:

$$HP^*(\mathbb{C}\Gamma) \simeq (\prod_{\langle\Gamma\rangle^{fin}} \bigoplus_k H^k(\Gamma_g, \mathbb{C})) \bigoplus T^*(\Gamma).$$

Theorem (CR-W-W)

One has a cohomological assembly map

$$H^*_{top}(\Gamma) \to HP_*(\mathbb{C}\Gamma)$$

where a cohomological formula is obtained from pairing with $\tau \in HP^*(\mathbb{C}\Gamma)$.

$$\langle ch^{top}([M,E]), \tau_g\rangle = \langle ch^g_M(E) \wedge Td^M_g, \pi^*_g(\tau_g)\rangle$$

where $\tau_g \in HP^*(\mathbb{C}\Gamma)$ is a cocycle in the g-component of the finite conjagacy class part and π^g_* is the composition

$$H^*(\Gamma) \to H^*(M \rtimes \Gamma) \to H^*(M^g \rtimes \Gamma_g).$$

Relation to the higher index

Remark

A commutative diagram of index formulas is expected

$$
\begin{array}{ccccc}
K^*_{top}(\Gamma) & \xrightarrow{\mu} & K_*(C^*_r(\Gamma)) & & \\
\downarrow{\scriptstyle ch^{top}} & & & \searrow & \\
H^*_{top}(\Gamma) & \xrightarrow[\mu]{} & HP_*(\mathbb{C}\Gamma) & \xrightarrow[\tau]{} & \mathbb{C}.
\end{array}
$$

The index formula assembles all information coming from the higher index of elliptic operators for proper cocompact actions.

Special cases

- When $g = e$, this index recovers the higher index formula by Connes-Moscovici,

$$\langle ch^{top}([M,E]), \alpha\rangle = \int_{M/\Gamma} ch(E/\Gamma) \wedge Td(M/\Gamma) \wedge \pi^*(\alpha), \forall [\alpha] \in H^*(\Gamma)$$

  for the classifying map $f : M/\Gamma \to B\Gamma$. (M/Γ is a manifold and M is the universal cover.)
  This was used to prove the Novikov conjecture for hyperbolic groups.
- Let $\mathrm{tr}_g : \mathbb{C}\Gamma \to \mathbb{C}$ be the delocalized trace associated to $g \in G$

$$\mathrm{tr}_g(\sum_{\gamma\in\Gamma} c_\gamma \gamma) := \sum_{\gamma\in\langle g\rangle} c_\gamma$$

  representing an element in $HP^0(\mathbb{C}\Gamma)$. Then

$$\langle [M,E], \mathrm{tr}_g\rangle$$

  recovers the L^2-Lefschetz fixed point formula for orbifolds by Wang-W.

Thanks for listening!

Happy and Prosperous to Prof Michèle Vergne!