Tempered Representations and Pseudodifferential Operators on Symmetric Spaces

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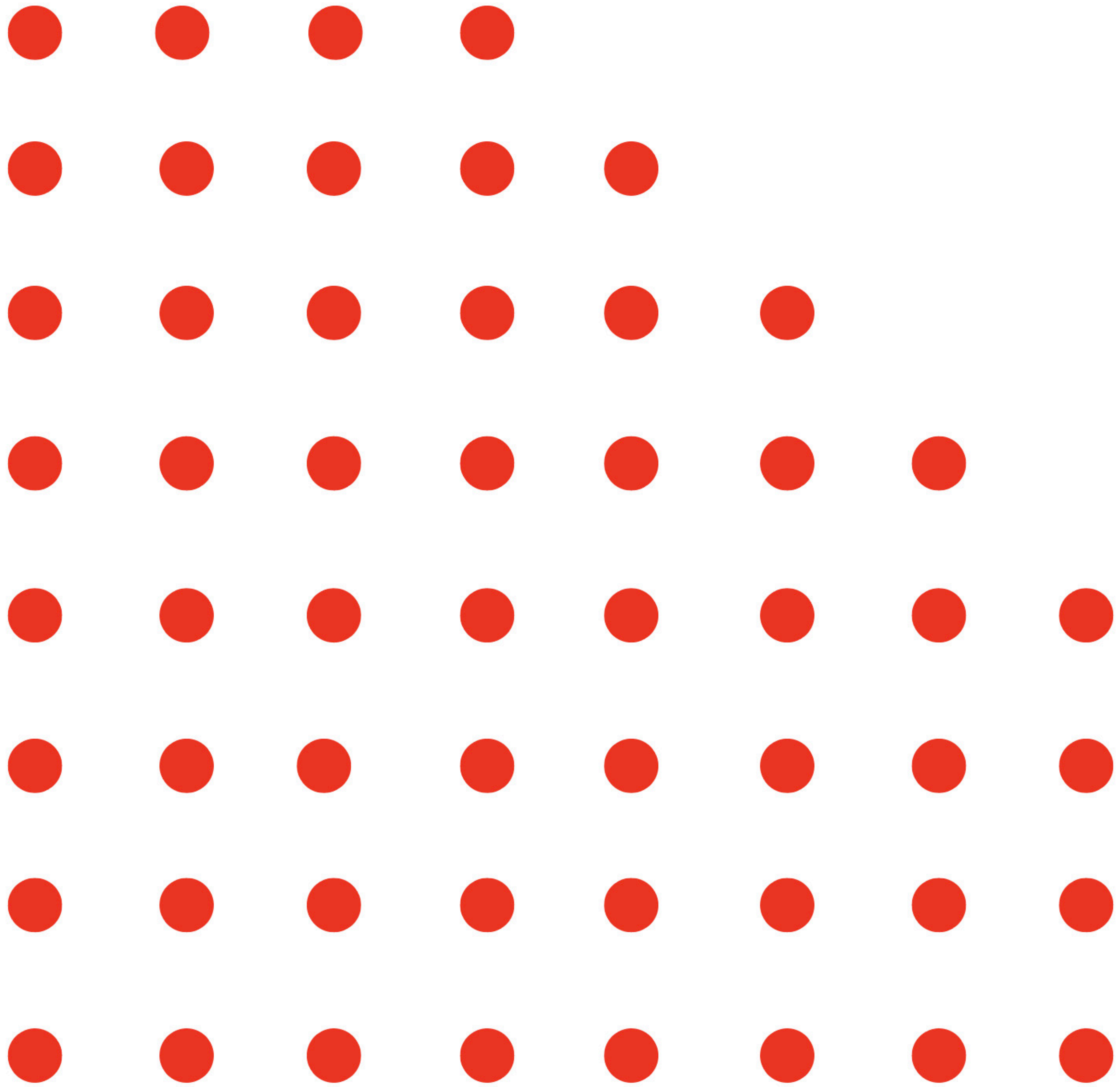
Connes-Kasparov Picture Book

I'll be examining the Connes-Kasparov isomorphism in C*-algebra K-theory …



Let's examine the real reductive group $G = Sp(1,1)$ — the group of 2×2 matrices g over the quarternions with

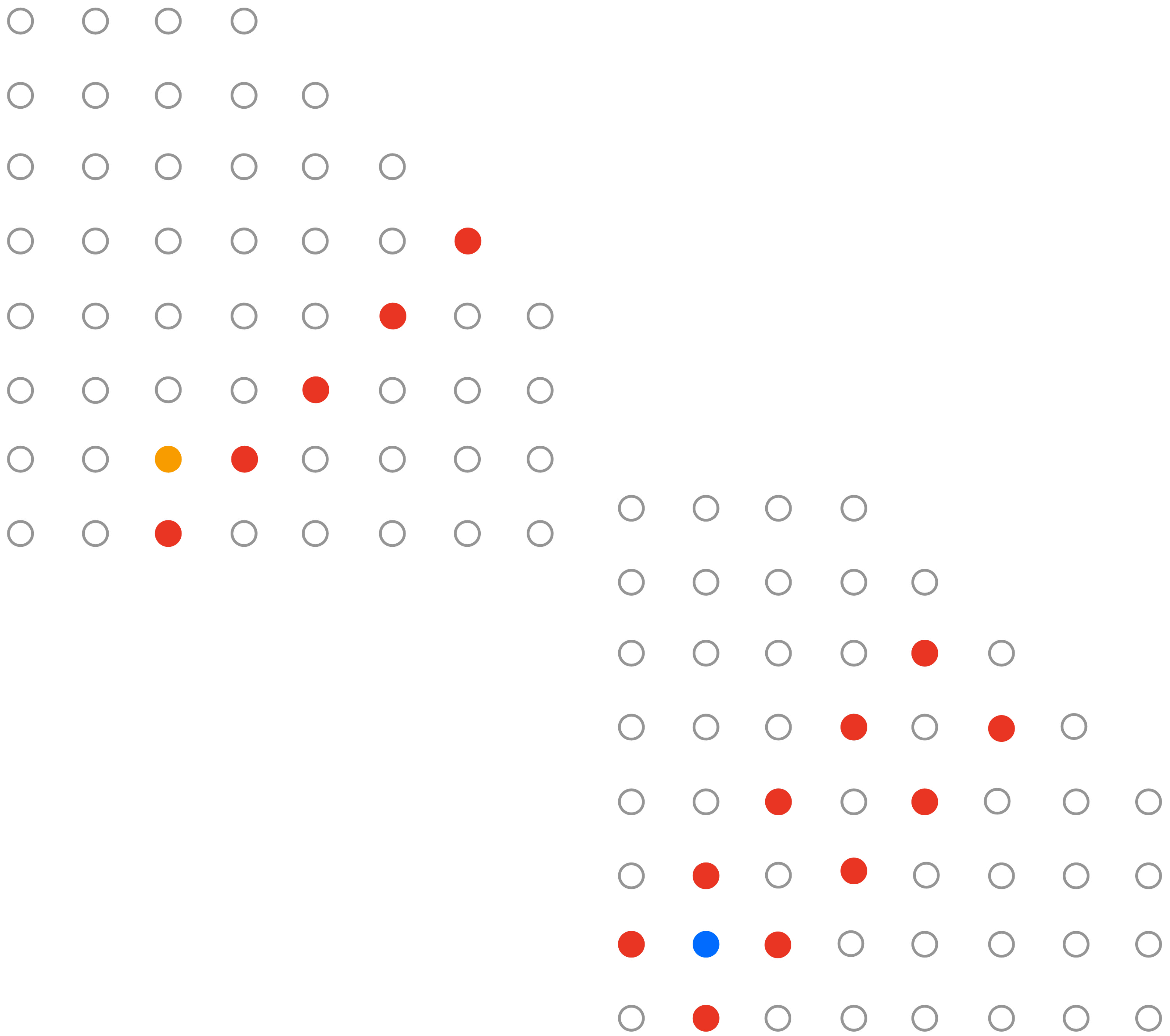
$$g \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} g^* = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

The diagonal subgroup

$$K = \left\{ \begin{bmatrix} a & \\ & d \end{bmatrix} \right\} \cong SU(2) \times SU(2)$$

is a maximal compact subgroup. *Its irreducible representations are parametrized by pairs (m, n) of nonnegative integers.*

K-Types and Minimal K-Types

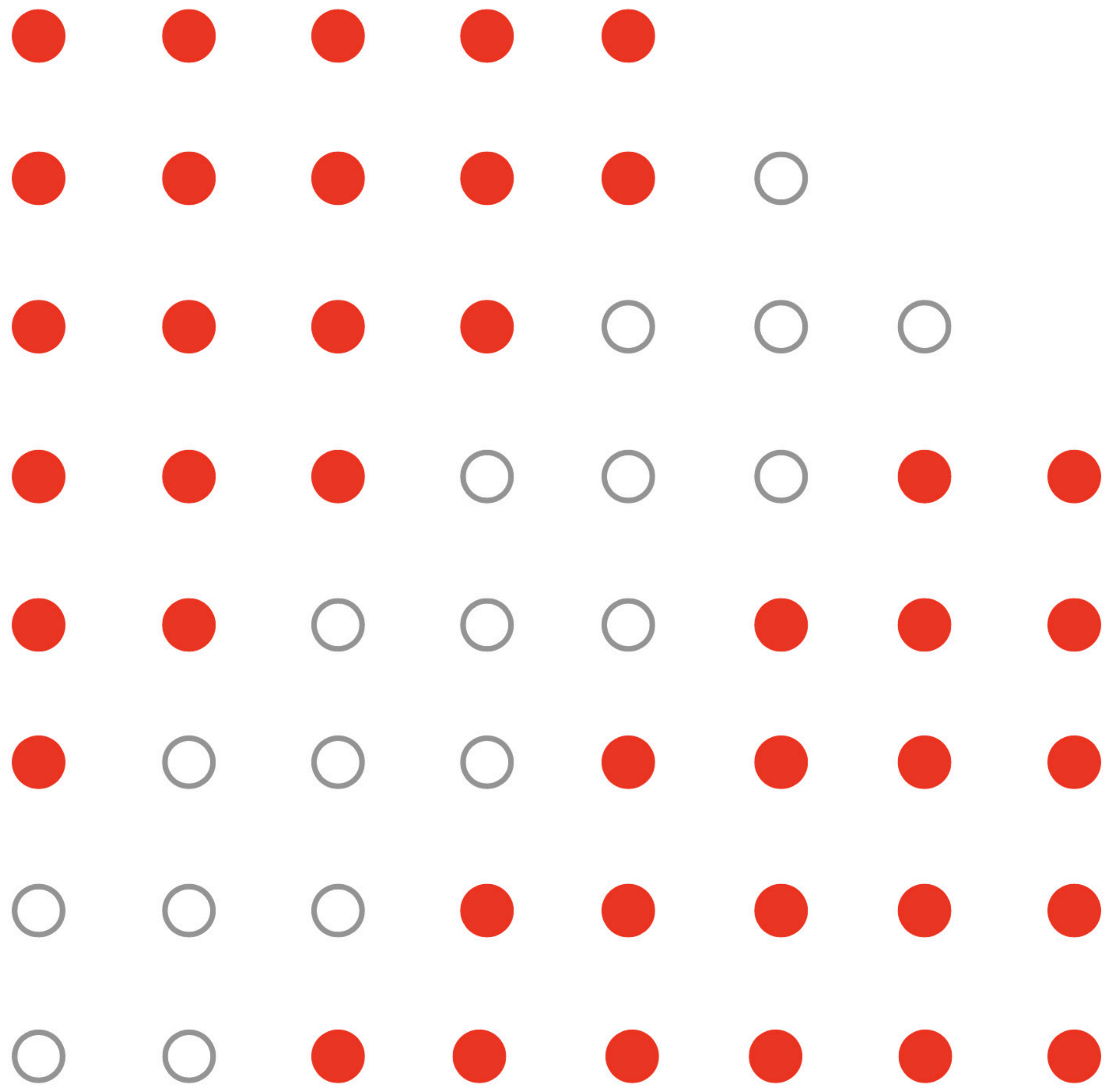


We're interested in understanding the tempered dual of G in terms of the dual of K. One way to try to do so is to restrict representations from G to K, and then decompose into irreducible representations of K.

(The orange and blue dots in the examples indicate which representations are involved, in the Connes-Kasparov parametrization that I'll discuss soon).

Definition. The minimal K-types of a representation of $G = Sp(1,1)$ are the K-types closest to the trivial representation $(0,0)$ of K.

Discrete Series

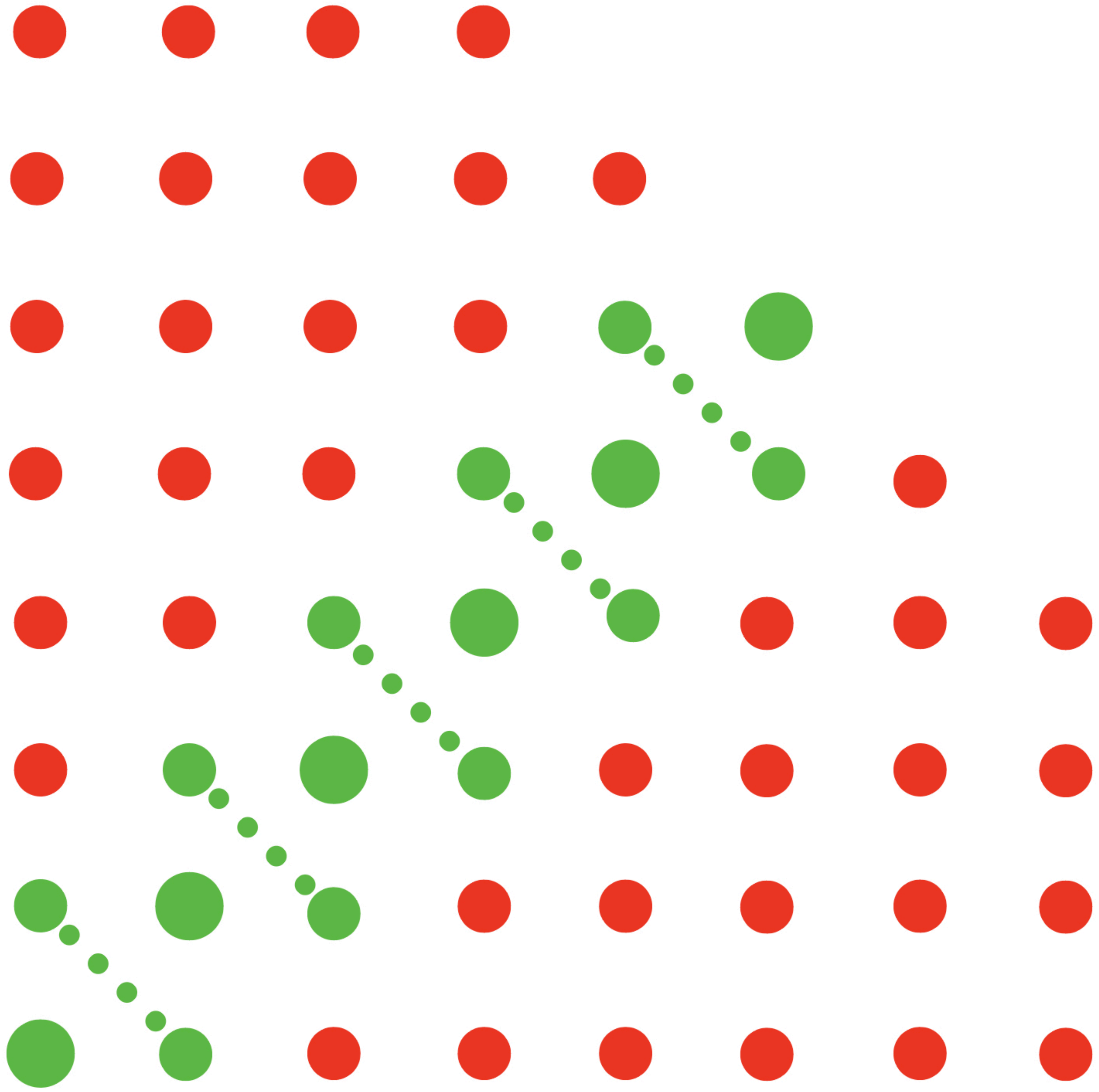


The diagram shows the minimal K-types of the discrete series of G (the irreducible and square-integrable representations). These are the isolated points in the tempered dual.

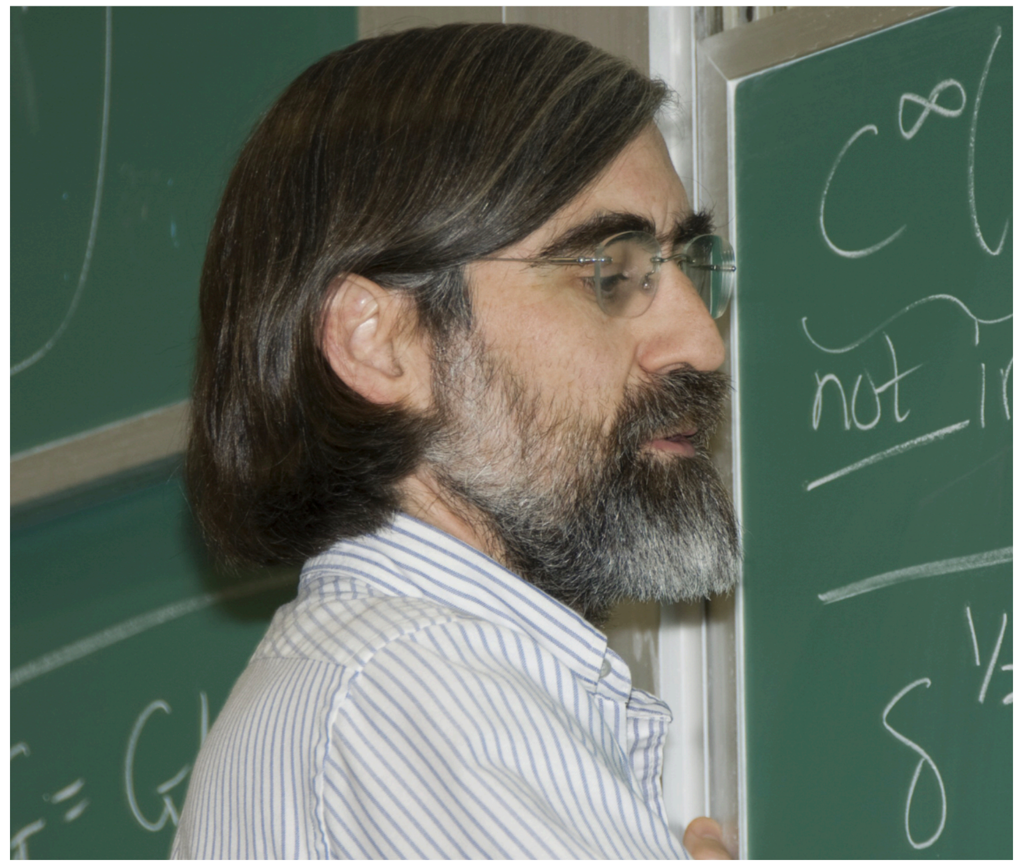
Theorem (Schmid). *Each discrete series has a unique minimal K-type, and all these minimal K-types are distinct from one another.*

Most irreducible representations of K occur as minimal K-types of the discrete series, but not all of them …

Vogan’s Theorem

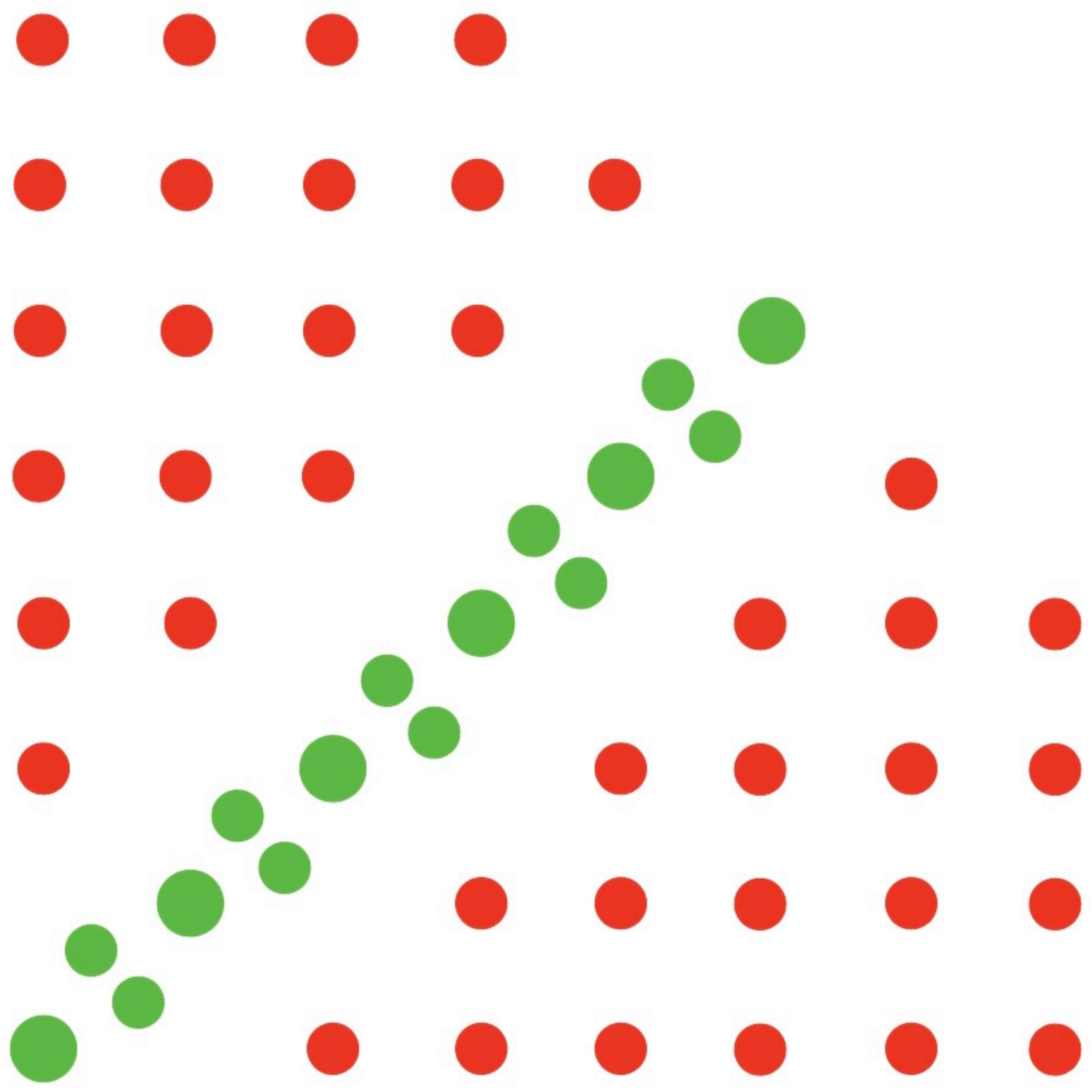


A remarkable discovery of David Vogan is that **every** irreducible representation of K arises as a minimal K-type somewhere in the tempered dual, and no irrep. of K is associated to more than one component of the tempered dual.



Some components have more than one minimal K-type. But this, however, is an additional feature of Vogan’s theorem …

The Tempered Dual from Vogan's Theorem

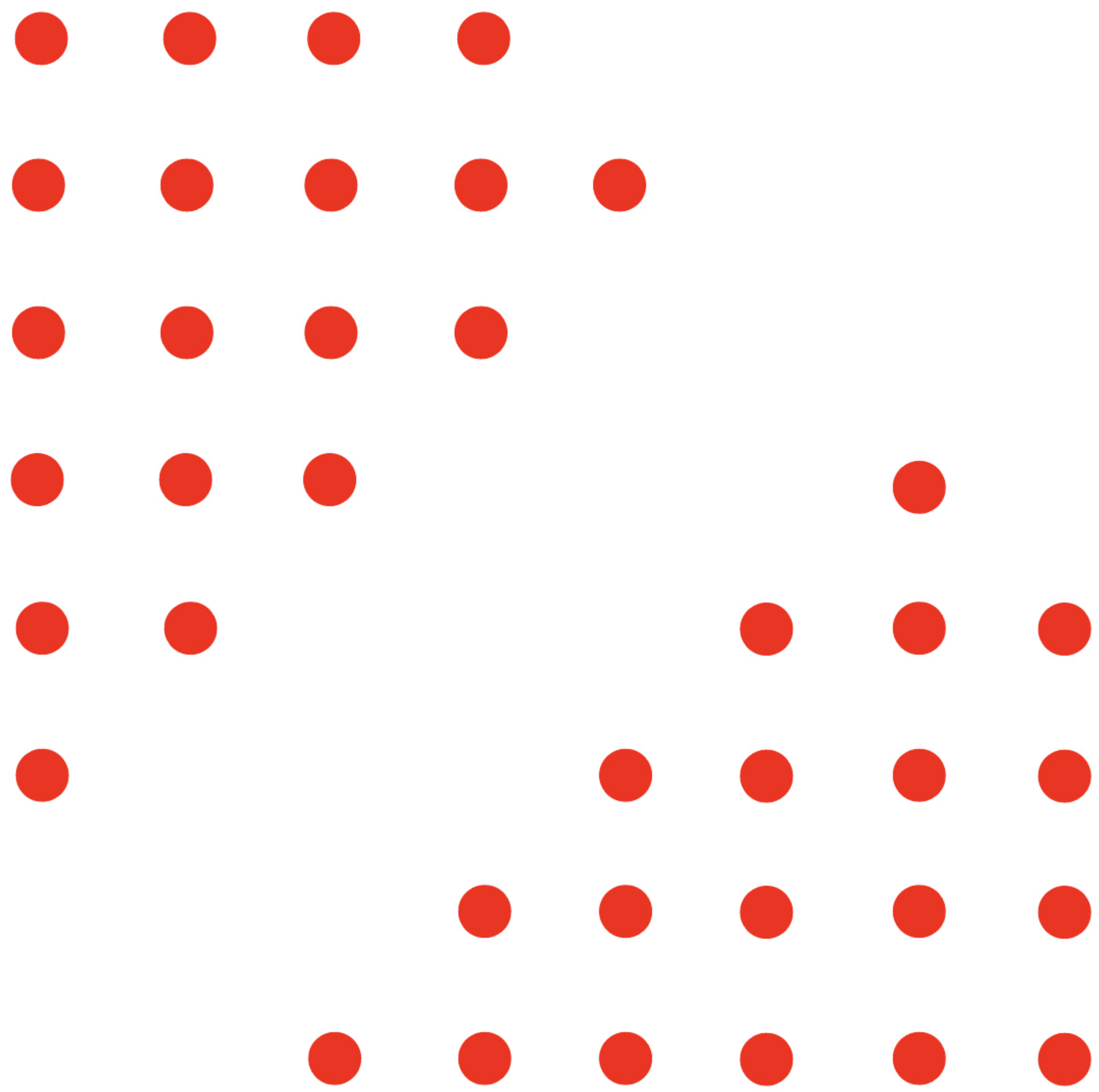


In fact, using these results of Vogan, plus a little bit more, one can completely describe the tempered dual.

First, I've distorted the diagram showing the irreducible representations of K to indicate that the continuous series representations have alternately one or two minimal K-types.

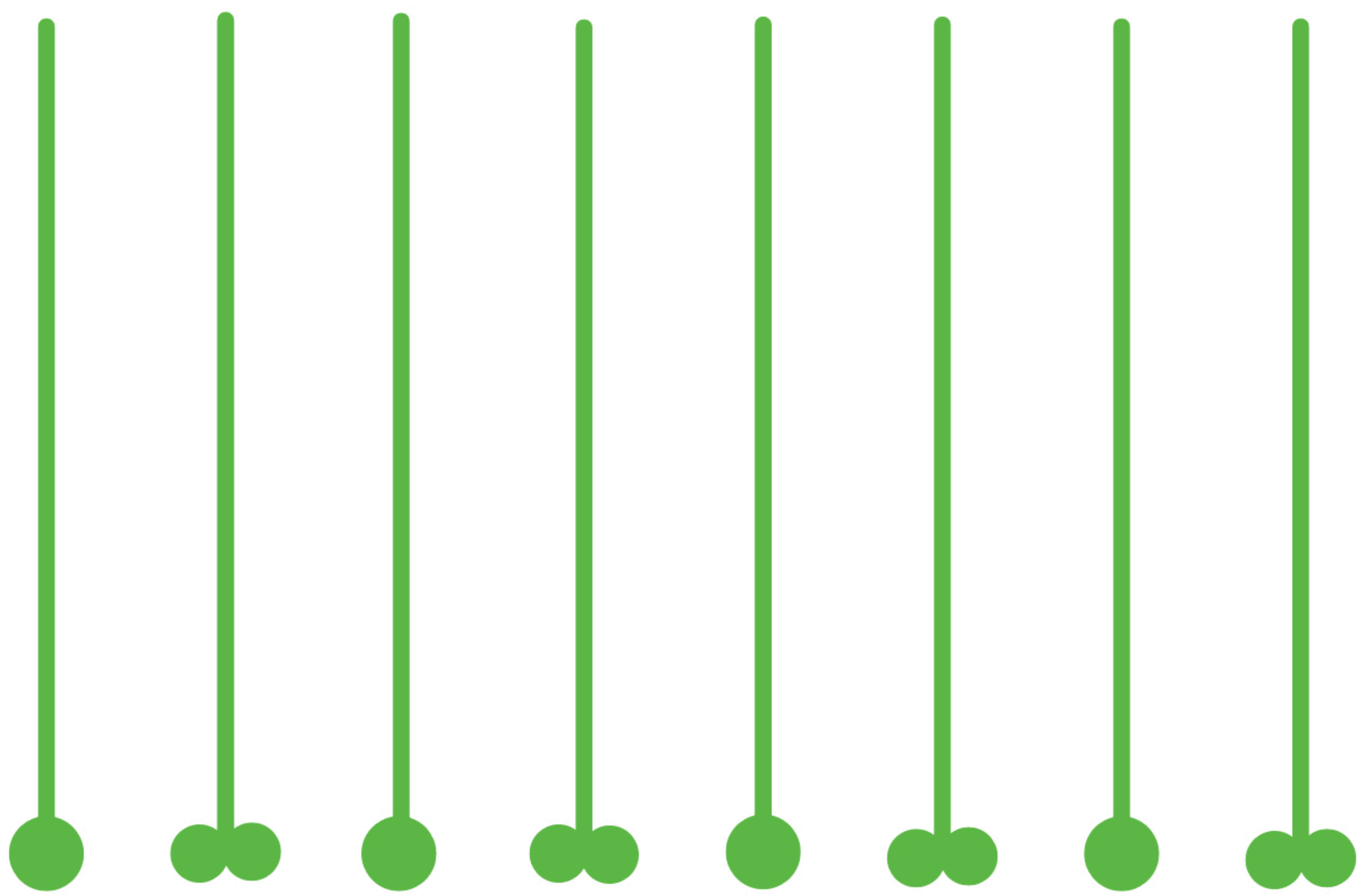
According to Vogan's theorem, the base representations in these continuous series are alternately irreducible, or decompose into two irreducible representations.

The Tempered Dual

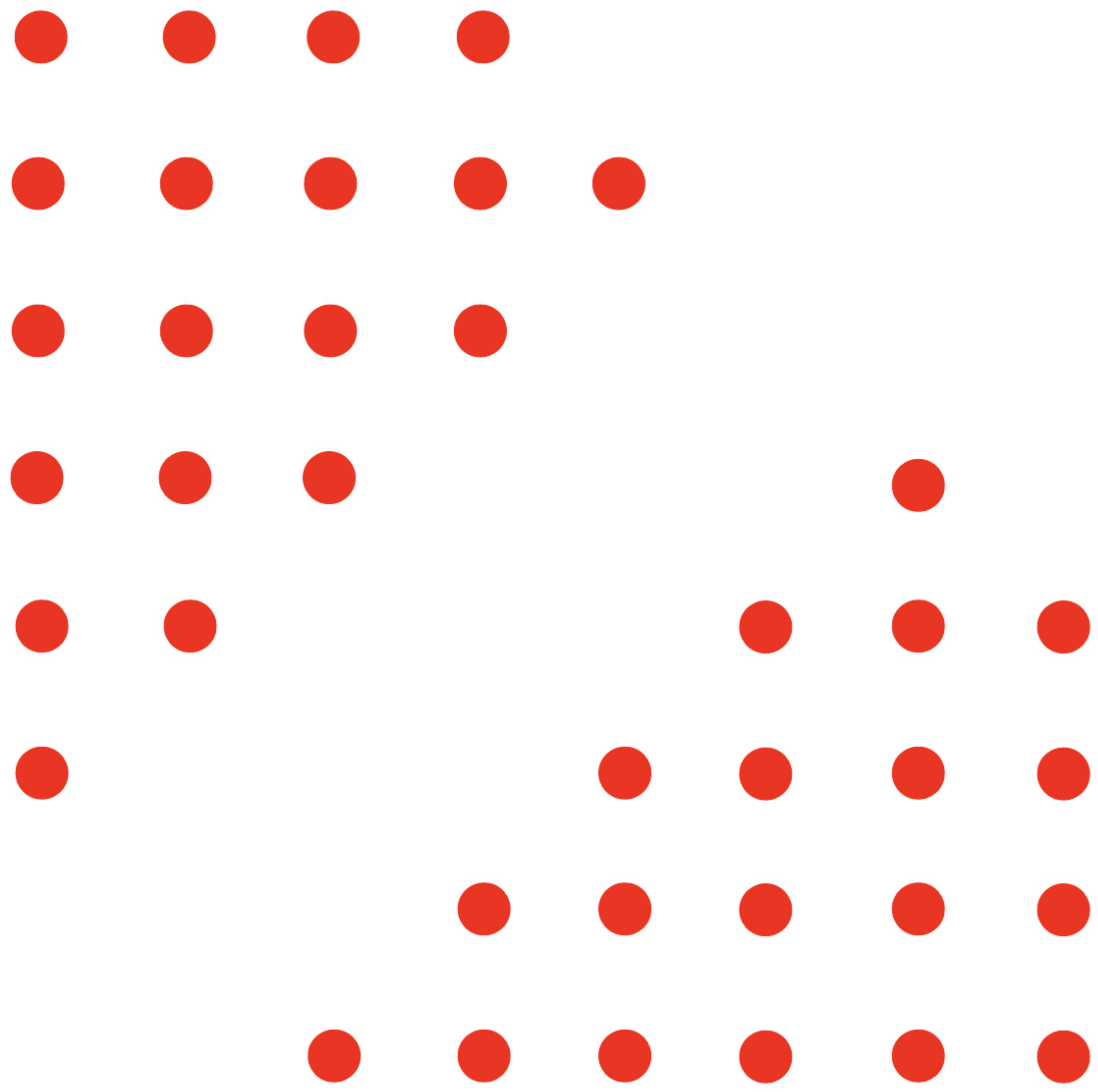


The remaining representations in the continuous series — above the base representations — are all irreducible.

The tempered dual is the disjoint union of the discrete series with the continuous series, as indicated.

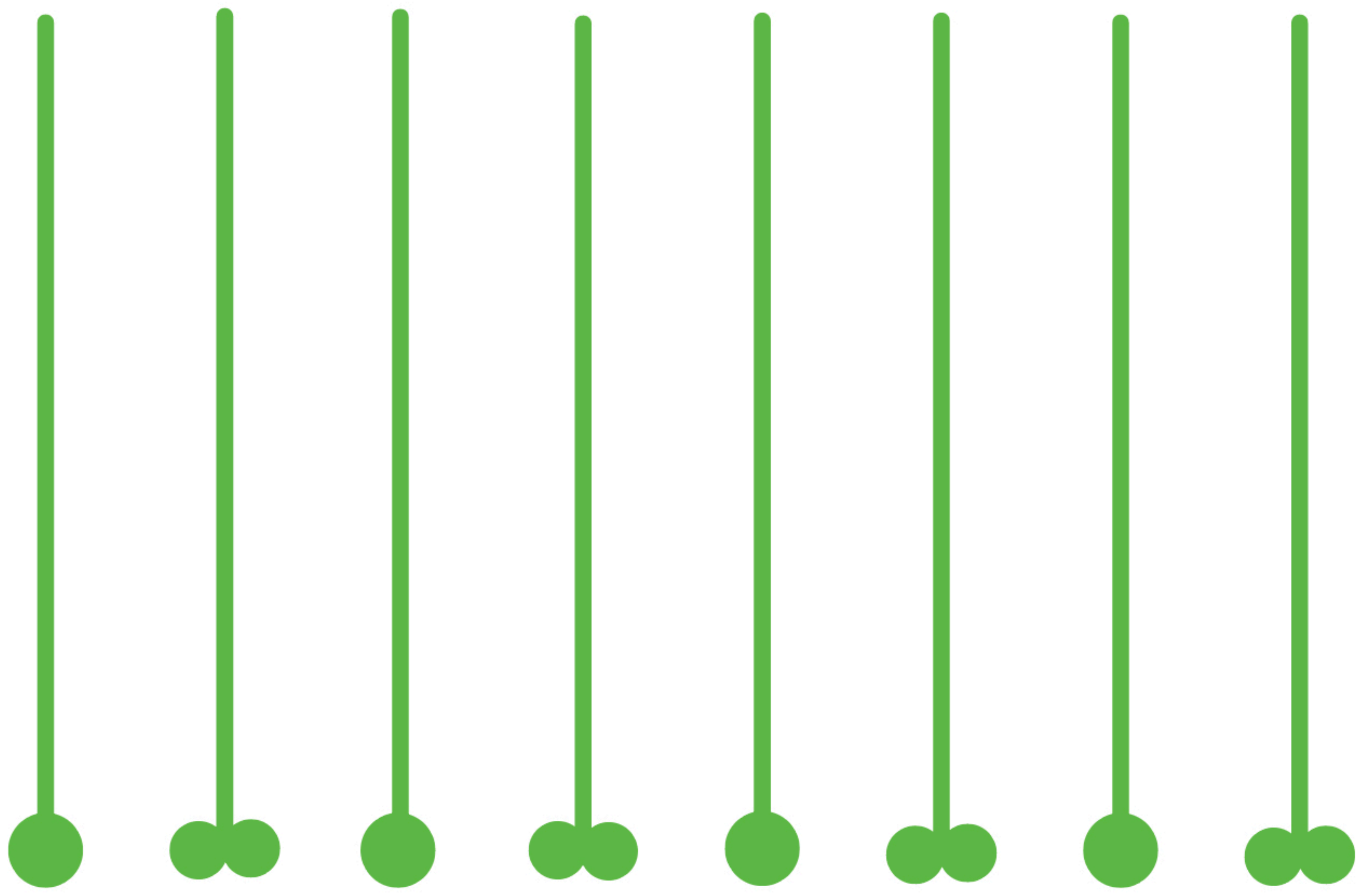


Tempiric Representations and Vogan's Theorem



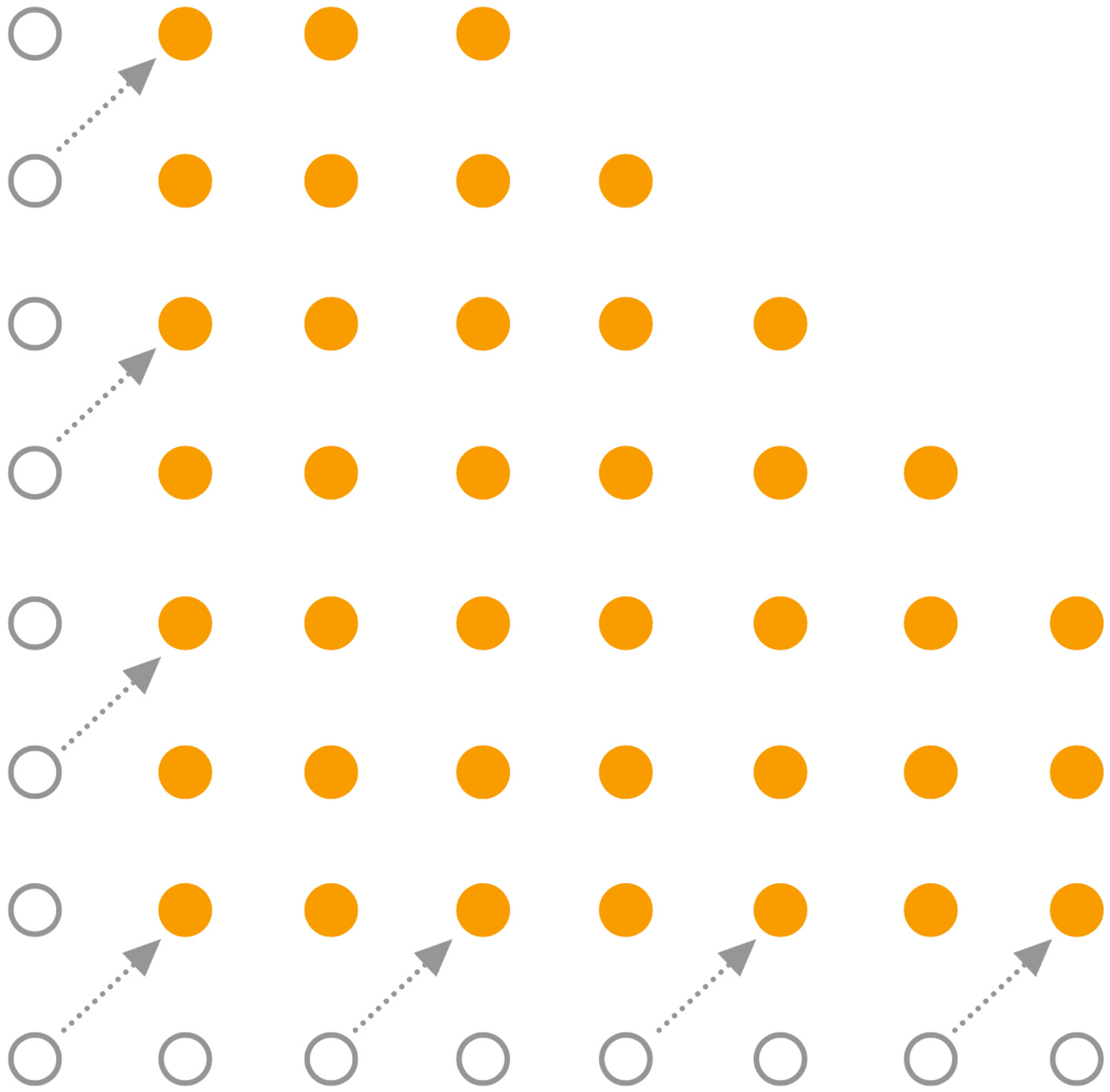
Definition. A representation is *tempiric* [terminology of Alexandre Afgoustidis] if it is tempered, irreducible and has real infinitesimal character.

These are the discrete series and the bases of the continuous series.



Theorem (Vogan). *The irreducible representations of K and the tempiric representations of G are in bijection via minimal K-type, and each minimal K-type has multiplicity one.*

Connes-Kasparov Theory

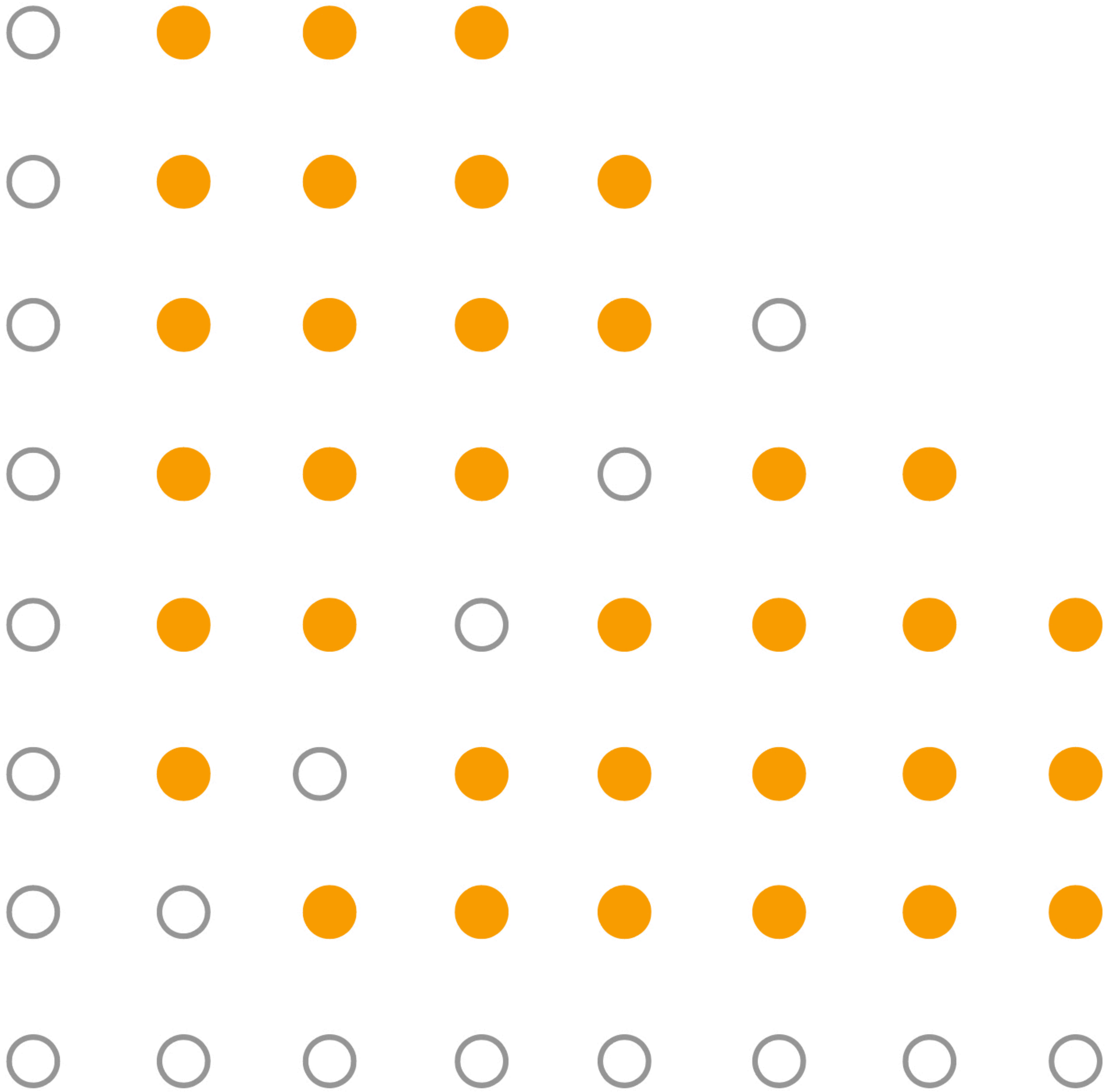


The Connes-Kasparov theory uses a shifted version of the set of irreducible representations of K. These shifted representations correspond to Dirac-type operators on G/K.

The Connes-Kasparov isomorphism is, in effect, a bijection from above the shifted representations of K to the set of (**nearly all of**) the components of the tempered dual of G.

More about this later, but roughly we are talking here about the K-theory (after Atiyah and Hirzebruch) of the tempered dual, considered as a topological space.

Discrete Series from the Connes-Kasparov Point of View

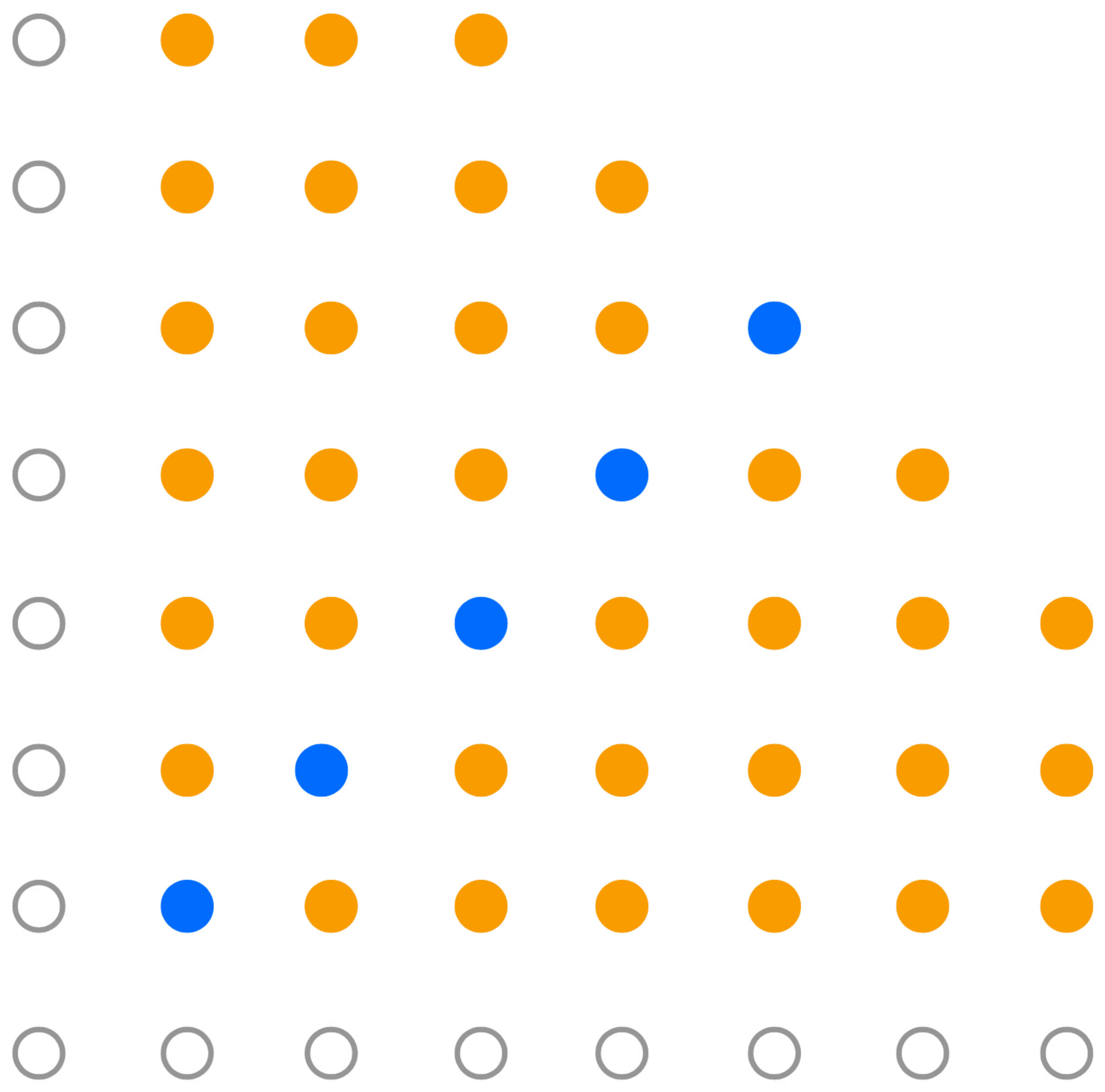


Most of the K-theory generators are in bijection with the discrete series of G (in an equal rank example such as $G = Sp(1,1)$).

(The shifted representation of K that is attached to a given discrete series coincides with the Harish-Chandra parameter of the discrete series representation. This is one of the reasons for making the shift.)

We obtain a picture of the discrete series that is is reminiscent of the minimal K-type picture, but it is not the same.

Essential Continuous Series



The remaining shifted representations are in bijection with **some** of the continuous series components of the tempered dual.

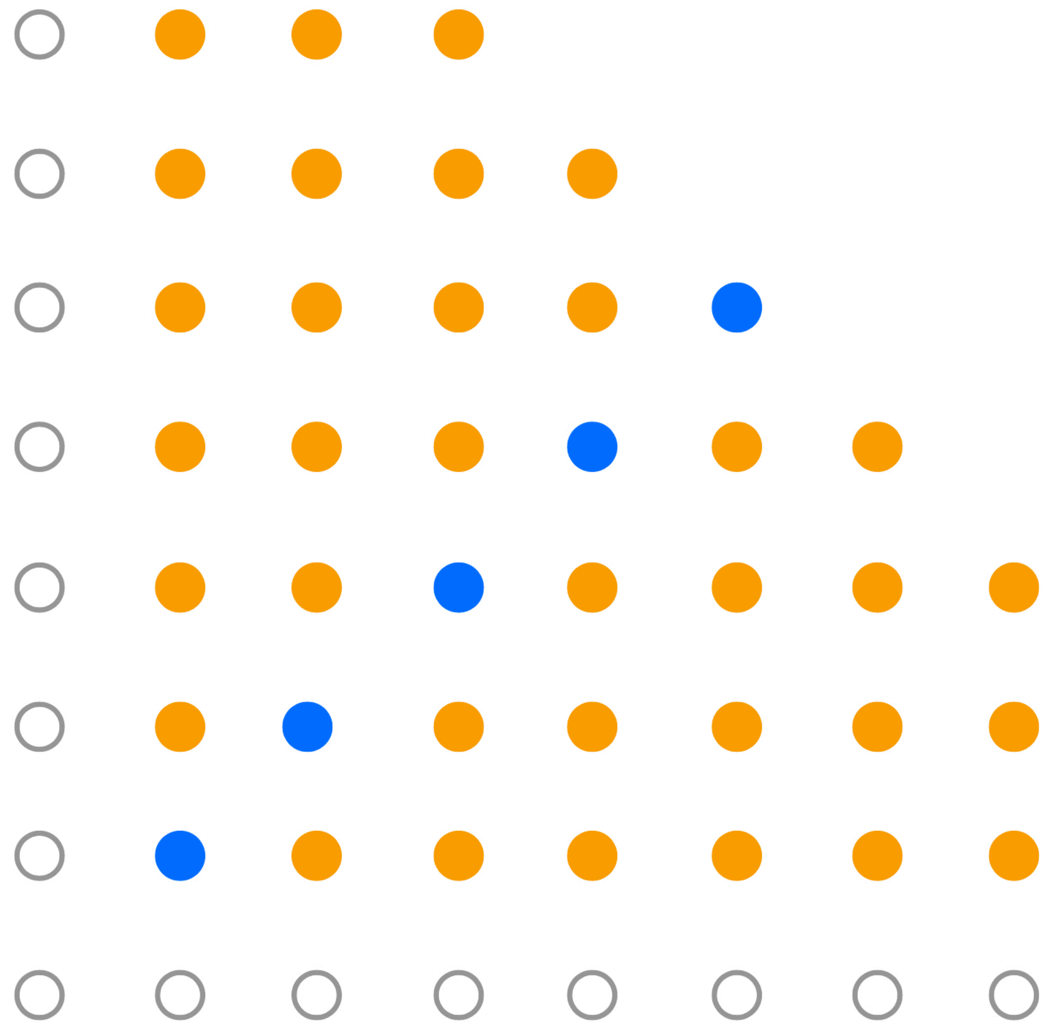
The picture is again reminiscent of Vogan's theory, but not the same.

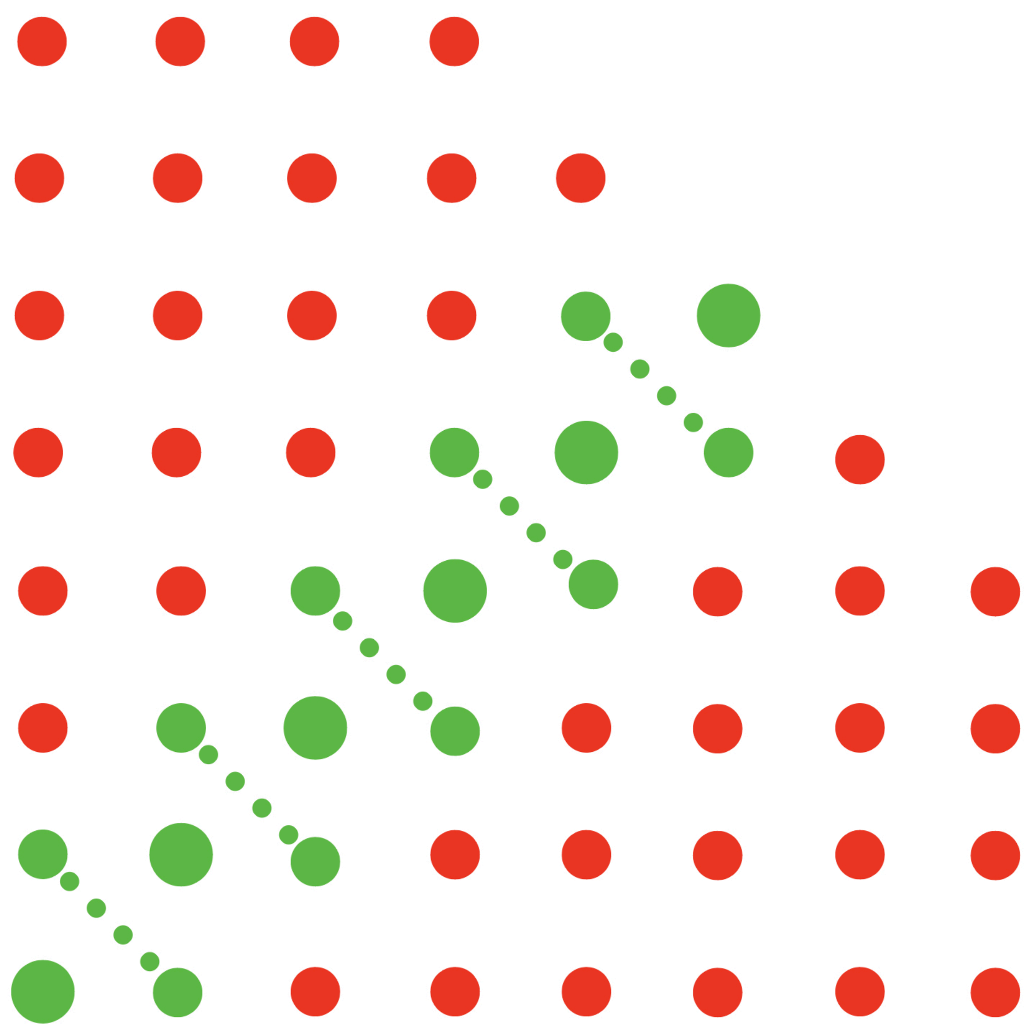
An important differences: not all continuous series contribute to K-theory.

Example. The spherical principal series.

That is, some parts of the tempered representation theory seem to be invisible to K-theory, in contrast with Vogan's theory.

Connes-Kasparov Bijection versus Vogan's Bijection





Our goal is to remedy this, and at the same time to put Vogan's bijection into a K-theoretic context.

This is in response to David Vogan, who has complained to me that in RTNCG we are studying the wrong C*-algebra of G ...

The hardest part of Vogan's theorem is the **existence** of a representation of G with a given minimal K-type. Can K-theory be used for this?

Tempered Representations and the Reduced Group C*-Algebra

G = Real reductive (connected, linear) Lie group (like $Sp(1,1)$ or $SL(n,\mathbb{R})$ or …).

From the C*-algebra point of view, tempered (admissible) unitary representations of G are the same thing as representations of the reduced group C*-algebra (valued in the C*-algebra of compact operators).

$$\pi\colon G \longrightarrow U(H_\pi) \quad \leftrightarrow \quad \pi\colon C_r^*(G) \longrightarrow \mathfrak{K}(H_\pi)$$

What is this good for? The tempered dual is constructed from families of representations

$$\pi_{\delta,\nu}\colon G \to U(H_\delta)$$

With $\{\delta\}$ a countable family of discrete parameters and $\nu \in \mathfrak{a}^*$ continuous parameters (here $\mathfrak{a}_\delta$ is a real vector space), leading to C*-algebra morphisms

$$\pi_\delta\colon C_r^*(G) \longrightarrow C_0(\mathfrak{a}_\delta^*, \mathfrak{K}(H_\delta))$$

Tempered Representations and the Reduced Group C*-Algebra

In fact these families of representations combine into a C*-algebra isomorphism

$$\oplus_\delta \pi_\delta \colon C^*_r(G) \xrightarrow{\cong} \oplus_\delta C_0(\mathfrak{a}^*_\delta, \mathfrak{K}(H_\delta))^{W_\delta}$$

that neatly summarizes work of Harish-Chandra and Langlands.

The W_δ are finite groups acting as intertwining operators, reflecting the facts that not all $\pi_{\delta,\nu}$ are in equivalent, and not all $\pi_{\delta,\nu}$ are irreducible. The full story of these intertwiners is complicated, but Wassermann pointed out that the Knapp-Stein theory of intertwining operators implies

$$K_*\Big(C_0(\mathfrak{a}^*_\delta, \mathfrak{K}(H_\delta))^{W_\delta}\Big) \;=\; 0 \;\text{ or }\; \mathbb{Z}\,.$$

As for the full story (and also the story at the level of K-theory) this may told using the work of Knapp and Zuckerman, and independently the work of Vogan.

K-Theory and Representation Theory

For most components of the tempered dual (that is to say, for most of the discrete parameters δ),

$$K_*\Big(C_0(\mathfrak{a}^*_\delta, \mathfrak{K}(H_\delta))^{W_\delta}\Big) \cong \mathbb{Z}\,.$$

But not for all. For instance the K-theory of the spherical dual is zero.

This is not optimal from the point of view of representation theory. But given the overall similarity between David Vogan's theorem about tempiric representations and the Connes-Kasparov isomorphism, it is natural to ask (as Vogan did) if the Connes-Kasparov theory can be "adjusted" so as to "see" all components of the tempered dual?

More ambitiously, it is natural to ask (as Vogan did not) if the Connes-Kasparov isomorphism can be "adjusted" so as to "include," or be equivalent to, Vogan's theorem?

Smoothing Operators and the Reduced Group C*-Algebra

It will be convenient to reorganize the information included within the reduced group C*-algebra, roughly speaking by breaking it into a collection of matrix parts …

V_1, V_2 = finite-dimensional unitary representations of K

$\mathrm{C}^*(V_1, V_2)$ = norm-closure of the G-equivariant, properly supported smoothing operators $L^2(G/K, V_1) \to L^2(G/K, V_2)$

Lemma. $\mathrm{C}^*(V_1, V_2) \cong \left[C^*_r(G) \otimes \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)\right]^{K\times K}$

These operator spaces constitute the morphisms in a C*-category C^*_G (whose objects are the finite-dimensional unitary representations of K).

Lemma. $K_*(\mathrm{C}^*_G) \cong K_*(C^*_r(G))$

Lemma. $\mathrm{C}^*(V_1, V_2) \cong \oplus_\delta\, C_0\Big(\mathfrak{a}_\delta, \mathfrak{K}\big([H_\delta \otimes V_1]^K, [H_\delta \otimes V_2]^K\big)\Big)^{W_\delta}$

finite direct sum

finite-dimensional

Pseudodifferential Operators on the Symmetric Space

Recall that a pseudodifferential operator (on euclidean space, to begin with) is an operator of the form

$$(Af)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,\xi)\hat{f}(\xi)e^{ix\xi}\, d\xi$$

for an appropriate symbol function $a(x,\xi)$ (including for instance $(1+\xi^2)^{1/2}$, which is an example of an order zero symbol, producing an order zero operator).

We aim to study the following spaces of operators, constituting a new C*-category, P^*_G :

V_1, V_2 = finite-dimensional unitary representations of K

$\mathrm{P}^*(V_1, V_2)$ = norm-closure of the G-equivariant, properly supported order zero pseudodifferential operators $L^2(G/K, V_1) \to L^2(G/K, V_2)$ (all of which are L^2-bounded)

Why Pseudodifferential Operators?

Recently, a new perspective on pseudodifferential operators, involving Alain Connes' tangent groupoid has come into view in noncommutative geometry (work of Claire Debord, Georges Skandalis, Robert Yuncken and Erik van Erp).

The deformations to the normal cone for the inclusion of the basepoint into G/K, and for the inclusion of K into G, are the smooth families of symmetric spaces and Lie groups

$$X_t = \begin{cases} G/K & t \neq 0 \\ \mathfrak{p} & t = 0 \end{cases} \qquad G_t = \begin{cases} G & t \neq 0 \\ K \ltimes \mathfrak{p} & t = 0 \end{cases}$$

Cartan motion group

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition.

Equivariant (classical) pseudodifferential operators on G/K arise naturally from these spaces …

Pseudodifferential Operators and the DNC

Theorem (Debord & Skandalis, van Erp & Yuncken). *Each* (*classical*) *equivariant, properly supported, order zero PSDO A on $G/K = X_1$ extends to a smooth family of equivariant operators*

$$A_t \colon C_c^\infty(X_t, V_1) \longrightarrow C_c^\infty(X_t, V_2) \qquad (t \in \mathbb{R})$$

*which is moreover invariant under rescalings $X_t \to X_{\lambda t}$ $(\lambda > 0)$, modulo smoothing operators. **And vice versa**.*

- The family $\{A_t\}$ is determined by A, modulo smoothing operators.
- The operator at $t = 0$ is a version of the principal symbol of the operator $A = A_1$.

Let's now pass to norm completions, C*-algebras and C*-categories, and K-theory …

K-Theory of Order Zero Pseudodifferential Operators: Technicalities

- The leftwards maps (functors) below are isomorphisms in K-theory for simple reasons.
- The bottom maps (functors) isomorphisms on the nose.
- The top right map (functor) is an isomorphism in K-theory; this is one formulation of the Connes-Kasparov (a.k.a. Baum-Connes) isomorphism.

eval. at t=0

Families over [0,1]

eval. at t=1

$$C^*_{G_0}(V_1, V_2) \longleftarrow C^*_{\mathbb{G}}(V_1, V_2) \longrightarrow C^*_{G_1}(V_1, V_2)$$

$$P^*_{G_0}(V_1, V_2) \longleftarrow P^*_{\mathbb{G}}(V_1, V_2) \longrightarrow P^*_{G_1}(V_1, V_2)$$

$$P^*_{G_0}/C^*_{G_0}(V_1, V_2) \longleftarrow P^*_{\mathbb{G}}/C^*_{\mathbb{G}}(V_1, V_2) \longrightarrow P^*_{G_0}/C^*_{G_1}(V_1, V_2)$$

K-Theory of Order Zero Pseudodifferential Operators: Summary

- Denote by Rep_K the C*-category of finite-dimensional unitary representations of K.
- And remember that P^*_G is the C*-category generated by order zero, equivariant pseudodifferential operators (with proper support), acting between homogeneous vector bundles on G/K.

Theorem. *The obvious functor from* Rep_K *to* P^*_G *is an isomorphism in K-theory*.

Remark. So $K_0(\mathsf{P}^*_G) = R(K)$ and $K_1(\mathsf{P}^*_G) = 0$. This is contingent on the Connes-Kasparov isomorphism, and in fact is equivalent to the Connes-Kasparov isomorphism.

Proof. Contingent on the Connes-Kasparov isomorphism, to prove the theorem for G, it suffices to check it for G_0.

Multiplicities

Denote by Fin the C*-category of finite-dimensional Hilbert spaces.

If π is tempered, admissible unitary representation of G, and if $A \in \mathsf{P}^*_G(V_1, V_2)$, then there is an induced

$$A_\pi \colon [H_\pi \otimes V_1]^K \longrightarrow [H_\pi \otimes V_2]^K$$

and the formulas

$$V \longmapsto [H_\pi \otimes V]^K \quad \text{and} \quad A \mapsto A_\pi$$

define a functor $\mathsf{mult}_\pi \colon \mathsf{P}^*_G \to \mathsf{Fin}$

Lemma. *The composite morphism of abelian groups*

$$R(K) = K_*(\mathsf{Rep}_\mathsf{K}) \longrightarrow \mathsf{K}_0(\mathsf{P}^*_\mathsf{G}) \longrightarrow \mathsf{K}_0(\mathsf{Fin}) \cong \mathbb{Z}$$

takes an irrep. of K to the multiplicity of the dual irrep. in π

Vogan's Tempiric Representations

Theorem. *The composition*

$$R(K) = K_*(\mathrm{Rep}_{\mathrm{K}}) \longrightarrow \mathrm{K}_0(\mathrm{P}^*_{\mathrm{G}}) \longrightarrow \bigoplus_\pi \mathrm{K}_0(\mathrm{Fin}) \cong \bigoplus_\pi \mathbb{Z}$$

obtained from the multiplicities of all tempiric representations is an isomorphism of abelian groups.

Proof. This is Vogan's theorem.

Since the first arrow is an isomorphism — by virtue of Connes-Kasparov — it is obviously of interest to find a direct proof that the second arrow is an isomorphism …

Spectral Picture of Pseudodifferential Operators

On this page assume that the real reductive group G has real rank one.

Form the radial compactification $\overline{\mathfrak{a}_\delta^*}$ of the vector space $\mathfrak{a}_\delta^*$.

Theorem. *In real rank one, the multiplicity construction defines a Fourier transform map (functor)*

$$P_G^*(V_1, V_2) \longrightarrow \oplus_\delta \, C_0\Big(\overline{\mathfrak{a}_\delta^*}, \mathfrak{K}\big([H_\delta \otimes V_1]^K, [H_\delta \otimes V_2]^K\big)\Big)^{W_\delta}$$

extending the Fourier transform isomorphism for smoothing operators, and this is an isomorphism.

Theorem. *In real rank one, the multiplicity functor* $\mathrm{P}_\mathrm{G}^* \to \bigoplus_\pi$ Fin (*direct sum over Vogan's tempiric representations*) *is a homotopy equivalence.*

Proof. Vogan's representations occur precisely at all $0 \in \mathfrak{a}_\delta^*$. Use a homotopy argument.

Thank You!
and
Happy Birthday, Michèle!