A Riemann-Roch formula for singular symplectic reductions

Louis IOOS
joint work with B. Delarue and P. Ramacher

In honour of Michèle Vergne

05/09/2023

Plan

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1. **Quantization commutes with Reduction**

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2. **Description of the Main result**

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- Then the action of G on (M, ω) is **Hamiltonian** : there is a G-equivariant $\mu : M \to \mathfrak{g}^*$, called **moment map**, satisfying

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- $\mu : M \to \mathfrak{g}^*$ is defined by the **Kostant formula**

$$\langle \mu, X \rangle := \frac{i}{2\pi} (\nabla_{\tilde{X}} - L_X) .$$

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The **symplectic reduction** is the smooth manifold $M_0 := \mu^{-1}(0)/G$ endowed with the unique symplectic form ω_0 satisfying

$$\pi_0^* \omega_0 = \omega|_{\mu^{-1}(0)} ,$$

where $\pi_0 : \mu^{-1}(0) \to M_0$ quotient map.

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- (M_0, ω_0) is prequantized by the line bundle L_0 caracterized by

$$\mathcal{C}^\infty(M_0, L_0) \simeq \mathcal{C}^\infty(\mu^{-1}(0), L|_{\mu^{-1}(0)})^G \,,$$

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$$\mathcal{C}^\infty(M_0, L_0) \simeq \mathcal{C}^\infty(\mu^{-1}(0), L|_{\mu^{-1}(0)})^G ,$$

  with induced Hermitian metric and connection.
- If (M, ω) admits a G-invariant compatible complex structure, then L_0 holomorphic line bundle over (M_0, ω_0) Kähler.

Quantization commutes with Reduction

Theorem ([Q,R]=0, Guillemin-Sternberg,'82)

Let G be a compact Lie group acting holomorphically on a holomorphic Hermitian line bundle (L, h^L) prequantizing a compact Kähler manifold (M, ω), and assume that G acts freely on $\mu^{-1}(0)$. Then the natural map

$$\begin{aligned} H^0(M, L)^G &\longrightarrow H^0(M_0, L_0) \\ s &\longmapsto s|_{\mu^{-1}(0)} \,, \end{aligned}$$

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- Teleman, Braverman, Zhang,'00 : For all $j > 0$,

$$\dim H^j(M, L)^G = \dim H^j(M_0, L_0) \,,$$

where $H^j(M, L)$ is the j-th **Dolbeault cohomology group** of L.

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  where $H^j(M, L)$ is the j-th **Dolbeault cohomology group** of L.
- Setting $RR^G(M, L) := \sum_{j=0}^{n}(-1)^j \dim H^j(M, L)^G$, this implies

$$RR^G(M, L) = RR(M_0, L_0)\,.$$

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This last statement extends to the **symplectic case** :

$$RR^G(M,L) = \dim(\mathrm{Ker} D_L^+)^G - \dim(\mathrm{Coker} D_L^+)^G \,,$$

where $D_L^+ : \Omega^{0,+}(M,L) \to \Omega^{0,-}(M,L)$ **spin^c Dirac operator** induced by a G-invariant almost complex structure $J \in \mathrm{End}(TM)$ compatible with ω.

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Assume that 0 is a regular value of μ. Then

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Theorem ([Q,R]=0 in the symplectic case)

Assume that 0 is a regular value of μ. Then

$$RR^G(M,L) = RR(M_0,L_0) .$$

By the **Hirzebruch-Riemann-Roch formula** (HRR), this implies

$$RR^G(M,L) = \int_{M_0} e^{\omega_0}\, \mathrm{Td}(M_0) ,$$

where $[\mathrm{Td}(M_0)] \in H(M_0,\mathbb{R})$ symplectic invariant.

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- For CR-manifolds : Hsiao-Ma-Marinescu,'19.

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- For 0 singular value of μ, the results of Meinrenken-Sjamaar, Zhang and Paradan establish

$$RR^G(M, L) = RR(\widetilde{M}_\varepsilon, \widetilde{L}_\varepsilon)\,,$$

for various desingularizations $(\widetilde{M}_\varepsilon, \widetilde{\omega}_\varepsilon)$ of (M_0, ω_0), depending on the choice of $\varepsilon > 0$.

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Question (Sjamaar,'95)

Can the right-hand side be expressed purely in terms of symplectic invariants of M_0 as a stratified symplectic space ?

Quantization commutes with Reduction

Theorem (Delarue-I.-Ramacher,'23)

Explicit Riemann-Roch type formula for $RR^G(M, L)$ when $G = S^1$ and 0 singular value of μ, expressed purely in terms of symplectic invariants of M_0 as a stratified symplectic space.

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Definition (Cartan)

The **equivariant cohomology** $H_G(M, \mathbb{C}) := H(\Omega_G(M), d_\mathfrak{g})$ of G acting on M is the cohomology of

$$\Omega_G(M) := \Omega(M, \mathbb{C})^G \otimes S(\mathfrak{g}^*)\,,$$

endowed with the differential

$$(d_\mathfrak{g}\alpha)(X) := d\alpha(X) + 2i\pi\ \iota_{\widetilde{X}}\alpha(X)\,.$$

for all $\alpha \in \Omega_G(M)$ and $X \in \mathfrak{g}$ inducing $\widetilde{X} \in \mathcal{C}^\infty(M, TM)$.

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Proposition

If G acts freely M, then $H_G(M) \simeq H(M/G)$.

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Definition

For G acting freely on $\mu^{-1}(0)$, the **Kirwan map** $\kappa : H_G(M) \to H(M_0, \mathbb{C})$ is given by

$$\kappa : H_G(M) \xrightarrow{\text{inc}^*} H_G(\mu^{-1}(0), \mathbb{C}) \simeq H(M_0, \mathbb{C}) \,.$$

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Proposition

The Kirwan map is characterized for all $\alpha \in \Omega_G(M)$ and $\beta \in \Omega(M_0, \mathbb{C})$ by

$$\int_{M_0} \beta \wedge \kappa(\alpha) = \int_{\mu^{-1}(0)} \pi_0^* \beta \wedge \alpha\Big(\frac{i}{2\pi} d\theta\Big) \wedge \theta \,,$$

where $\theta \in \Omega^1(\mu^{-1}(0), \mathbb{R})$ is a **connection** over the S^1-principal bundle $\pi_0 : \mu^{-1}(0) \xrightarrow{S^1} M_0$, so that $\theta(\tilde{X}) = x$ for all $X \in \mathfrak{g}$ identified with $x \in \mathbb{R}$.

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Definition (Berline-Vergne)

E complex vector bundle over M with G-invariant Hermitian connection ∇^E, the **equivariant curvature** is

$$R^E_{\mathfrak{g}} := R^E + 2i\pi\,\mu^E \in \Omega^\bullet(M, \mathrm{End}(E))^G \otimes S(\mathfrak{g}^*)\,,$$

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Let $E = (TM, J)$ be equipped with the Chern connection ∇^{TM}.

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Let $E = (TM, J)$ be equipped with the Chern connection ∇^{TM}.

Proposition (Chern-Weil theory, Berline-Vergne)

The equivariant forms $c_{1,\mathfrak{g}}(L) := \omega + 2i\pi\,\mu \in \Omega_G(M)$ and

$$\mathrm{Td}_{\mathfrak{g}}(M) := \det\left(\frac{R^{TM}_{\mathfrak{g}}/2i\pi}{\exp R^{TM}_{\mathfrak{g}}/2i\pi - \mathrm{Id}}\right) \in \Omega_G(M)$$

are $d_{\mathfrak{g}}$-closed and their classes in $H_G(M)$ are independent of $J \in \mathrm{End}(TM)$.

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$$RR^G(M, L) = \int_G \chi(g)\, dg\,.$$

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Theorem (equivariant index formula, Atiyah-Bott-Segal-Singer,'68)

For all $g \in G$, writing $M^g := \{x \in M \mid g.x = x\}$, we have

$$\chi(g) = \int_{M^g} \mathrm{Tr}[g^{-1}|_L] \frac{e^\omega\, \mathrm{Td}(M^g)}{D^g(M/M^g)}\, ,$$

where $D^g(M/M^g) = \det_{N^g}(\mathrm{Id} - g\exp(R^{N^g})) \in \Omega^\bullet(M, \mathbb{C})$ with $N^g := TM/TM^g$ normal bundle of $M^g \subset M$.

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- For $g = e^X$ with $X \in \mathfrak{g}$, we get $\mathrm{Tr}[g^{-1}|_L] = e^{2i\pi\langle \mu, X\rangle}$.

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Theorem (Kirillov formula, Berline-Vergne,'82)

For all $X \in \mathfrak{g}$ small enough, we have

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Theorem (Meinrenken,'96)

If $G = S^1$ acts freely on $\mu^{-1}(0)$, then

$$RR^G(M,L) = \int_{M_0} e^{\omega_0} \kappa(\operatorname{Td}_{\mathfrak{g}}(M)) = RR(M_0,L_0)\,.$$

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Theorem (Duistermaat-Guillemin-Meinrenken-Wu,'96)

If 0 is a minimum/maximum of the moment map, then

$$RR^G(M,L) = \operatorname{Res}_{z=0/\infty} \frac{\int_{M_0} z^{-1} e^{\omega_0} \operatorname{Td}(M_0)}{D^z(M_0/M)} = RR(M_0,L_0)\,.$$

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Proposition (local normal form, Guillemin-Sternberg,'84)

There exists a chart $U \subset \mathbb{C}^n$ around $F \subset M$ such that for all $v \in U$,

$$\langle \mu(v), X \rangle = x \sum_{k \in \mathbb{Z}} k \, |\pi_k(v)|^2$$

for all $X \in \mathfrak{g}$ sent to $x \in \mathbb{R}$ via $G \simeq \mathbb{R}/\mathbb{Z}$, and where for any $k \in \mathbb{Z}$,

$$\pi_k : \mathbb{C}^n \to \left\{ v \in \mathbb{C}^n \mid e^X.v = e^{2i\pi kx} v \right\}.$$

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- In particular,

$$(\mu^{-1}(0) \cap U) \backslash F \simeq S^+ \times S^- \times]0, \varepsilon[\, ,$$

where $S^{\pm}$ ellipsoids inside the subspaces of $\pm$ weights inside $\mathbb{C}^n$.

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Theorem (Delarue-I.-Ramacher,'23)

Assume 0 singular value of μ and G acts on $\mu^{-1}(0)\backslash F$ freely. Then

$$RR^G(M,L) = \int_{M_0} e^{\omega_0}\kappa(\mathrm{Td}_{\mathfrak{g}}(M)) + \int_{\mathrm{Exc}} e^{\pi^*\omega}\kappa_{\mathrm{Exc}}(\mathrm{Td}_{\mathfrak{g}}(M)) + \mathrm{Res}_{z=0,\infty} \frac{\int_F z^{-1} e^{\omega}\,\mathrm{Td}(F)}{D^z(M/F)} .$$

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- Under a natural condition on the weights of the S^1-action around F, $\kappa : H_G(M) \to H(\widetilde{M}_0, \mathbb{C})$ with $\pi : \widetilde{M}_0 \to M_0$ partial resolution of the singularities.

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- $\mathrm{Res}_{z=0,\infty}$ is the average of the residues at 0 and ∞.

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Theorem (Delarue-I.-Ramacher,'23)

Assume 0 singular value of μ and no orbifold points. Then

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- The exceptional divisor $\pi|_{\mathrm{Exc}} : \mathrm{Exc} \to F$ of the resolution is an $S^+/S^1 \times S^-/S^1$-principal bundle.

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- $\kappa_{\mathrm{Exc}} : H_G(M) \to H(\mathrm{Exc},\mathbb{C})$ is defined for all $\alpha \in \Omega_G(M)$ by

$$\kappa_{\mathrm{Exc}}(\alpha) = \frac{\frac{1}{2}(\alpha(\frac{i}{2\pi}d\theta^+) + \alpha(\frac{i}{2\pi}d\theta^-)) - \alpha(\frac{i}{2\pi}\frac{d\theta^+ + d\theta^-}{2})}{d\theta^+ - d\theta^-}$$

where $\theta^\pm \in \Omega(S^\pm,\mathbb{R})$ connections for the S^1-actions on $S^\pm$.

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- To simplify : $G = S^1$ acts freely on $M \backslash M^G$.

$$RR^G(M, L^m) = \int_G \chi^{(m)}(g)\, dg$$

$$= \int_G \chi^{(m)}(g)\phi(g)\, dg + \int_G \chi^{(m)}(g)(1 - \phi(g))\, dg$$

$$= \int_{\mathfrak{g}} \int_M e^{2i\pi m\langle\mu, X\rangle} e^{m\omega}\, \mathrm{Td}_{\mathfrak{g}}(M, X)\phi(e^X)\, dX$$

$$+ \int_G \int_{M^G} \mathrm{Tr}[g^{-1}|_{L^m}] \frac{e^{m\omega}\, \mathrm{Td}(M^G)}{D^g(M/M^G)}(1 - \phi(g))\, dg\,.$$

by the Kirillov and equivariant index formulas.

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- For any neighborhood $U \subset M$ of $\mu^{-1}(0)$, we have

$$\int_{\mathfrak{g}}\int_M e^{2i\pi m\langle\mu,X\rangle}e^{m\omega}\,\mathrm{Td}_{\mathfrak{g}}(M,X)\phi(e^X)\,dX$$
$$= \int_{\mathfrak{g}}\int_U e^{2i\pi m\langle\mu,X\rangle}e^{m\omega}\,\mathrm{Td}_{\mathfrak{g}}(M,X)\phi(e^X)\,dX + O(m^{-\infty})\,.$$

Elements of proof

- Duistermaat-Heckman,'82 : For $U = \mu^{-1}(I)$ with $0 \in I \subset \mathbb{R}$ small enough, there is a connection $\theta \in \Omega^1(\mu^{-1}(0), \mathbb{R})$ such that, in a trivialization $U \simeq \mu^{-1}(0) \times I$ with $q \in I$, we have

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- We get as $m \to +\infty$,

$$\int_{\mathfrak{g}} \int_M e^{2i\pi m\langle \mu, X\rangle} e^{m\omega} \operatorname{Td}_{\mathfrak{g}}(M, X)\phi(X)\, dX$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mu^{-1}(0)} e^{2i\pi mxr} e^{m(\omega + d(q\theta))} \operatorname{Td}_{\mathfrak{g}}(M, x)\phi(x)\phi(q)\, dx\, dq$$

$$+ O(m^{-\infty})\,.$$

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- **stationary phase lemma** : for all $\psi, \rho \in \mathcal{C}_c^\infty(\mathbb{R})$,

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- Taking $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\phi \equiv 1$ around 0, we get as $m \to +\infty$,

$$\begin{aligned} & RR^G(M, L^m) \\ & = m \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mu^{-1}(0)} e^{2i\pi mxq} e^{m(\omega + q d\theta)} \operatorname{Td}_{\mathfrak{g}}(M, x) \wedge \theta\, \phi(x)\phi(q)\, dx\, dq \\ & \qquad + O(m^{-\infty}) \\ & = \int_{\mu^{-1}(0)} e^{m\omega} \operatorname{Td}_{\mathfrak{g}}\left(M, \frac{i}{2\pi} d\theta\right) \wedge \theta + O(m^{-\infty}) \\ & = \int_{M_0} e^{m\omega_0} \kappa(\operatorname{Td}_{\mathfrak{g}}(M)) + O(m^{-\infty}) \\ & = RR(M_0, L_0^m) + O(m^{-\infty}),\ \text{since } \kappa(\operatorname{Td}_{\mathfrak{g}}(M)) = \operatorname{Td}(M_0). \end{aligned}$$

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There exists $k \in \mathbb{N}$ such that for all $0 \leqslant j \leqslant k-1$, the functions $m \mapsto RR(M, L^{km-j})$ are polynomials in $m \in \mathbb{N}$.

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- Uses the equivariant index formula for $\chi^{(m)}$ and a result of Erhart,'77 on the polynomiality of the number of integer points inside polytopes.
- Then $RR^G(M, L^m) = RR(M_0, L_0^m) + O(m^{-\infty})$ implies $RR^G(M, L^m) = RR(M_0, L_0^m)$ for all $m \in \mathbb{N}$, and setting $m = 1$, we get

$$RR^G(M, L) = RR(M_0, L_0)\,.$$

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will contribute to the residue term of the Main result.
- Delarue-I.-Ramacher,'23 : Compute the asymptotics as $m \to +\infty$ of

$$\int_{\mathfrak{g}} \int_M e^{2i\pi m\langle \mu, X\rangle} e^{m\omega}\,\mathrm{Td}_{\mathfrak{g}}(M,X)\phi(e^X)\,dX$$
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using explicit local coordinates for $U \subset M$ around F.

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- To simplify : F reduced to one point. We use the coordinates

$$\Psi : S^+ \times S^- \times]0, \varepsilon[\times \mathbb{R} \to U \subset \mathbb{C}^n$$

$$(w^+, w^-, r, q) \mapsto \left(\sqrt{\sqrt{r^4 + q^2} + q}\, w^+, \sqrt{\sqrt{r^4 + q^2} - q}\, w^-\right)$$

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- In these coordinates, the symplectic form becomes

$$\omega = \omega|_{\mu^{-1}(0)} + d(q\theta + (\sqrt{r^4+q^2} - r^2)\overline{\theta}),$$

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- The integral picks up a boundary term on $S^+ \times S^- \times \{0\}$ due to Stokes, leading to the two last terms of the Main result.
- As $\sqrt{r^4 + q^2} - r^2 \xrightarrow{r \to 0} |q|$, the amplitudes of oscillating integrals contain a factor of $|q|$, leading to **Cauchy principal values**.

Elements of proof

- We get an explicit formula of the form

$$\int_{\mathfrak{g}} \int_U e^{2i\pi m\langle \mu, X\rangle} e^{m\omega}\, \mathrm{Td}_{\mathfrak{g}}(M, X)\phi(e^X)\, dX$$
$$= \langle \delta\text{-term}, \phi\rangle + \langle \text{p.v.-term}, \phi\rangle + O(m^{-\infty})$$

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- In particular, if $e \notin \mathrm{Supp}\,\phi$, then

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- To conclude, we use Meinrenken,'96 on the polynomial behavior of $RR^G(M,L^m)$ in $m\in\mathbb{N}$, compared to our polynomial formula ■

The End

Thank you !