Einstein-Bogomol'nyi equation and Gravitating Vortex equations on Riemann surfaces

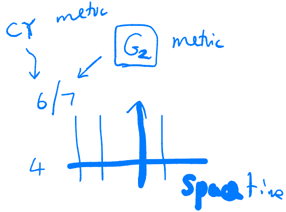
Chengjian Yao

@Institute of Mathematical Sciences, ShanghaiTech University

Talk at Chern Institute of Mathematics, April 8, 2021

1. I. Motivation and Background
   - moment map interpretation
   - GV equations
   - EB equation
2. II. Gravitating Vortex equations
3. III. Main Theorems

I. Motivations and Backgrounds



- Kähler-Yang-Mills theory;
  Dimension reduction solution: Gravitating vortices
- Einstein-Maxwell-Higgs theory.
  String-like solution: Einstein-Bogomol'nyi equation

static

$$\mathrm{Ric}\, g - \tfrac{1}{2} R g = \Lambda g + T \to T_{\mu\nu}$$

- $T = 0.$ $\Rightarrow$ $\mathrm{Ric}\, g = 0.$
- $T \neq 0.$ $U(1)$, $SU(2)$, $SU(3)$.

$A.$

Kähler-Yang-Mills Equations

Introduced by Álvarez-Cónsul, Garcia-Fernandez and García-Prada

Let $E \to X$ be a holomorphic vector bundle over a Kählerian manifold, try to find (H, ω) where H is a Hermitian metric on E and ω a Kähler metric on X, s.t.

$$\begin{cases} \Lambda_\omega F_H &= z \\ \alpha_0 S_\omega + \alpha_1 \Lambda_\omega^2 (F_H \wedge F_H) &= c. \end{cases}$$

$\boxed{\text{HYM}}$ Donaldson, Uhlenbeck Yau.

where F_H is the curvature of the Chern connection for H, and S_ω is the scalar curvature of ω.

Moment map interpretation

There are two well-known problem::

A. Hermitian-Yang-Mills connection is zero of a moment map for $\mathcal{G} \curvearrowright (\mathcal{A}^{1,1}, \omega_{\mathcal{A}})$ by [Atiyah-Bott, Donaldson];

B. Constant scalar curvature Kähler metric is zero of a moment map for $\mathcal{H} \curvearrowright (\mathcal{J}^{int}, \omega_{\mathcal{J}})$ by [Fujiki, Donaldson]; "YTD"

$\mathcal{G}^c$

$\mathcal{H}^c$

Coupling together these two, Álvarez-Cónsul, Garcia-Fernandez and Garcia-Prada studied $\widetilde{\mathcal{G}} \curvearrowright (\mathcal{P}, \omega_\alpha)$, where

- $\mathcal{P} = \{(J, A) \in \mathcal{J}^{int} \times \mathcal{A} | F_A^{0,2} = 0\}$;
- $\omega_\alpha = \alpha_0 \omega_{\mathcal{J}} + \alpha_1 \omega_{\mathcal{A}}$;
- $\widetilde{\mathcal{G}}$ is the extended gauge group,

full gauge group.

$\widetilde{\mathcal{G}}^c$

$$1 \to \mathcal{G} \to \widetilde{\mathcal{G}} \to \mathcal{H} \to 1.$$

The zero moment map equation of this action is KYM equation.

Gravitating Vortex equations

X

Assume the rank 2 holomorphic vector bundle $E \to \Sigma \times \mathbb{P}^1$ comes from extension of holomorphic line bundles, i.e. asssume E is an extension of L on Σ and $\mathcal{O}_{\mathbb{P}^1}(2)$ determined by $\phi \in H^0(\Sigma, L)$:

$$0 \to p_1^* L \to E \to p_2^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0,$$

$\phi = 0 \Rightarrow E = L \oplus \mathcal{O}(2).$

then the $SU(2)$-invariant KYM solution is equivalent to **Gravitating Vortex equations**, which therefore also has a moment map interpretation too [AC-GF-GP, 17'].

Here,

Σ (ω, ϕ).

$$\omega_\tau = p_1^* \omega + \frac{4}{\tau} \omega_{FS}.$$

Einstein-Maxwell-Higgs Model

Physical meaning: describe how gravity and electro-magnetic field interacts

Let M be a 4-manifold, $L \to M$ a Hermitian line bundle. Consider the action

$$S(g, A, \phi) = \int_M \left(\frac{R_g}{16\pi G} + \mathcal{L} \right) \mathrm{dvol}_g ,$$

Hilbert-Einstein

where

- g is Lorentzian of signature $(-,+,+,+)$ on M, A is a unitary connection on L, and ϕ is a section of L;
- $$\mathcal{L} = \frac{1}{4}|F_A|^2 + \frac{1}{2}|D_A\phi|^2 + \frac{1}{8}\left(|\phi|^2 - \tau\right)^2 .$$

Potential energy.

Euler-Lagrange equation: **Einstein-Maxwell-Higgs equations.**

Bogomol'nyi reduction

ASD

$M = \mathbb{R}^{1,1} \times \Sigma$, L, A, ϕ are pulled back from Σ, and $g = -\mathrm{d}t^2 + \mathrm{d}z^2 + g_\Sigma$. The EMH equations are equivalent [Comtet-Gibbons, Linet, Yang] to a system of Bogomol'nyi self-dual equations. It admits vortex like solutions. (Physically known as cosmic strings, gives a potential explanation of galaxy formation in the early universe proposed by physicst T. Kibble.)

cosmic string

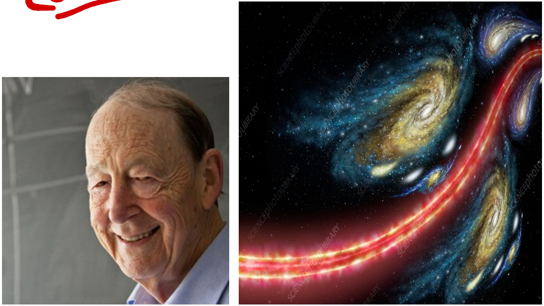


Figure: T. Kibble & Cosmic strings, Photo source: website

Einstein-Bogomol'nyi equation

The mathematical treatment started from Y. Yang in 1990's. It becomes the following PDEs for (g, u):

Surface

$$\begin{cases} \Delta_g u = (e^u - \tau) + 4\pi \sum_{j=1}^{d} n_j \delta_{p_j}, \\ K_g = -\alpha \left[\tau(e^u - \tau) - \Delta_g e^u \right], \end{cases} \tag{1}$$

and could be combined to one single semilinear PDE (with $\alpha = \frac{1}{\tau N}$) with an undetermined parameter (see below). $\Rightarrow$ 1 equation

- α is the coupling constant;
- τ is the symmetry-breaking scale;
- p_j's are the location of the strings, and n_j are local string numbers.

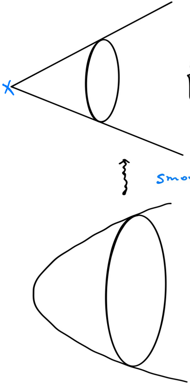
Some model solutions

A singular solution, and its smoothing obtained by [Chen-Hastings-Mcleod-Yang, 94]:

Conical metric

$$\begin{cases} g = dr^2 + \beta^2 r^2 d\theta^2 \\ h = \dfrac{1}{|z^n|^2}, \quad z \in \mathbb{C} \end{cases}$$

Singular Hermitian metric



smoothing

$(g_\varepsilon, h_\varepsilon)$ solution by [CHMY, 94']

... asymptotically conical

... rotationally symmetric.

EB equation recast as GV equations

The above EB equation fits into the following more general PDE system:

(L, h)
$\downarrow$
(Σ, ω)

vortex

$$\begin{cases} iF_h + \frac{1}{2}(|\phi|_h^2 - \tau)\omega = 0, \\ S_\omega + \alpha(\Delta_\omega + \tau)(|\phi|_h^2 - \tau) = c_\alpha, \end{cases} \tag{2}$$

called the GV equations (when $c_\alpha = 0$, we recover $\boxed{\text{EB equation}}$).

The unknowns are (ω, h) where ω is a Kähler metric on Σ and h is a Hermitian metric on the holomorphic line bundle L, ϕ is a holomorphic section (called the Higgs field).

KEY DIFFERENCE: EB is one PDE while GV is a system of two PDEs for $c_\alpha \neq 0$!

Vortex Equation

The first equation in the above system is called the **Vortex equation**, studied throughly by [Jaffe-Taubes, Witten, Noguchi, Bradlow, Garcia-Prada,....].

On $\mathbb{C}$, the unique solution obtained by Li [JGA. 2018]

$$\Delta w = e^{w} - |\phi|^2 e^{-(k-1)w}$$

gives Vortex solution $h = e^{-kw}$ for the data $((\mathbb{C}, \sigma = e^{w}|dz|^2), L = K_{\mathbb{C}}^{k}, \phi = \phi(dz)^{\otimes k})$.

It is proved on compact Riemann surface Σ, for any given Kähler metric ω with $\boxed{\mathrm{Vol}_{\omega} > \frac{4\pi c_1(L)}{\tau}}$, there exists a unique h solving it for any $\phi \in H^0(\Sigma, L)$.

fixed

$=$

$\Rightarrow \phi = 0$

The solvability does not depend on ϕ. (In Garcia-Prada's proof of relating this vortex equation to Hermitian-Yang-Mills connections, the boxed numerical condition is the *slope stability condition*

II. Gravitating Vortex equations

Previous known results: About $c_\alpha \geqslant 0$

The constant $c_\alpha = \frac{2\pi(\chi(\Sigma) - 2\alpha\tau N)}{\mathrm{Vol}_\omega}$, is topologically determined.
And, $c_\alpha \geqslant 0$ implies $\Sigma = \mathbb{P}^1$.

Theorem (Yang)

1. *(97') Let ϕ be strictly polystable. Then, $\forall\ V > \frac{4\pi c_1(L)}{\tau}$, $\exists$ a solution (ω, h) to the EB equation with $Vol_\omega = V$.*
2. *(95') Let ϕ be stable. Then, $\forall\ V > \frac{4\pi N}{\tau}$, $\exists$ a solution (ω, h) to the EB equation satisfying $Vol_\omega > V$.*

A converse was proved recently.

Theorem (AC-GF-GP-P, 20')

The existence of solution to GV equations with $c_\alpha \geqslant 0$ implies ϕ is polystable.

Stability of binary quantics

The holomorphic section ϕ is homogeneous polynomial of degree N in variable z_0, z_1. It is one of the main themes of classical invariant theory, in 19th century.

It is also the most basic example of Mumford's Geometric Invariant Theory. Let $(\phi = 0) = \sum_{j=1}^{d} n_j p_j \in \mathrm{Sym}^N(\mathbb{P}^1)$,

- ϕ is strictly polystable if $d = 2$ and $n_1 = n_2 = \frac{N}{2}$;
- ϕ is stable if $n_j < \frac{N}{2}$ for $j = 1, 2, \cdots, d$.

$SL(2, \mathbb{C}) \curvearrowright S^N(\mathbb{P}^1)$

$S^N(\mathbb{P}^1) / SL(2, \mathbb{C})$?

Questions left

The stability condition was a technical assumption in Yang's proof, its PDE is

$$\Delta f_\lambda = \frac{1}{2\lambda}(\tau - |\phi|^2 e^{2f_\lambda})e^{4\alpha\tau f_\lambda - 2\alpha|\phi|^2 e^{2f_\lambda}} - N,$$

where the assumption on the multiplicities of zeros of ϕ enables one to construct super/subsolutions (cf. also [Han-Sohn, 19']).

Using the moment map picture, [AC-GF-GP-P, 20'] showed the necessity! Some *questions* are left:

a) Existence of solution to EB equation for arbitrary admissible volume $V \in (\frac{4\pi N}{\tau}, +\infty)$;

b) Existence of solutions to GV equations for $\alpha \in (0, \frac{1}{\tau N}]$;

$C_\alpha = 0$

$C_\alpha > 0$

c) Uniqueness of solutions.

III. Main Theorems

The first result strengthens Yang's existence theorem, confirming question *a*) above:

Theorem (Garcia-Fernandez, Pingali & Y., 21')

Let ϕ be polystable. Then $\forall$ $V > \frac{4\pi N}{\tau}$, $\exists$ a solution to the EB equation with $Vol_\omega = V$.

Then, we prove a similar existence result for $c_\alpha > 0$, answering *b*):

Theorem (ibid.)

Let ϕ be polystable, $\alpha \in (0, \frac{1}{\tau N}]$. Then, $\forall$ $V > \frac{4\pi N}{\tau}$, $\exists$ a solution (ω, h) to the GV equations with $Vol_\omega = V$.

Outline of proof

The two theorems are proved via the same strategy, only two crucial differences in a priori estimates.

Step1 (set up the continuity method): Starting from one of Yang's solution (ω_0, h_0) for EB equation. Then $(\widetilde{\omega}_0, \widetilde{h}_0) = \left(\frac{2\pi}{\mathrm{Vol}_{\omega_0}}\omega_0, h_0\right)$ is a solution to the following rescaled system with parameter $\varepsilon = \varepsilon_0 = \frac{2\pi}{\mathrm{Vol}_{\omega_0}}$:

Volume

$$
\begin{cases}
iF_{\widetilde{h}} + \frac{1}{2\varepsilon}(|\phi|^2_{\widetilde{h}} - \tau)\widetilde{\omega} = 0, \\
S_{\widetilde{\omega}} + \alpha(\Delta_{\widetilde{\omega}} + \frac{\tau}{\varepsilon})(|\phi|^2_{\widetilde{h}} - \tau) = 0.
\end{cases}
\tag{3}
$$

$\mathrm{Vol} > \frac{4\pi N}{\tau}$

Solve the system for $(\widetilde{\omega}, \widetilde{h}) \in (\mathcal{H}_{\omega_{FS}}, \mathcal{H}_L)$ and $\varepsilon \in (0, \frac{\tau}{2N})$.

Step2 (openness): Kernal of the linearized operator $\mathfrak{L} : (C^\infty/\mathbb{R}) \times C^\infty \to C^\infty \times C^\infty$ corresponds to

$$\mathrm{Aut}(\mathbb{P}^1, L, \phi)$$

which is $\mathbb{C}^* \times \mathbb{C}^*$ in case ϕ is strictly polystable, and is $\{1\}$ in case ϕ is stable.

$\mathbb{P}^1$

Step 3 (closedness): Let $\Phi = |\phi|_h^2$ and $k = e^{2\alpha\Phi}g$ be conformally rescaled metric, then there holds a priori estimates:

- $0 \leqslant \Phi \leqslant \tau$,
- $c_\alpha e^{-2\alpha\tau} \leqslant S_k \leqslant c_\alpha + \alpha\tau^2$, notice the following formula for a solution (ω, h):

$$S_g = 2\alpha|\mathrm{d}_A\phi|^2 + \alpha(\tau - |\phi|_h^2)^2,$$

- $|\nabla_k S_k|_k^2 \leqslant \frac{3}{2}\alpha\tau^2\left(2c_\alpha + 2\alpha\tau^2 + \tau\right)^2$,
- $\mathrm{Diam}(g) \leq C$ for uniform $C > 0$ (the proofs about EB equation and GV equations diverge at this point).

ODE

For a family of solutions (g_n, h_n), we can get a subsequential $C^{2,\beta}$ Cheeger-Gromov limit, i.e. $\exists$ diffeomorphism $\varphi_n : S^2 \to S^2$ s.t. $\varphi_n^* k_n \longrightarrow k_\infty$, in $C^{2,\beta}$ sense.

inj. radius

N odd

5.

Using uniqueness of almost complex structure on S^2 (i.e. any two $C^{2,\beta}$ almost complex structure on S^2 are related by an $C^{3,\beta}$ diffeomorphism) to improve the sequence φ_n to **holomorphic automorphism** $\sigma_n \in PSL(2,\mathbb{C})$, and

State function

$$k'_n = \sigma_n^* k_n \longrightarrow k'_\infty, \text{ in } C^{2,\beta} \text{ sense.}$$

- $\|\log \Phi'_n - 4\pi G'_n\|_{C^0(\mathbb{P}^1)} \leqslant C$, where $\Phi'_n = \sigma_n^*\Phi_n$ and G'_n is the Green's function for the metric k'_n with poles $\sigma_n^*[\phi = 0]$.

This results in $\sigma_n^*(g_n, \Phi_n) \rightarrow_{C^{1,\beta}} (g'_\infty, \Phi'_\infty)$.

Moreover, Φ'_∞ satisfies

$$\begin{aligned} \Delta_{g'_\infty} \log \Phi'_\infty &= (\tau - \Phi'_\infty) - 4\pi \sum_j n_j \delta_{p'_{j,\infty}} \\ S_{g'_\infty} + \alpha(\Delta_{g'_\infty} + \tau)(\Phi'_\infty - \tau) &= c_\alpha . \end{aligned} \tag{4}$$

Another regularity result shows that $(g'_\infty, \Phi'_\infty)$ is solution to the GV equations on $\mathbb{P}^1$ with the Higgs field ϕ'_∞ determined by $\sum_j n_j p'_{j,\infty}$.

Finally, ϕ'_∞ is polystable by the above mentioned result of AC-GF-GP-P, and $\phi'_\infty = \lim_{n\to\infty} \sigma_n^* \phi \in \overline{PSL(2,\mathbb{C}) \cdot \phi}$. Since ϕ is also polystable, we conclude that $\phi'_\infty \in PSL(2,\mathbb{C}) \cdot \phi$.

σ

Further results and directions

Vol $\to \frac{4\pi N}{\tau}$ $\quad\quad$ $\to \infty$

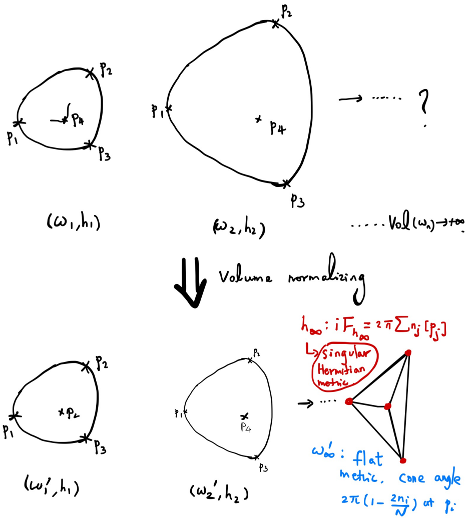
Recent progresses: fixing ϕ, look at how does the solution to EB equation behave as

i. $V \to \frac{4\pi N}{\tau}$, the family exhibits a Bradlow/Dissolving limit feature as in the study of Vortex equations:

$$h = h_0 e^{2f} \to 0, \quad i.e. \quad f \to -\infty \text{ uniformly;}$$
$$\omega \to \frac{2N}{\tau}\omega_{FS} \quad \text{in some sense.}$$

ii. $V \to +\infty$, the family of rescaled solution converges to flat conical metric on $\mathbb{P}^1$ (polyhedron metrics).

Large Volume Limit



Directions: study *uniqueness* and "Weil-Petersen type" metric on the **conjectured** moduli space of Einstein-Bogomol'nyi solutions/Gravitating Vortices

$$\mathfrak{M}_\alpha = \mathrm{Sym}^N(\mathbb{P}^1)//PSL(2,\mathbb{C}).$$

"GIT"

Thank you for your attention!