Directed Anosov and weakly positive representations

Tengren Zhang
(w/ Sungwoon Kim and Ser Peow Tan)

National University of Singapore

matzt@nus.edu.sg

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Singular values and symmetric spaces

- Let V be an n-dimensional real vector space.

Fact

Choose an inner product on V. For any linear map $L : V \to V$, there are orthonormal bases

$$(e_1, \dots, e_n) \text{ and } (f_1, \dots, f_n)$$

of V such that for all $k = 1, \dots, n$, $L(e_k) = c_k f_k$ for some real number $c_k \geq 0$, and $c_1 \geq \cdots \geq c_n$.

- We refer to c_k as the *k-th singular value* of L.
- If L is invertible, then $c_n > 0$.

Singular values and symmetric spaces

- Let $\mathrm{PGL}(V) := \mathrm{GL}(V)/\mathbb{R}^\times$
- The $\mathrm{PGL}(V)$-Riemanian symmetric space is

$$X := \{\text{inner products on } V\}/\mathbb{R}^\times.$$

  Let $d_X : X \times X \to \mathbb{R}$ be the distance function on X.
- For any point $o \in X$, choose an inner product on V representing o.
- For any $g \in \mathrm{PGL}(V)$, let $\sigma_k(g)$ denote the k-th singular value of a volume-preserving representative of g.

Fact

For any $g \in \mathrm{PGL}(V)$,

$$d_X(g \cdot o, o) = \sqrt{\sum_{k=1}^{n-1} \left(\log \frac{\sigma_k}{\sigma_{k+1}}(g) \right)^2}.$$

Anosov representations

- Let Γ be a group generated by a finite set $S \subset \Gamma$.
- Equip Γ with the word metric associated to $S \cup S^{-1}$.
- A geodesic ray $(\gamma_i)_{i \geq 0}$ in Γ is *rooted* if $\gamma_0 = \mathrm{id}$.

$d(\gamma_1, \gamma_2)$ = word length of $\gamma_2^{-1}\gamma_1$ w.r.t. $S \cup S^{-1}$

Definition $d(\gamma_i, \gamma_j) = |j - i|$

Fix a point $o \in X$. A representation $\rho : \Gamma \to \mathrm{PGL}(V)$ is *(Borel) Anosov* if there are constants $\kappa, \kappa' > 0$ such that

$$\boxed{\log \frac{\sigma_k}{\sigma_{k+1}} \rho(\gamma_i) \geq \kappa i - \kappa'} \leftarrow \text{Anosov condition.}$$

for all $k = 1, \ldots, n-1$ and all rooted, geodesic rays $(\gamma_i)_{i \geq 0}$ in Γ.

- Anosovness of a representation does not depend on S or o.
- There is a constant $\kappa'' > 0$ such that

$$\log \frac{\sigma_k}{\sigma_{k+1}} \rho(\gamma_i) \leq \kappa'' i.$$

  Thus, we are requiring $\log \frac{\sigma_k}{\sigma_{k+1}} \rho(\gamma_i)$ to grow uniformly linearly along rooted geodesic rays.

Anosov representations

- If $V = \mathbb{R}^2$, then $X = \mathbb{H}^2$ and

$$d_X(g \cdot o, o) = \log \frac{\sigma_1}{\sigma_2}(g)$$

  for all $g \in \mathrm{PGL}(2, \mathbb{R})$.
- As such, $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{R})$ is Anosov if and only if the orbit map $\Gamma \to X$ given by $\gamma \mapsto \rho(\gamma) \cdot o$ sends every rooted geodesic ray in Γ to a uniform quasi-geodesic ray in X.
- This is in turn equivalent to requiring that the orbit map is a quasi-isometric embedding.
- More generally, if G is a semisimple rank 1 Lie group of non-compact type, then $\rho : \Gamma \to G$ is Anosov if and only if the orbit map $\Gamma \to X$ is a quasi-isometric embedding. Classically, these are known as *convex-cocompact representations*.

Anosov representations

(Labourie, Guichard-Wienhard, Kapovich-Leeb-Porti, GGKW, Bochi-Potrie-Sambarino)

- If $\rho : \Gamma \to PGL(V)$ is Anosov, then → diagonalizable w/ e-values having pairwise distinct absolute values
  1. the orbit map $\Gamma \to X$ is a quasi-isometric embedding.
  2. $\rho(\gamma)$ is loxodromic for every infinite order $\gamma \in \Gamma$.
  3. Γ is a hyperbolic group.
- Anosov representations form an open subset of $\mathrm{Hom}(\Gamma, PGL(V))$.
- Examples of Anosov representations include Hitchin representations, Barbot representations, ping-pong lemma type constructions...
- If we require the Anosov property to hold only for some $k = 1, \ldots, n-1$, then examples include maximal representations, Benoist representations...

$\begin{pmatrix}1 & 1 & \\ & 1 & \\ & & 1\end{pmatrix}$ ✗

$\begin{pmatrix}1 & & \\ & 1 & \\ & & 2\end{pmatrix}$ ✗

$\begin{pmatrix}1 & & \\ & -1 & \\ & & 2\end{pmatrix}$ ✗

Directed Anosov representations

- Recall that S is a generating set of Γ. A geodesic ray $(\gamma_i)_{i\geq 0}$ in Γ is *S-directed* if $\gamma_i^{-1}\gamma_{i+1} \in S$ for all i.

Example

If $\Gamma = F_2 = \langle a, b\rangle$ and $S = \{a, b\}$, then

$$(a, ab, aba, abab, ababa, \dots)$$

is S-directed, but

$$(a, ab, aba^{-1}, aba^{-1}b, aba^{-1}ba, \dots)$$

is not S-directed.

- If $S = S^{-1}$, then every geodesic ray is S-directed.

Directed Anosov representations

Definition

Fix a point $o \in X$. A representation $\rho : \Gamma \to \mathrm{PGL}(V)$ is *(Borel) S-directed Anosov* if there are constants $\kappa, \kappa' > 0$ such that

$$\log \frac{\sigma_k}{\sigma_{k+1}} \rho(\gamma_i) \geq \kappa i - \kappa'$$

for all $k = 1, \ldots, n-1$ and all rooted, S-directed or S^{-1}-directed, geodesic rays $(\gamma_i)_{i \geq 0}$ in Γ.

- For any generating set S, every Anosov representation is S-directed Anosov, and the converse is true if $S = S^{-1}$.
- If $\rho : \Gamma \to \mathrm{PGL}(V)$ is S-directed Anosov, then
  1. the orbit map $\Gamma \to X$ sends rooted, S-directed and S^{-1}-directed geodesic rays to uniform quasi-geodesic rays in X.
  2. $\rho(\gamma)$ is loxodromic for every infinite order element $\gamma \in \Gamma$ that is a product of elements in S.
- S-directed Anosov representations might not have discrete image.

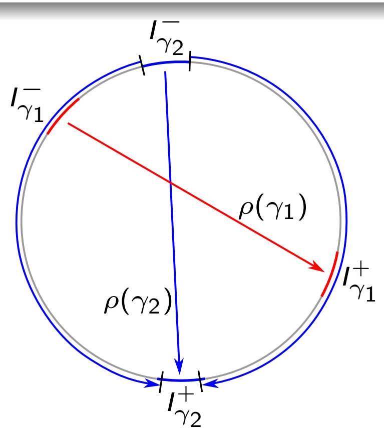
The ping-pong lemma in $\mathbb{RP}^1$.

Proposition (The ping-pong lemma in $\mathbb{RP}^1$)

Let $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{R})$ be a representation, and let $S \subset \Gamma$ be a finite generating set. Suppose that for each $\gamma \in S$, there are open intervals $I_\gamma^+, I_\gamma^- \subset \partial\mathbb{H}^2 = \mathbb{RP}^1$ such that

- *the intervals $\bigcup_{\gamma \in S} \{I_\gamma^+, I_\gamma^-\}$ have pairwise disjoint closures, and*
- $\rho(\gamma) \cdot \left(\mathbb{RP}^1 - I_\gamma^-\right) \subset I_\gamma^+$ *for all $\gamma \in S$.*

Then Γ is the free group generated by S, and ρ is an Anosov representation.



S = ⟨γ₁, γ₂⟩

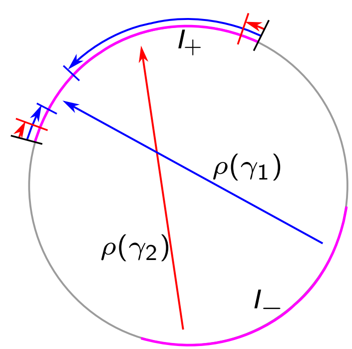
Special case of main theorem

Proposition (Main theorem specialized to $\mathbb{RP}^1$)

Let $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{R})$ be a representation, and let $S \subset \Gamma$ be a finite generating set. Suppose that there are open intervals $I^+, I^- \subset \partial \mathbb{H}^2$ such that

- *I^+ and I^- have disjoint closures,*
- *$\rho(\gamma) \cdot \overline{I^+} \subset I^+$ and $\rho(\gamma^{-1}) \cdot \overline{I^-} \subset I^-$ for all $\gamma \in S$.*

Then ρ is S-directed Anosov.



$\Gamma = F_2 = \langle \gamma_1, \gamma_2 \rangle$

$S = \{\gamma_1, \gamma_2\}$.

Fock-Goncharov positivity

Definition (Lusztig)

Let $\mathcal{B}$ denote an (ordered) basis of V. Let $U_{>0}(\mathcal{B})$ denote the set of unipotent elements $u \in \mathrm{PGL}(V)$ that are represented in $\mathcal{B}$ by an upper triangular matrix M_u whose minors are positive unless they are forced to be zero by virtue of M_u being upper triangular.

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Fock-Goncharov positivity

- A *(complete) flag* F of V is a nested sequence of subspaces

$$0 = F^{(0)} \subsetneq F^{(1)} \subsetneq \cdots \subsetneq F^{(n-1)} \subsetneq F^{(n)} = V.$$

  Denote the set of flags in V by $\mathcal{F}(V)$.
- A pair of flags (F_1, F_2) in $\mathcal{F}(V)$ are *transverse* if $F_1^{(i)} \cap F_2^{(n-i)} = \{0\}$ for all $i = 1, \ldots, n-1$.
- A basis $(e_1, \ldots, e_n)$ of V is *associated* to a pair of transverse flags (F_1, F_2) if $e_i \in F_1^{(i)} \cap F_2^{(n-i+1)}$ for all $i = 1, \ldots, n$.

Definition (Fock-Goncharov)

A tuple of flags $(F_1, \ldots, F_k)$ is *positive* if (F_1, F_k) is a transverse pair of flags, and there is

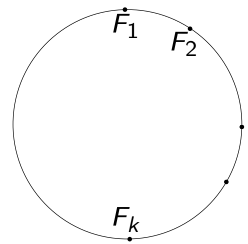
- a basis $\mathcal{B}$ associated to (F_1, F_k), and
- elements $u_2, \ldots, u_{k-1} \in U_{>0}(\mathcal{B})$

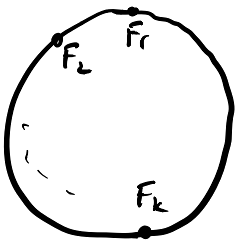
such that $F_i = u_i \ldots u_{k-1} \cdot F_k$ for all $i = 2, \ldots, k-1$.

Fock-Goncharov positivity

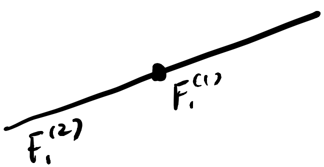
Example

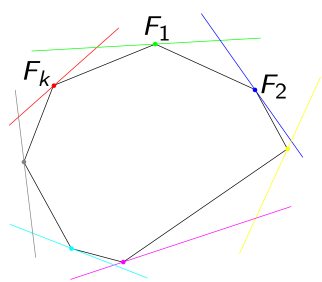
- $(F_1, \ldots, F_k) \in \mathcal{F}(\mathbb{R}^2) = \mathbb{RP}^1$ is positive if and only if $F_1 < F_2 < \cdots < F_k$ in one of the two cyclic orders on $\mathbb{RP}^1$.





- $(F_1, \ldots, F_k) \in \mathcal{F}(\mathbb{R}^3)$ is positive if and only if there is a pair of convex k-gons P_1, P_2 in an affine chart $\mathbb{A}^2 \subset \mathbb{RP}^2$ such that P_1 is inscribed in P_2, the vertices of P_1 are $F_1^{(1)}, \ldots, F_k^{(1)}$, and the edges of P_2 are $F_1^{(2)}, \ldots, F_k^{(2)}$ in either cyclic order.



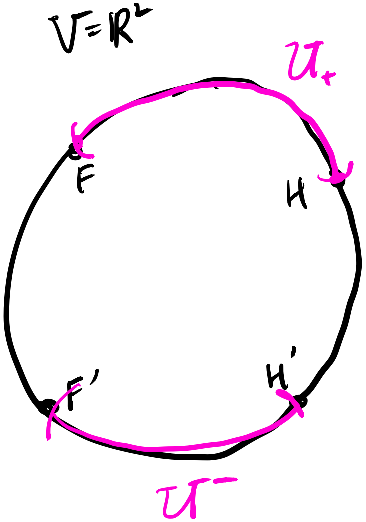


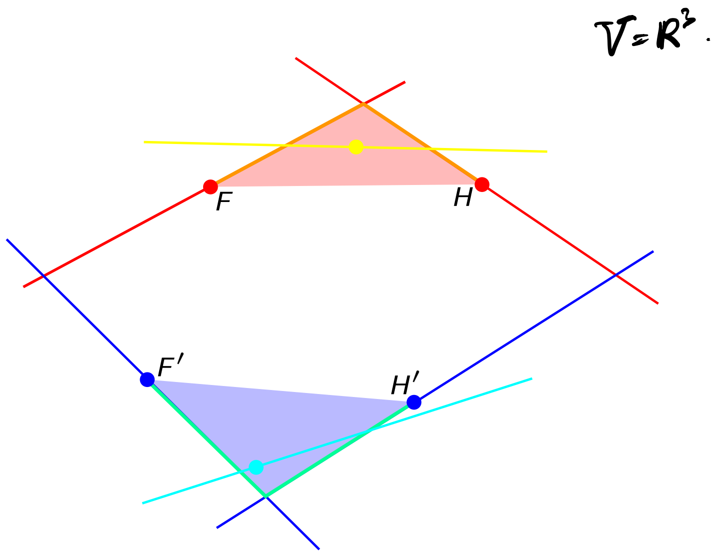
Fock-Goncharov positivity

- The forward domain (resp. backward domain) of a positive quadruple of flags (F', F, H, H') is

$$\mathfrak{U}_+ := \{K \in \mathcal{F}(V) : (F', F, K, H, H') \text{ is positive}\}$$

$$(\text{resp. } \mathfrak{U}_- := \{K \in \mathcal{F}(V) : (K, F', F, H, H') \text{ is positive}\}).$$





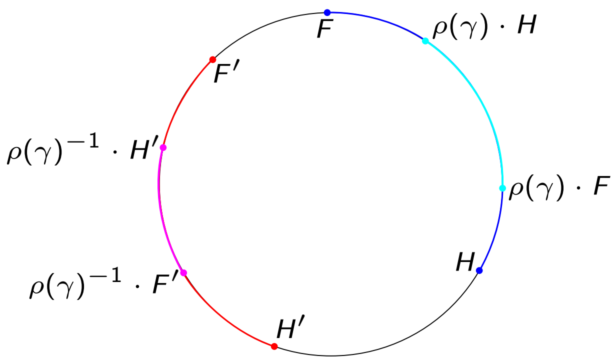
- These are open sets in $\mathcal{F}(V)$.

Weakly positive representations

Definition

A representation $\rho : \Gamma \to \mathrm{PGL}(V)$ is *S-weakly positive* if there is a positive quadruple of flags (F', F, H, H') such that:

- For all $\gamma \in S$, $\rho(\gamma) \cdot \overline{\mathfrak{U}^+} \subset \mathfrak{U}^+$ and $\rho(\gamma)^{-1} \cdot \overline{\mathfrak{U}^-} \subset \mathfrak{U}^-$,
- For all $\gamma \in S$, the tuple $(F', F, \rho(\gamma) \cdot F, \rho(\gamma) \cdot H, H, H')$ is positive up to switching $\rho(\gamma) \cdot F$ and $\rho(\gamma) \cdot H$,
- For all $\gamma \in S$, the tuple $(\rho(\gamma)^{-1} \cdot F', F', F, H, H', \rho(\gamma)^{-1} \cdot H')$ is positive up to switching $\rho(\gamma)^{-1} \cdot F'$ and $\rho(\gamma)^{-1} \cdot H'$.



We refer to (F', F, H, H') as a *separator* of ρ.

Weakly positive representations

Theorem (Main Theorem)

If $\rho : \Gamma \to \mathrm{PGL}(V)$ *is* S*-weakly positive, then it is* S*-directed Anosov.*

- Suppose $S \subset \Gamma$ and the elements in $\rho(S)$ are given.
- Verifying that ρ is S-directed Anosov requires checking infinitely many conditions.
- However, given a candidate separator, verifying that ρ is S-weakly positive with respect to the given separator requires checking only finitely many conditions.

Primitive stable representations

- Let $S \subset F_d$ be a minimal generating set, and equip F_d with the metric associated to $S \cup S^{-1}$.
- An element $\gamma \in F_d$ is *primitive* if it is a member of a minimal generating set of F_d.
- A geodesic in F_d is *primitive* if it is invariant under the left action by a primitive element in F_d.
- A geodesic ray in F_d is *primitive* if it lies in a primitive geodesic in F_d.

$\mathrm{Out}(F_d) \curvearrowright \mathrm{Hom}(F_d, \mathrm{PGL}(V)) \underset{\text{open}}{\supseteq} \{\text{primitive stable}\}$

Definition (Minsky, Guichard-Gueritaud-Kassel-Wienhard)

Fix a point $o \in X$. A representation $\rho : F_d \to \mathrm{PGL}(V)$ is *(Borel) primitive stable* if there are constants $\kappa, \kappa' > 0$ such that

$$\log \frac{\sigma_k}{\sigma_{k+1}} \rho(\gamma_i) \geq \kappa i - \kappa'$$

for all $k = 1, \ldots, n-1$ and all rooted, primitive geodesic rays $(\gamma_i)_{i \geq 0}$ in F_d.

Primitive stable representations

- Let $S = \{\gamma_1, \gamma_2\}$ generate F_2, and let $\gamma_3 := \gamma_2^{-1}\gamma_1^{-1}$.
- Let $S' := \{\gamma_1^{-1}, \gamma_2\}$, $S'' := \{\gamma_2^{-1}, \gamma_3\}$ and $S''' := \{\gamma_3^{-1}, \gamma_1\}$.

Proposition (Using Cohen-Metzler-Zimmermann)

If $\rho : F_2 \to \mathrm{PGL}(V)$ is (S, S')-weakly positive or (S', S'', S''')-weakly positive, the ρ is primitive stable.

Proposition (Using Goldman-McShane-Stantchev-Tan)

If $\rho : F_2 \to \mathrm{PGL}(2, \mathbb{R})$ is primitive stable, then it is (S', S'', S''')-weakly positive for some generating pair S of F_2.

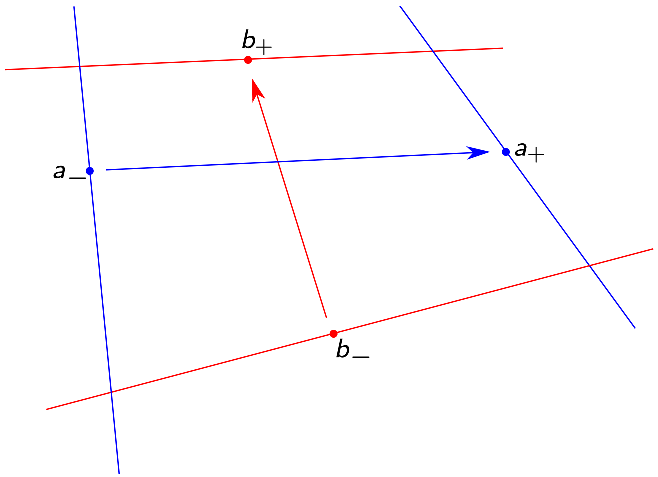
Conj: $\exists\, \rho : F_3 \longrightarrow PGL_2(\mathbb{R})$ that is primitive stable but not convex cocompact.

Primitive stable representations

- An element $g \in \mathrm{PGL}(V)$ is *positive loxodromic* if it is loxodromic and all its eigenvalues have the same sign.
- For any loxodromic $g \in \mathrm{PGL}(V)$, let g_+ and g_- respectively denote the attracting and repelling fixed flag in $\mathcal{F}(V)$ of g.

Proposition

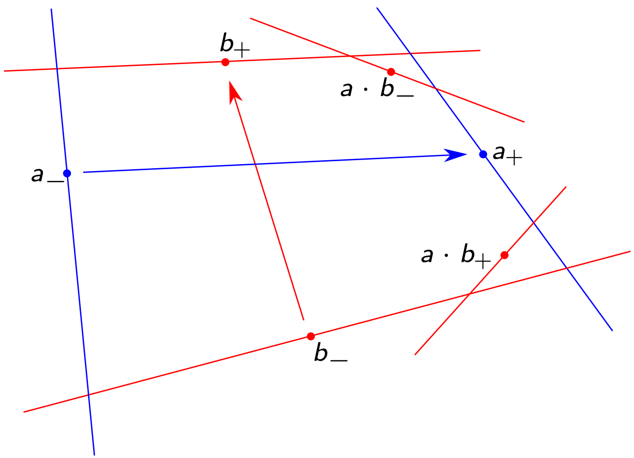
Let $\{\gamma_1, \gamma_2\}$ be a generating set for F_2, and $\rho : F_2 \to \mathrm{PGL}_3(\mathbb{R})$ be a representation. If $a := \rho(\gamma_1)$, $b := \rho(\gamma_2)$ are positive loxodromic, and (b_-, a_+, b_+, a_-) is positive, then ρ is primitive stable.



Primitive stable representations

Proposition

Let $\{\gamma_1, \gamma_2\}$ *be a generating set for* F_2*, and let* $\rho : F_2 \to \mathrm{PGL}(V)$ *be a representation. If* $a := \rho(\gamma_1)$ *is loxodromic,* $b := \rho(\gamma_2)$ *is positive loxodromic, and* $(b_-, a \cdot b_-, a_+, a \cdot b_+, b_+, a_-)$ *is positive up to switching* $a \cdot b_-$ *and* $a \cdot b_+$*, then* ρ *is primitive stable.*



- This gives many new, explicit examples of primitive stable representations from F_2 to $\mathrm{PGL}(V)$, including non-discrete and non-faithful examples.

Proof of Main Theorem

The proof has two parts that are related by the following definition.

Definition

Let $\mathcal{W}$ be a collection of sequences in $\mathrm{PGL}(V)$, and fix $o \in X$.

1. $\mathcal{W}$ is *uniformly well-behaved* if there is a constant $C > 0$ such that for any sequence $(g_i)_{i \geq 0} \in \mathcal{W}$,
   - there is a maximal flat $\mathrm{F} \subset X$ such that $d_X(g_i \cdot o, \mathrm{F}) < C$ for all $i \geq 0$.
   - $d_X(g_i \cdot o, g_{i+1} \cdot o) < C$ for all $i \geq 0$.
2. $\mathcal{W}$ is *regulated* if for every $D > 0$, there is an integer $N(D) > 0$ such that

$$\log \frac{\sigma_k}{\sigma_{k+1}}(g_i) \geq D$$

for all sequences $(g_i)_{i \geq 0} \in \mathcal{W}$ and all integers $i \geq N(D)$, and $k = 1, \ldots, n-1$.

Proof of Main Theorem (Symmetric space part)

Theorem (Symmetric space part)

Fix $o \in X$. If $\mathcal{W}$ is a collection of sequences in $\mathrm{PGL}(V)$ that is uniformly well-behaved and regulated, then there exists constants $\kappa, \kappa' > 0$ such that

$$\log \frac{\sigma_k}{\sigma_{k+1}}(g_i) \geq \kappa i - \kappa'$$

for all $k = 1, \ldots, n-1$ and all sequences $(g_i)_{i \geq 0}$ in $\mathcal{W}$.

- The proof uses ideas developed by Kapovich-Leeb-Porti in their study of Anosov representations $\rho : \Gamma \to \mathrm{PGL}(V)$ via their induced actions on X.
- It is now sufficient to show that if $\rho : \Gamma \to \mathrm{PGL}(V)$ is an S-weakly positive representation, then

  $$\mathcal{W}_S := \{(\rho(\gamma_i))_{i \geq 0} : (\gamma_i)_{i \geq 0} \text{ is a rooted } S\text{-directed geodesic ray}\}$$

  is uniformly well-behaved and regulated.

Proof of Main Theorem (Positivity part)

Let $\rho : \Gamma \to \mathrm{PGL}(V)$ be an S-weakly positive representation, let (F', F, H, H') be a separator for ρ with forward and backward domain $\mathfrak{U}_+$ and $\mathfrak{U}_-$ respectively.

Step 1: Show that if $(\gamma_i)_{i\geq 0}$ is a rooted S-directed geodesic, then

$$(F', F, \rho(\gamma_1)\cdot F, \dots, \rho(\gamma_k)\cdot F, \rho(\gamma_k)\cdot H, \dots, \rho(\gamma_1)\cdot H, H, H')$$

is positive for all $k \geq 0$. (Consequence of semigroup property of $U_{>0}(\mathcal{B})$.)

Step 2: Show that $\lim_{i\to\infty} \rho(\gamma_i)\cdot F = \lim_{i\to\infty} \rho(\gamma_i)\cdot H$. (Uses a cross ratio argument.)

Step 3: Use this to deduce that $\bigcap_{i=0}^{\infty} \overline{\rho(\gamma_i)\cdot \mathfrak{U}_+}$ is a point. Hence, $\mathcal{W}_S$ is regulated.

Step 4: Show that $\overline{\mathfrak{U}_+} \times \overline{\mathfrak{U}_-}$ lies in the set of transverse pairs of flags in $\mathcal{F}(V)$ to deduce that $\mathcal{W}_S$ is uniformly well-behaved.

The End