Non commutative cluster coordinates for Higher Teichmüller Spaces

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May 27, 2021

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Important in several areas of Geometry and Theoretical Physics.

Higher Teichmüller-Thurston Theory
Theory of Higgs Bundles
Geometric Quantization
SUSY Quantum Field Theories
Gauge Theory
Knot Theory
Integrable Systems

Higher Teichmüller-Thurston Theory

We borrow ideas from
the classical **Teichmüller-Thurston Theory**
to study some special subsets of $\text{Rep}(\pi_1(S), G)$.

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The other components don't have the same nice geometry.

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More subtle definitions are needed, there is a hierarchy of special representations.

Special representations

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   e.g. $G = PSL(n, \mathbb{R}), PSp(2n, \mathbb{R}), SO(p, p+1), SO(p, p)$.

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4. Discrete and faithful representations.

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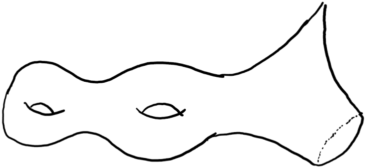
$$\mathcal{T}(S) = \left\{ \text{hyp. str. on } \mathrm{int}(S) \;\middle|\; \begin{array}{l}\text{every end is a cusp or a}\\ \text{geodesic boundary component}\end{array} \right\} / \sim$$



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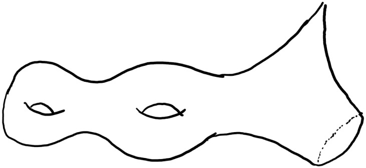

$$\mathcal{T}_0(S) = \{ \text{hyp. str. on } \mathrm{int}(S) \mid \text{every end is a cusp} \} / \sim \; \subset \; \mathcal{T}(S)$$

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It generalizes to higher rank giving **positive representations**.

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Fock-Goncharov's work is for this positive structure.

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$P = \mathrm{Stab}_G(L)$ — the parabolic subgroup.

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We want to study them using ideas from Teichmüller-Thurston theory.

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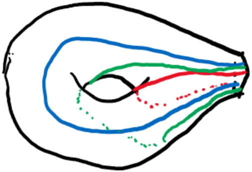

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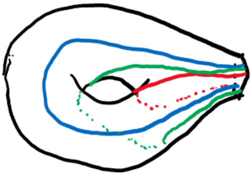

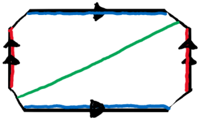

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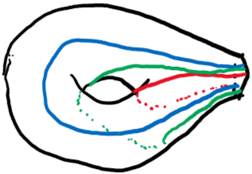

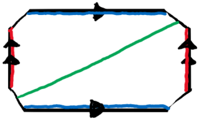

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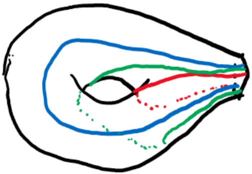

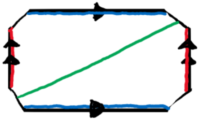

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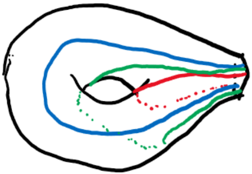

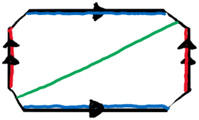

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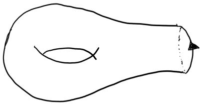

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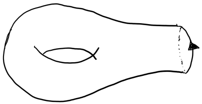

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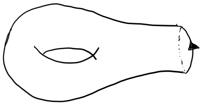

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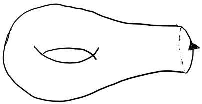

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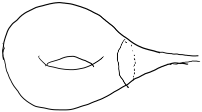

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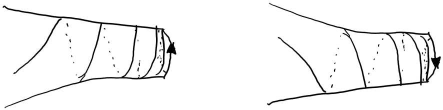

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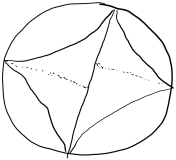

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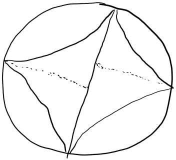

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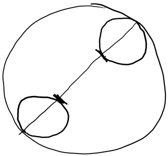

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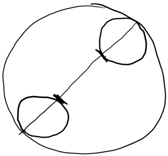

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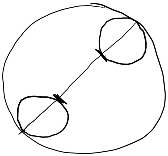

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Space of framed representations. Our X-type moduli space.