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Differential Geometry

Bergman kernels and symplectic reduction

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Abstract

We present several results concerning the asymptotic expansion of the invariant Bergman kernel of the spin^c Dirac operator associated with high tensor powers of a positive line bundle on a compact symplectic manifold. ***To cite this article: X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Noyaux de Bergman et réduction symplectique. Nous annonçons des résultats sur le développement asymptotique du noyau de Bergman G-invariant de l'opérateur de Dirac spin^c associé à une puissance tendant vers l'infini d'un fibré en droites positif sur une variété symplectique compacte. ***Pour citer cet article : X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soit (X,ω) une variété symplectique compacte, et soit (L,h^L) un fibré en droites hermitien muni d'une connexion hermitienne ∇^L telle que $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2=\omega$. Soit (E,h^E) un fibré vectoriel hermitien sur X muni d'une connexion hermitienne ∇^E. Soit g^{TX} une métrique riemannienne sur X, et soit J une structure presque complexe compatible séparément à g^{TX} et ω. Alors les données géométriques ci-dessus définissent canoniquement un opérateur de Dirac spin^c D_p agissant sur $\Omega^{0,\bullet}(X,L^p\otimes E)$, l'espace de $(0,\bullet)$-formes à valeurs dans $L^p\otimes E$.

Soit G un groupe de Lie compact connexe et soit $\mathfrak{g}$ son algèbre de Lie. On suppose que G agit sur X, et que son action se relève à L et E en préservant J, les métriques et les connexions ci-dessus. Alors $\operatorname{Ker} D_p$ est une G-représentation de dimension finie. Soit $(\operatorname{Ker} D_p)^G$ la partie G-invariante de $\operatorname{Ker} D_p$. Le noyau de Bergman

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G-invariant $P_p^G(x,x')$ $(x,x'\in X)$ est le noyau $\mathscr{C}^\infty$ de la projection orthogonale P_p^G de $\Omega^{0,\bullet}(X,L^p\otimes E)$ sur $(\operatorname{Ker} D_p)^G$ associé à la forme de volume riemannienne $dv_X(x')$.

Dans cette Note, nous annonçons des résultats sur le développement asymptotique de $P_p^G(x,x')$ quand p tend vers l'infini. Le détail des démonstrations et des applications de nos résultats est donné dans [6].

1. Introduction

Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$, where $R^L=(\nabla^L)^2$ is the curvature of (L,∇^L). Let (E,h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Let g^{TX} be a Riemannian metric on X. Let $\mathbf{J}: TX\to TX$ be the skew-adjoint linear map which satisfies the relation $\omega(u,v)=g^{TX}(\mathbf{J}u,v)$ for $u,v\in TX$. Let J be an almost complex structure such that $g^{TX}(Ju,Jv)=g^{TX}(u,v)$, $\omega(Ju,Jv)=\omega(u,v)$, and that $\omega(\cdot,J\cdot)$ defines a metric on TX. Then J commutes with $\mathbf{J}$, thus $J=\mathbf{J}(-\mathbf{J}^2)^{-1/2}$. Let ∇^{TX} be the Levi-Civita connection on (TX,g^{TX}) with curvature R^{TX} and scalar curvature r^X. Then ∇^{TX} induces a natural connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ with curvature $R^{\det}$, and the Clifford connection ∇^{Cliff} on the Clifford module $\Lambda(T^{*(0,1)}X)$ with curvature R^{Cliff}. The spin^c Dirac operator D_p acts on $\Omega^{0,\bullet}(X,L^p\otimes E)=\bigoplus_{q=0}^n\Omega^{0,q}(X,L^p\otimes E)$, the direct sum of spaces of $(0,q)$-forms with values in $L^p\otimes E$. We denote by D_p^+ the restriction of D_p on $\Omega^{0,\mathrm{even}}(X,L^p\otimes E)$. By [4, Theorem 2.5], when p is large enough, $\operatorname{Coker} D_p^+$ vanishes.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\dim G=n_0$. Suppose that G acts on X and its action on X lifts on L,E. Moreover, we assume the G-action preserves the above connections and metrics on TX, L, E and J. Then $\operatorname{Ker} D_p$ is a finite dimensional representation space of G.

The action of G on L induces naturally a moment map $\mu: X\to\mathfrak{g}^*$. Now we assume that $0\in\mathfrak{g}^*$ is a regular value of μ. Then the Marsden–Weinstein symplectic reduction $(X_G=\mu^{-1}(0)/G,\omega_{X_G})$ is a symplectic manifold when G acts freely on $\mu^{-1}(0)$. Moreover, (L,∇^L), (E,∇^E) descend to (L_G,∇^{L_G}), (E_G,∇^{E_G}) over X_G so that the corresponding curvature condition $\frac{\sqrt{-1}}{2\pi}R^{L_G}=\omega_G$ holds (cf. [3]). The G-invariant almost complex structure J also descends to an almost complex structure J_G on TX_G, and h^L, h^E, g^{TX} descend to $h^{L_G}, h^{E_G}, g^{TX_G}$ respectively. Thus we can construct the corresponding spin^c Dirac operator $D_{G,p}$ on X_G.

Let $(\operatorname{Ker} D_p)^G$ denote the G-invariant part of $\operatorname{Ker} D_p$. Let P_p^G be the orthogonal projection from $\Omega^{0,\bullet}(X,L^p\otimes E)$ on $(\operatorname{Ker} D_p)^G$. The G-invariant Bergman kernel is $P_p^G(x,x')$ $(x,x'\in X)$, the smooth kernel of P_p^G with respect to the Riemannian volume form $dv_X(x')$. Let pr_1 and pr_2 be the projections from $X\times X$ onto the first and second factor X respectively. Then $P_p^G(x,x')$ is a smooth section of $\mathrm{pr}_1^*E_p\otimes\mathrm{pr}_2^*E_p^*$ on $X\times X$ with $E_p=\Lambda(T^{*(0,1)}X)\otimes L^p\otimes E$. In particular, $P_p^G(x,x)\in\operatorname{End}(E_p)_x=\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_x$.

The G-invariant Bergman kernel $P_p^G(x,x')$ is a local analytic version of $(\operatorname{Ker} D_p)^G$.

In this Note, we present several results concerning the asymptotic expansions of $P_p^G(x,x')$ as $p\to+\infty$. More details will appear in [6].

2. Main results

The first result shows that one can reduce our problem to a problem near $\mu^{-1}(0)$.

Theorem 2.1. *For any open G-neighborhood U of $\mu^{-1}(0)$ in X, $\varepsilon_0>0$, $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ (depend on U, ε_0) such that for $p\geqslant 1$, $x,x'\in X$, $d(Gx,x')\geqslant\varepsilon_0$ or $x,x'\in X\setminus U$,*

$$\left|P_p^G(x,x')\right|_{\mathscr{C}^m} \leqslant C_{l,m} p^{-l}, \tag{1}$$

where $\mathscr{C}^m$ is the $\mathscr{C}^m$-norm induced by ∇^L, ∇^E, ∇^{TX}, h^L, h^E, g^{TX}.

Assume for simplicity that G acts freely on $\mu^{-1}(0)$. Let U be an open neighborhood of $\mu^{-1}(0)$ such that G acts freely on U. For any G-equivariant bundle (F,∇^F) on U, we denote by F_B the bundle on $U/G=B$ induced naturally by G-invariant sections of F on U. The connection ∇^F induces canonically a connection ∇^{F_B} on F_B. Let R^{F_B} be its curvature. We denote also by $\mu^F(K)=\nabla^F_K - L_K \in \operatorname{End}(F)$ for $K\in\mathfrak{g}$. Note that $P_p^G \in (\mathscr{C}^\infty(U\times U, \mathrm{pr}_1^* E_p \otimes \mathrm{pr}_2^* E_p^*))^{G\times G}$, thus we can view $P_p^G(x,x')$ as a smooth section of $(\mathrm{pr}_1^* E_p)_B \otimes (\mathrm{pr}_2^* E_p^*)_B$ on $B\times B$.

Let g^{TB} be the Riemannian metric on $U/G=B$ induced by g^{TX}. Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}) with curvature R^{TB}. Let N_G be the normal bundle to X_G in B. We identify N_G with the orthogonal complement of TX_G in $(TB|_{X_G}, g^{TB})$. Let P^{TX_G}, P^{N_G} be the orthogonal projections from $TB|_{X_G}$ on TX_G, N_G respectively. Set

$$\nabla^{N_G} = P^{N_G}\big(\nabla^{TB}|_{X_G}\big)P^{N_G}, \quad {}^0\nabla^{TB} = P^{TX_G}\big(\nabla^{TB}|_{X_G}\big)P^{TX_G} \oplus \nabla^{N_G}, \quad A = \nabla^{TB} - {}^0\nabla^{TB}. \tag{2}$$

Then ∇^{N_G}, ${}^0\nabla^{TB}$ are Euclidean connections on N_G, $TB|_{X_G}$ on X_G, and A is the associated second fundamental form. We denote by $\operatorname{vol}(Gx)$ $(x\in U)$ the volume of the orbit Gx equipped with the metric induced by g^{TX}. Following [9, (3.10)], let $h(x)$ be the function on U defined by

$$h(x) = \big(\operatorname{vol}(Gx)\big)^{1/2}. \tag{3}$$

Then h reduces to a function on B. We denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X)\otimes E$ onto $\mathbb{C}\otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X) = \mathbb{C}\oplus \Lambda^{>0}(T^{*(0,1)}X)$, and $I_{\mathbb{C}\otimes E_B}$ the corresponding projection on B.

For any $x_0\in X_G$, $Z\in T_{x_0}B$, we write $Z=Z^0+Z^\perp$, with $Z^0\in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$. Let $\tau_{Z^0}Z^\perp \in N_{G,\exp^{X_G}_{x_0}(Z^0)}$ be the parallel transport of $Z^\perp$ with respect to the connection ∇^{N_G} along the geodesic in X_G, $[0,1]\ni t\to \exp^{X_G}_{x_0}(tZ^0)$. For $\varepsilon_0>0$ small enough, we identify $Z\in T_{x_0}B$, $|Z|<\varepsilon_0$ with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp)\in B$, then for $x_0\in X_G$, $Z,Z'\in T_{x_0}B$, $|Z|,|Z'|<\varepsilon_0$, the map

$$\Psi : TB|_{X_G}\times TB|_{X_G} \to B\times B, \quad \Psi(Z,Z') = \big(\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp), \exp^B_{\exp^{X_G}_{x_0}(Z'^0)}(\tau_{Z'^0}Z'^\perp)\big)$$

is well defined. We identify $(E_p)_{B,Z}$ to $(E_p)_{B,x_0}$ by using parallel transport with respect to $\nabla^{(E_p)_B}$ along $[0,1]\ni u\to uZ$. Let $\pi_B : TB|_{X_G}\times TB|_{X_G}\to X_G$ be the natural projection from the fiberwise product of $TB|_{X_G}$ on X_G onto X_G. From Theorem 2.1, we only need to understand $P_p^G\circ\Psi$, and under our identification, $P_p^G\circ\Psi(Z,Z')$ is a smooth section of $\pi_B^*(\operatorname{End}(E_p)_B) = \pi_B^*(\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B)$ on $TB|_{X_G}\times TB|_{X_G}$. Let $|\ \ |_{\mathscr{C}^{m'}(X_G)}$ be the $\mathscr{C}^{m'}$-norm on $\mathscr{C}^\infty(X_G, \operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B)$ induced by $\nabla^{\mathrm{Cliff}_B}$, ∇^{E_B}, h^E and g^{TX}. The norm $|\ \ |_{\mathscr{C}^{m'}(X_G)}$ induces naturally a $\mathscr{C}^{m'}$-norm along X_G on $\mathscr{C}^\infty(TB|_{X_G}\times TB|_{X_G}, \pi_B^*(\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B))$, we still denote it by $|\ \ |_{\mathscr{C}^{m'}(X_G)}$.

Let g^{TX_G}, g^{N_G} be the metric on TX_G, N_G induced by g^{TX}. Let $\mathrm{d}v_{X_G}$, $\mathrm{d}v_{N_G}$ be the Riemannian volume forms on (X_G, g^{TX_G}), (N_G, g^{N_G}). Let $\kappa\in\mathscr{C}^\infty(TB|_{X_G},\mathbb{R})$, with $\kappa=1$ on X_G, be defined by that for $Z\in T_{x_0}B$, $x_0\in X_G$,

$$\mathrm{d}v_B(x_0,Z) = \kappa(x_0,Z)\,\mathrm{d}v_{T_{x_0}B}(Z) = \kappa(x_0,Z)\,\mathrm{d}v_{X_G}(x_0)\,\mathrm{d}v_{N_{G,x_0}}. \tag{4}$$

Theorem 2.2. *Assume that G acts freely on $\mu^{-1}(0)$ and $\mathbf{J}=J$ on $\mu^{-1}(0)$. Then there exist $\mathcal{Q}_r(Z,Z')\in$* $\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}$ $(x_0\in X_G, r\in\mathbb{N})$*, polynomials in Z,Z' with the same parity as r, whose coefficients are polynomials in A, R^{TB}, R^{Cliff_B}, R^{E_B}, μ^E, μ^{Cliff}* (*resp.* r^X, $R^{\det}$, R^E; *resp.* h, R^L, R^{L_B}; *resp.* μ) *and their derivatives at x_0 to order $r-1$* (*resp.* $r-2$; *resp.* r, *resp.* $r+1$)*, such that if we denote by*

$$P^{(r)}_{x_0}(Z,Z') = \mathcal{Q}_r(Z,Z')P(Z,Z'), \quad \mathcal{Q}_0(Z,Z') = I_{\mathbb{C}\otimes E_B}, \tag{5}$$

with

$$P(Z,Z') = \exp\Big(-\frac{\pi}{2}\big|Z^0 - Z'^0\big|^2 - \pi\sqrt{-1}\big\langle J_{x_0}Z^0, Z'^0\big\rangle\Big)2^{n_0/2}\exp\big(-\pi\big(|Z^\perp|^2 + |Z'^\perp|^2\big)\big), \tag{6}$$

then there exists $C'' > 0$ *such that for any* $k, m, m', m'' \in \mathbb{N}$, *there exists* $C > 0$ *such that for* $x_0 \in X_G$, $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| \leqslant \varepsilon_0$,

$$\begin{aligned}&\big(1 + \sqrt{p}\,|Z^\perp| + \sqrt{p}\,|Z'^\perp|\big)^{m''} \sup_{|\alpha|+|\alpha'|\leqslant m}\bigg|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\\ &\qquad\bigg(p^{-n+n_0/2}\big(h\kappa^{1/2}\big)(Z)\big(h\kappa^{1/2}\big)(Z')P_p^G \circ \Psi(Z,Z') - \sum_{r=0}^{k} P_{x_0}^{(r)}(\sqrt{p}\,Z, \sqrt{p}\,Z')p^{-r/2}\bigg)\bigg|_{\mathscr{C}^{m'}(X_G)}\\ &\quad\leqslant Cp^{-(k+1-m)/2}\big(1 + \sqrt{p}\,\big|Z^0\big| + \sqrt{p}\,\big|Z'^0\big|\big)^{2(n+k+2)+m}\exp\big(-\sqrt{C''}\sqrt{p}\,|Z - Z'|\big) + \mathscr{O}\big(p^{-\infty}\big).\end{aligned} \tag{7}$$

Furthermore, the expansion is uniform in the following sense: for any fixed $k, m, m', m'' \in \mathbb{N}$, *assume that the derivatives of* g^{TX}, h^L, ∇^L, h^E, ∇^E, *and* J *with order* $\leqslant 2n + 2k + m + m' + 4$ *run over a set bounded in the* $\mathscr{C}^{m'}$*-norm taken with respect to the parameters and, moreover,* g^{TX} *runs over a set bounded below. Then the constant* C *is independent of* g^{TX}; *and the* $\mathscr{C}^{m'}$*-norm in* (7) *includes also the derivatives on the parameters.*

In (7), the term $\mathscr{O}(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1}p^{-l}$. The kernel $P(Z,Z')$ is the product of two kernels: along $T_{x_0}X_G$, it is the classical Bergman kernel on $T_{x_0}X_G$ with complex structure J_{x_0}, while along N_G, it is the kernel of a harmonic oscillator on N_{G,x_0}.

Remark 1. (i) When $G = \{1\}$, Theorem 2.2 has been proved in [2, Theorem 3.18′].

(ii) If we take $Z = Z' = 0$ in (7), then we get for $x_0 \in X_G$,

$$P_{x_0}^{(0)}(0,0) = 2^{n_0/2} I_{\mathbb{C}\otimes E_B}, \tag{8}$$

and

$$\bigg|p^{-n+n_0/2}h^2(x_0)P_p^G(x_0,x_0) - \sum_{r=0}^{k} P_{x_0}^{(2r)}(0,0)p^{-r}\bigg|_{\mathscr{C}^{m'}(X_G)} \leqslant Cp^{-k-1}. \tag{9}$$

In fact, (8) and (9) can be obtained as direct consequences of the full off-diagonal asymptotic expansion of the Bergman kernel proved in [2, Theorem 3.18′].

Remark 2. Assume that (X,ω) is a Kähler manifold and $\mathbf{J} = J$ on X. Assume also that (L, ∇^L), (E, ∇^E) are holomorphic vector bundles with holomorphic Hermitian connections. Then D_p^2 preserves the $\mathbb{Z}$-graduation of $\Omega^{0,\bullet}(X, L^p \otimes E)$ and $\operatorname{Ker} D_p = H^0(X, L^p \otimes E)$ when p is large enough, and so $P_p^G(x_0,x_0) \in \operatorname{End}(E)$. In particular $P_{x_0}^{(0)}(0,0) = 2^{n_0/2}\mathrm{Id}_E$ in (8). In the special case of $E = \mathbb{C}$, $P_p^G(x_0,x_0)$ is a function on X_G, (9) has been proved in [7, Theorem 1] without knowing the informations on $P_{x_0}^{(2r)}(0,0)$, while in [8, Theorem 1], it was claimed that $P_{x_0}^{(0)}(0,0) = 1$.

Let $\mathscr{I}_p$ be a section of $\operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B$ on X_G defined by

$$\mathscr{I}_p(x_0) = \int_{Z\in N_G,\ |Z|\leqslant\varepsilon_0} h^2(x_0,Z)P_p^G \circ \Psi\big((x_0,Z),(x_0,Z)\big)\kappa(x_0,Z)\,\mathrm{d}v_{N_G}(Z). \tag{10}$$

By Theorem 2.1, modulo $\mathscr{O}(p^{-\infty})$, $\mathscr{I}_p(x_0)$ does not depend on ε_0, and

$$\dim(\operatorname{Ker} D_p)^G = \int_X \operatorname{Tr}\bigl[P_p^G(y,y)\bigr]\,dv_X(y) = \int_{X_G} \operatorname{Tr}\bigl[\mathscr{I}_p(x_0)\bigr]\,dv_{X_G}(x_0) + \mathscr{O}\bigl(p^{-\infty}\bigr). \tag{11}$$

A direct consequence of Theorem 2.2 is the following corollary.

Corollary 2.3. *Taken $Z = Z' \in N_{G,x_0}$, $m = 0$ in* (7)*, we get*

$$\left| p^{-n+n_0/2} h^2(Z) P_p^G(Z,Z) - \sum_{r=0}^{k} P_{x_0}^{(r)}(\sqrt{p}\,Z, \sqrt{p}\,Z) p^{-r/2} \right|_{\mathscr{C}^{m'}(X_G)} \leqslant C p^{-(k+1)/2}\bigl(1 + \sqrt{p}\,|Z|\bigr)^{-m''} + \mathscr{O}\bigl(p^{-\infty}\bigr). \tag{12}$$

In particular, there exist $\Phi_r \in \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}$ ($r \in \mathbb{N}$) which are polynomials in A, R^{TB}, R^{Cliff_B}, R^{E_B}, μ^E, μ^{Cliff}, (resp. r^X, $R^{\det}$, R^E; resp. h, R^{L_B}, R^L; resp. μ), and their derivatives at x_0 to order $2r-1$ (resp. $2r-2$; resp. $2r$; resp. $2r+1$), and $\Phi_0 = I_{\mathbb{C}\otimes E_B}$, such that for any $k, m' \in \mathbb{N}$, there exists $C_{k,m'} > 0$ such that for any $x_0 \in X_G$, $p \in \mathbb{N}$,

$$\left| p^{-n+n_0} \mathscr{I}_p(x_0) - \sum_{r=0}^{k} \Phi_r(x_0) p^{-r} \right|_{\mathscr{C}^{m'}} \leqslant C_{k,m'} p^{-k-1}. \tag{13}$$

Theorem 2.4. *If (X,ω) is a Kähler manifold and L, E are holomorphic vector bundles with holomorphic Hermitian connections ∇^L, ∇^E, $\mathbf{J} = J$ on U, and G acts freely on $\mu^{-1}(0)$, then in* (13)*, $\Phi_r(x_0) \in \operatorname{End}(E_G)_{x_0}$ are polynomials in A, R^{TB}, R^{E_B}, μ^E, R^E (resp. h, R^{L_B}; resp. μ) and their derivatives at x_0 to order $2r-1$ (resp. $2r$; resp. $2r+1$), and $\Phi_0 = \mathrm{Id}_{E_G}$. Moreover*

$$\Phi_1(x_0) = \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{3}{4\pi} \Delta_{X_G} \log(h|_{X_G}) + \frac{\sqrt{-1}}{4\pi} R_{x_0}^{E_G}\bigl(e_j^0, J_G e_j^0\bigr). \tag{14}$$

Here r^{X_G} is the Riemannian scalar curvature of (TX_G, g^{TX_G}), Δ_{X_G} is the Bochner–Laplacian on X_G, and $\{e_j^0\}$ is an orthonormal basis of TX_G.

Still making the same assumptions as in Theorem 2.4, let $\tilde{h}$ denote the restriction to X_G of the function h defined in (3). In view of [9, (3.11), (3.54)], set $\tilde{h}^{E_G} = \tilde{h}^2 h^{E_G}$.

Let $\widetilde{P}_p^{X_G}$ denote the orthogonal projection from $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ onto $H^0(X, L_G^p \otimes E_G)$ associated to the metrics h^{L_G}, $\tilde{h}^{E_G}$, g^{TX_G}. Let $\widetilde{P}_p^{X_G}(x_0, x_0')$ ($x_0, x_0' \in X_G$) denote the corresponding Bergman kernel with respect to $dv_{X_G}(x_0')$.

Then by [2, Theorem 1.3], we have the following theorem.

Theorem 2.5. *Under the assumption of Theorem 2.4, there exist smooth coefficients $\widetilde{\Phi}_r(x_0) \in \operatorname{End}(E_G)_{x_0}$ which are polynomials in R^{TX_G}, R^{E_G} (resp. h), and their derivatives at x_0 to order $2r-1$ (resp. $2r$), and $\widetilde{\Phi}_0 = \mathrm{Id}_{E_G}$, such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x_0 \in X_G$, $p \in \mathbb{N}$,*

$$\left| p^{-n+n_0} \widetilde{P}_p^{X_G}(x_0, x_0) - \sum_{r=0}^{k} \widetilde{\Phi}_r(x_0) p^{-r} \right|_{\mathscr{C}^l} \leqslant C_{k,l} p^{-k-1}. \tag{15}$$

Moreover, the following identity holds,

$$\widetilde{\Phi}_1(x_0) = \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{1}{2\pi} \Delta_{X_G} \log \tilde{h} + \frac{\sqrt{-1}}{4\pi} R_{x_0}^{E_G}\bigl(e_j^0, J_G e_j^0\bigr). \tag{16}$$

Remark 3. From (14) and (16), one sees that in general $\Phi_1 \neq \widetilde{\Phi}_1$, if $\widetilde{h}$ is not constant on X_G. This reflects a subtle difference between the Bergman kernel and the geometric quantization.

The proof of the above theorems uses techniques adapted from [1, §11], [2,5], along with a deformation of D_p^2 by the Casimir operator (i.e., to consider $D_p^2 - p\mathrm{Cas}$, which plays a key role in proving Theorems 2.1, 2.2). We refer to [6] for more details.

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