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ORIGINAL ARTICLE

Remark on the Off-Diagonal Expansion of the Bergman Kernel on Compact Kähler Manifolds

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Abstract In this short note, we compare our previous work on the off-diagonal expansion of the Bergman kernel and the preprint of Lu–Shiffman (arXiv:1301.2166). In particular, we note that the vanishing of the coefficient of $p^{-1/2}$ is implicitly contained in Dai–Liu–Ma's work (J. Differ. Geom. 72(1), 1–41, 2006) and was explicitly stated in our book (Holomorphic Morse inequalities and Bergman kernels. Progress in Math., vol. 254, 2007).

Keywords Kähler manifold · Bergman kernel of a positive line bundle

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In this short note we revisit the calculations of some coefficients of the off-diagonal expansion of the Bergman kernel from our previous work [4, 5].

Let (X, ω) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$ with Kähler form ω. Let (L, h^L) be a holomorphic Hermitian line bundle on X, and let (E, h^E) be a holomorphic Hermitian vector bundle on X. Let ∇^L, ∇^E be the holomorphic Hermitian connections on (L, h^L), (E, h^E) with curvatures $R^L = (\nabla^L)^2$, $R^E = (\nabla^E)^2$, respectively. We assume that (L, h^L, ∇^L) is a prequantum line bundle, i.e., $\omega = \frac{\sqrt{-1}}{2\pi} R^L$.

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Let $P_p(x,x')$ be the Bergman kernel of $L^p \otimes E$ with respect to h^L, h^E and the Riemannian volume form $dv_X = \omega^n/n!$. This is the integral kernel of the orthogonal projection from $\mathscr{C}^\infty(X, L^p \otimes E)$ to the space of holomorphic sections $H^0(X, L^p \otimes E)$ (cf. [4, §4.1.1]).

We fix $x_0 \in X$. We identify the ball $B^{T_{x_0}X}(0,\varepsilon)$ in the tangent space $T_{x_0}X$ to the ball $B^X(x_0,\varepsilon)$ in X by the exponential map (cf. [4, §4.1.3]). For $Z \in B^{T_{x_0}X}(0,\varepsilon)$ we identify (L_Z, h^L_Z), (E_Z, h^E_Z) to $(L_{x_0}, h^L_{x_0})$, $(E_{x_0}, h^E_{x_0})$ by parallel transport with respect to the connections ∇^L, ∇^E along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$. Then $P_p(x,x')$ induces a smooth section $(Z,Z') \mapsto P_{p,x_0}(Z,Z')$ of $\pi^* \operatorname{End}(E)$ over $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on x_0, with $\pi : TX \times_X TX \to X$ the natural projection. If dv_{TX} is the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, there exists a smooth positive function $\kappa_{x_0} : T_{x_0}X \to \mathbb{R}$, defined by

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1. \tag{1}$$

For $Z \in T_{x_0}X \cong \mathbb{R}^{2n}$, we denote $z_j = Z_{2j-1} + \sqrt{-1} Z_{2j-1}$ its complex coordinates, and set

$$\mathscr{P}(Z,Z') = \exp\left(-\frac{\pi}{2}\sum_{i=1}^n \left(|z_i|^2 + |z'_i|^2 - 2 z_i \overline{z}'_i\right)\right). \tag{2}$$

The near off-diagonal asymptotic expansion of the Bergman kernel in the form established [4, Theorem 4.1.24] is the following.

Theorem 1 *Given $k, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $p \geqslant 1$, $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma/\sqrt{p}$,*

$$\left| \frac{1}{p^n} P_p(Z,Z') - \sum_{r=0}^k \mathscr{F}_r(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') p^{-\frac{r}{2}} \right|_{\mathscr{C}^{m'}(X)} \leqslant C p^{-\frac{k+1}{2}}. \tag{3}$$

where $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$-norm with respect to the parameter x_0,

$$\mathscr{F}_r(Z,Z') = J_r(Z,Z') \mathscr{P}(Z,Z'), \tag{4}$$

$J_r(Z,Z') \in \operatorname{End}(E)_{x_0}$ are polynomials in Z, Z' with the same parity as r and $\deg J_r(Z,Z') \leqslant 3r$, whose coefficients are polynomials in R^{TX} (the curvature of the Levi-Civita connection on TX), R^E and their derivatives of order $\leqslant r-2$.

Remark 2 For the above properties of $J_r(Z,Z')$ see [4, Theorem 4.1.21 and end of §4.1.8]. They are also given in [2, Theorem 4.6, (4.107) and (4.117)]. Moreover, by [4, (1.2.19) and (4.1.28)], κ has a Taylor expansion with coefficients the derivatives of R^{TX}. As in [4, (4.1.101)] or [5, Lemma 3.1 and (3.27)] we have

$$\kappa(Z)^{-1/2} = 1 + \frac{1}{6}\operatorname{Ric}(z,\overline{z}) + \mathscr{O}(|Z|^3) = 1 + \frac{1}{3} R_{\ell\bar{k}k\bar{q}} z_\ell \overline{z}_q + \mathscr{O}(|Z|^3). \tag{5}$$

Note that a more powerful result than the near-off diagonal expansion from Theorem 1 holds. Namely, by [2, Theorem 4.18′] and [4, Theorem 4.2.1], the full off-diagonal expansion of the Bergman kernel holds (even for symplectic manifolds), i.e., an analogous result to (3) for $|Z|, |Z'| \leqslant \varepsilon$. This appears naturally in the proof of the diagonal expansion of the Bergman kernel on orbifolds in [2, (5.25)] or [4, (5.4.14), (5.4.23)].

Proposition 3 *The coefficient* $\mathscr{F}_1$ *vanishes identically*: $\mathscr{F}_1(Z, Z') = 0$ *for all* Z, Z'. *Therefore the coefficient of* $p^{-1/2}$ *in the expansion of* $p^{-n} P_p(p^{-1/2}Z, p^{-1/2}Z')$ *vanishes, so the latter converges to* $\mathscr{F}_0(Z, Z')$ *at rate* p^{-1} *as* $p \to \infty$.

Proof This is [4, Remark 4.1.26] or [5, (2.19)], see also [2, (4.107), (4.117), (5.4)]. □

When $E = \mathbb{C}$ with trivial metric, the vanishing of $\mathscr{F}_1$ was recently rediscovered in [3, Theorem 2.1] ($b_1(u, v) = 0$ therein). In [3] an equivalent formulation [6] of the expansion (3) is used, based on the analysis of the Szegö kernel from [1]. In [3, Theorem 2.1] further off-diagonal coefficients $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4$ are calculated in the K-coordinates. From [5, (3.22)], we see that the usual normal coordinates are K-coordinates up to order at least 3. This shows that the vanishing of $\mathcal{F}_1$ given by Proposition 3 implies the vanishing of b_1 calculated with the help of in K-coordinates. We wish to point out that we calculated in [5] the coefficients $\mathscr{F}_1, \dots, \mathscr{F}_4$ on the diagonal, using the off-diagonal expansion (3) and evaluating $\mathscr{F}_r$ for $Z = Z' = 0$. Thus, off-diagonal formulas for $\mathscr{F}_1, \dots, \mathscr{F}_4$ are implicitly contained in [5]. We show below how the coefficient $\mathscr{F}_2$ can be calculated in the framework of [5].

We use the notation in [5, (3.6)], then $\boldsymbol{r} = 8R_{m\overline{q}q\overline{m}}$ is the scalar curvature.

Proposition 4 *The coefficient* J_2 *in* (4) *is given by*

$$\begin{aligned} J_2(Z, Z') = & -\frac{\pi}{12} R_{k\overline{m}\ell\overline{q}} \big(z_k z_\ell \overline{z}_m \overline{z}_q + 6 z_k z_\ell \overline{z}'_m \overline{z}'_q - 4 z_k z_\ell \overline{z}_m \overline{z}'_q \\ & - 4 z_k z'_\ell \overline{z}'_m \overline{z}'_q + z'_k z'_\ell \overline{z}'_m \overline{z}'_q\big) \\ & - \frac{1}{3} R_{k\overline{m}q\overline{q}} \big(z_k \overline{z}_m + z'_k \overline{z}'_m\big) + \frac{1}{8\pi} \boldsymbol{r} + \frac{1}{\pi} R^E_{q\overline{q}} \\ & - \frac{1}{2} \big(z_\ell \overline{z}_q - 2 z_\ell \overline{z}'_q + z'_\ell \overline{z}'_q\big) R^E_{\ell\overline{q}}. \end{aligned} \tag{6}$$

Remark 5 Setting $Z = Z' = 0$ in (6) we obtain the coefficient $\boldsymbol{b}_1(x_0) = J_2(0, 0) = \frac{1}{8\pi}\boldsymbol{r} + \frac{1}{\pi} R^E_{q\overline{q}}$ of p^{-1} of the (diagonal) expansion of $p^{-n} P_p(x_0, x_0)$, cf. [4, Theorem 4.1.2].

Moreover, in order to obtain the coefficient of p^{-1} in the expansion (3) we multiply $\mathscr{F}_2(\sqrt{p}Z, \sqrt{p}Z')$ to the expansion of $\kappa(Z)^{-1/2}\kappa(Z')^{-1/2}$ with respect to the variable $\sqrt{p}Z$ obtained from (5). If $E = \mathbb{C}$ the result is a polynomial which is the sum of a homogeneous polynomial of order four and a constant, similar to [3].

Proof of Proposition 4 Set

$$\begin{aligned} &b_i = -2\frac{\partial}{\partial z_i} + \pi \overline{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial \overline{z}_i} + \pi z_i, \quad \mathscr{L} = \sum_{i=1}^n b_i b_i^+, \\ &\widetilde{\mathscr{O}}_2 = \frac{b_m b_q}{48\pi} R_{k\overline{m}\ell\overline{q}} z_k z_l + \frac{b_q}{3\pi} R_{\ell\overline{k}k\overline{q}} z_\ell - \frac{b_q}{12} R_{k\overline{m}\ell\overline{q}} z_k z_\ell \overline{z}'_m. \end{aligned} \tag{7}$$

By [4, (4.1.107)] or [5, (2.19)], we have

$$\mathscr{F}_{2,x_0} = -\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathscr{O}_2\mathscr{P} - \mathscr{P}\mathscr{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp}. \tag{8}$$

By [5, (4.1a), (4.7)] we have

$$\big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathscr{O}_2\mathscr{P}\big)(Z,Z') = \big(\mathscr{L}^{-1}\mathscr{O}_2\mathscr{P}\big)(Z,Z') = \left\{\widetilde{\mathscr{O}}_2 + \frac{b_q}{4\pi} R^E_{\ell\overline{q}} z_\ell\right\}\mathscr{P}(Z,Z'). \tag{9}$$

By the symmetry properties of the curvature [5, Lemma 3.1] we have

$$R_{k\overline{m}\ell\overline{q}} = R_{\ell\overline{m}k\overline{q}} = R_{k\overline{q}\ell\overline{m}} = R_{\ell\overline{q}k\overline{m}}, \quad \overline{R_{k\overline{m}\ell\overline{q}}} = R_{m\overline{k}q\overline{\ell}}, \qquad \big(R^E_{k\overline{q}}\big)^* = R^E_{q\overline{k}}. \tag{10}$$

We use throughout that $[g(z,\overline{z}), b_j] = 2\frac{\partial}{\partial z_j} g(z,\overline{z})$ for any polynomial $g(z,\overline{z})$ (cf. [5, (1.7)]). Hence from (10), we get

$$\begin{aligned} &b_q R_{k\overline{k}\ell\overline{q}} z_\ell = R_{k\overline{k}\ell\overline{q}} z_\ell b_q - 2R_{k\overline{k}q\overline{q}}, \\ &b_q R_{k\overline{m}\ell\overline{q}} z_k z_\ell = -4R_{k\overline{m}q\overline{q}} z_k + R_{k\overline{m}\ell\overline{q}} z_k z_\ell b_q, \\ &b_m b_q R_{k\overline{m}\ell\overline{q}} z_k z_l = R_{k\overline{m}\ell\overline{q}} z_k z_l b_m b_q - 8R_{k\overline{k}\ell\overline{q}} z_\ell b_q + 8R_{m\overline{m}q\overline{q}}. \end{aligned} \tag{11}$$

Thus from (7) and (11), we get

$$\widetilde{\mathscr{O}}_2 = \frac{1}{48\pi} R_{k\overline{m}\ell\overline{q}} z_k z_l \big(b_m - 4\pi\overline{z}'_m\big) b_q + \frac{1}{6\pi} R_{k\overline{k}\ell\overline{q}} z_\ell b_q - \frac{1}{2\pi} R_{m\overline{m}q\overline{q}} + \frac{1}{3} R_{k\overline{m}q\overline{q}} z_k \overline{z}'_m. \tag{12}$$

Now, $(b_i\mathscr{P})(Z,Z') = 2\pi(\overline{z}_i - \overline{z}'_i)\mathscr{P}(Z,Z')$, see [4, (4.1.108)] or [5, (4.2)]. Therefore

$$\begin{aligned} (\widetilde{\mathscr{O}}_2\mathscr{P})(Z,Z') &= \bigg[\frac{\pi}{12} R_{k\overline{m}\ell\overline{q}} z_k z_l \big(\overline{z}_m - 3\overline{z}'_m\big)\big(\overline{z}_q - \overline{z}'_q\big) + \frac{1}{3} R_{k\overline{k}\ell\overline{q}} z_\ell \big(\overline{z}_q - \overline{z}'_q\big) \\ &\qquad - \frac{1}{2\pi} R_{m\overline{m}q\overline{q}} + \frac{1}{3} R_{k\overline{m}q\overline{q}} z_k \overline{z}'_m\bigg]\mathscr{P}(Z,Z') \\ &= \bigg[\frac{\pi}{12} R_{k\overline{m}\ell\overline{q}} z_k z_\ell \big(\overline{z}_m - 3\overline{z}'_m\big)\big(\overline{z}_q - \overline{z}'_q\big) + \frac{1}{3} R_{k\overline{m}q\overline{q}} z_k \overline{z}_m \\ &\qquad - \frac{1}{2\pi} R_{m\overline{m}q\overline{q}}\bigg]\mathscr{P}(Z,Z'). \end{aligned} \tag{13}$$

We know that for an operator T we have $T^*(Z, Z') = \overline{T(Z', Z)}$, thus

$$(\widetilde{\mathcal{O}}_2 \mathscr{P})^*(Z, Z') = \left[\frac{\pi}{12} R_{k\overline{m}\ell\overline{q}} \overline{z}'_m \overline{z}'_q (z'_k - 3z_k)(z'_\ell - z_\ell) + \frac{1}{3} R_{k\overline{m}q\overline{q}} \overline{z}'_m z'_k \right.$$
$$\left. - \frac{1}{2\pi} R_{m\overline{m}q\overline{q}} \right] \mathscr{P}(Z, Z'). \tag{14}$$

We have $(\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp)^* = \mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2\mathscr{P}$ by [4, Theorem 4.1.8], so from (13) and (14), we obtain the factor of $R_{k\overline{m}\ell\overline{q}}$ in (6).

Let us calculate the contribution of the last term (curvature of E). We have

$$-\left(\frac{b_q}{4\pi} R^E_{\ell\overline{q}} z_\ell \mathscr{P} \right)(Z, Z') = \left(\frac{1}{2\pi} R^E_{q\overline{q}} - \frac{1}{2} z_\ell (\overline{z}_q - \overline{z}'_q) R^E_{\ell\overline{q}} \right) \mathscr{P}(Z, Z') \tag{15}$$

and by (10), we also have

$$-\left(\frac{b_q}{4\pi} R^E_{\ell\overline{q}} z_\ell \mathscr{P} \right)^*(Z, Z') = \left(\frac{1}{2\pi} R^E_{q\overline{q}} - \frac{1}{2} \overline{z}'_\ell (z'_q - z_q) R^E_{q\overline{\ell}} \right) \mathscr{P}(Z, Z'). \tag{16}$$

The contribution to J_2 of the term on E is thus given by the last two terms in (6). □

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