**Xiaonan Ma**
**Weiping Zhang**

# BERGMAN KERNELS AND SYMPLECTIC REDUCTION

*Xiaonan Ma*

Centre de Mathématiques Laurent Schwartz, UMR 7640 du CNRS, Ecole Polytechnique, 91128 Palaiseau Cedex, France.

Current address : Université Denis Diderot - Paris 7, UFR de Mathématiques, Case 7012, Site Chevaleret, 75205 Paris Cedex 13, France.

*E-mail :* ma@math.jussieu.fr

*Weiping Zhang*

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China.

*E-mail :* weiping@nankai.edu.cn

# BERGMAN KERNELS AND SYMPLECTIC REDUCTION

**Xiaonan Ma, Weiping Zhang**

***Abstract.*** — We generalize several recent results concerning the asymptotic expansions of Bergman kernels to the framework of geometric quantization and establish an asymptotic symplectic identification property. More precisely, we study the asymptotic expansion of the $G$-invariant Bergman kernel of the spin$^c$ Dirac operator associated with high tensor powers of a positive line bundle on a symplectic manifold admitting a Hamiltonian action of a compact connected Lie group $G$. We also develop a way to compute the coefficients of the expansion, and compute the first few of them, especially, we obtain the scalar curvature of the reduction space from the $G$-invariant Bergman kernel on the total space. These results generalize the corresponding results in the non-equivariant setting, which have played a crucial role in the recent work of Donaldson on stability of projective manifolds, to the geometric quantization setting.

As another kind of application, we establish some Toeplitz operator type properties in semi-classical analysis in the framework of geometric quantization.

The method we use is inspired by Local Index Theory, especially by the analytic localization techniques developed by Bismut and Lebeau.

***Résumé.*** — Nous généralisons des résultats récents sur le développement asymptotique du noyau de Bergman au cadre de quantification géométrique, et établissons une propriété d'identification asymptotique symplectique. Plus précisement, nous étudions le développement asymptotique du noyau de Bergman $G$-invariant de l'opérateur de Dirac spin$^c$ associé à une puissance tendant vers l'infini d'un fibré en droites positif sur une variété symplectique compacte munie d'une action hamiltonienne d'un groupe de Lie compact connexe. Nous développons aussi une façon pour calculer les coéfficients du développement, et nous calculons les premiers termes, en particulier, nous obtennons la courbure scalaire de la réduction symplectique à partir du noyau de Bergman $G$-invariant sur l'espace total. Ces résultats généralisent les résultats correspondants dans le cas non-équivariant, qui ont joué un rôle crucial dans un travail récent de Donaldson sur la stabilité de variétés projectives, au cadre de quantification géométrique.

Comme une application de notre développement, nous établissons aussi des propriétés de type de l'opérateur de Toeplitz en limite semi-classique dans le cadre de quantification géométrique.

Notre méthode est inspirée par la théorie de l'indice local, en particulier les techniques de localisation analytique développé par Bismut-Lebeau.

Dedicated to our teacher Jean-Michel Bismut

# CONTENTS

# CHAPTER 0

# INTRODUCTION

The study of the Bergman kernel is a classical subject in the theory of several complex variables, where usually it concerns the kernel function of the projection operator to an infinite dimensional Hilbert space. The recent interest of the analogue of this concept in complex geometry mainly started with the paper of Tian [**43**], which was in turn inspired by a question of Yau [**46**]. Here, the projection concerned is, however, onto a finite dimensional space.

Since [**43**], the Bergman kernel has been studied extensively in [**38**], [**14**], [**47**], [**25**], where the diagonal asymptotic expansion properties for high powers of an ample line bundle were established. Moreover, the coefficients in the asymptotic expansion encode geometric information of the underlying complex projective manifolds. This asymptotic expansion plays a crucial role in the recent work of Donaldson [**18**], where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to the Chow-Mumford stability.

In [**17**], [**28**], [**30**], Dai, Liu, Ma and Marinescu studied the full off-diagonal asymptotic expansion of the (generalized) Bergman kernel of the spin$^c$ Dirac operator and the renormalized Bochner–Laplacian associated to a positive line bundle on a compact symplectic manifold. As a by product, they gave a new proof of the results mentioned in the previous paragraph. They found also various applications therein, especially as was pointed out in [**30**], the full off-diagonal asymptotic expansion implies Toeplitz operator type properties. This approach is inspired by the Local Index Theory, especially by the analytic localization techniques of Bismut-Lebeau [**7**, §11]. We refer to the above papers as well as the recent book [**31**] for detail informations of the Bergman kernel on compact symplectic manifolds.

In this paper, we generalize some of the results in [**17**], [**28**] and [**30**] to the framework of geometric quantization, by studying the asymptotic expansion of the $G$-invariant Bergman kernel for high powers of an ample line bundle on symplectic manifolds admitting a Hamiltonian group action of a compact Lie group $G$.

To start with, let $(X,\omega)$ be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle $L$ over $X$ endowed with a Hermitian connection $\nabla^L$ with the property that

$$\frac{\sqrt{-1}}{2\pi}R^L = \omega, \tag{0.1}$$

where $R^L = (\nabla^L)^2$ is the curvature of $\nabla^L$.

Let $(E, h^E)$ be a Hermitian vector bundle on $X$ equipped with a Hermitian connection $\nabla^E$ and let $R^E$ denote the associated curvature.

Let $g^{TX}$ be a Riemannian metric on $X$. Let $\mathbf{J} : TX \to TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u,v) = g^{TX}(\mathbf{J}u, v) \tag{0.2}$$

for $u, v \in TX$.

Let $J$ be an almost complex structure such that

$$g^{TX}(Ju, Jv) = g^{TX}(u,v), \quad \omega(Ju, Jv) = \omega(u,v) \tag{0.3}$$

and that $\omega(\cdot, J\cdot)$ defines a metric on $TX$. Then $J$ commutes with $\mathbf{J}$ and $J = \mathbf{J}(-\mathbf{J}^2)^{-1/2}$ (cf. (2.8)).

Let $\nabla^{TX}$ be the Levi-Civita connection on $(TX, g^{TX})$ with curvature $R^{TX}$ and scalar curvature $r^X$. The connection $\nabla^{TX}$ induces a natural connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ with curvature $R^{\det}$, and the Clifford connection $\nabla^{\mathrm{Cliff}}$ on the Clifford module $\Lambda(T^{*(0,1)}X)$ with curvature $R^{\mathrm{Cliff}}$ (cf. Section 2.2).

The spin$^c$ Dirac operator $D_p$ acts on $\Omega^{0,\bullet}(X, L^p \otimes E) = \bigoplus_{q=0}^n \Omega^{0,q}(X, L^p \otimes E)$, the direct sum of spaces of $(0,q)$–forms with values in $L^p \otimes E$. We denote by $D_p^+$ the restriction of $D_p$ on $\Omega^{0,\mathrm{even}}(X, L^p \otimes E)$. The index of $D_p^+$ is defined by

$$\mathrm{Ind}(D_p^+) = \mathrm{Ker}\, D_p^+ - \mathrm{Coker}\, D_p^+. \tag{0.4}$$

Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\dim_{\mathbb{R}} G = n_0$. Suppose that $G$ acts on $X$ and its action on $X$ lifts on $L$ and $E$. Moreover, we assume the $G$-action preserves the above connections and metrics on $TX, L, E$ and $J$. Then $\mathrm{Ind}(D_p^+)$ is a virtual representation of $G$. Denote by $(\mathrm{Ker}\, D_p)^G$, $\mathrm{Ind}(D_p^+)^G$ the $G$-trivial components of $\mathrm{Ker}\, D_p$, $\mathrm{Ind}(D_p^+)$ respectively.

The action of $G$ on $L$ induces naturally a moment map $\mu : X \to \mathfrak{g}^*$ (cf. (2.16)). We assume that $0 \in \mathfrak{g}^*$ is a regular value of $\mu$.

Set $P = \mu^{-1}(0)$. Then the Marsden-Weinstein symplectic reduction $(X_G = P/G, \omega_{X_G})$ is a symplectic orbifold ($X_G$ is smooth if $G$ acts freely on $P$).

Moreover, $(L, \nabla^L)$, $(E, \nabla^E)$ descend to $(L_G, \nabla^{L_G})$, $(E_G, \nabla^{E_G})$ over $X_G$ so that the corresponding curvature condition $\frac{\sqrt{-1}}{2\pi}R^{L_G} = \omega_G$ holds (cf. [**21**]). The $G$-invariant almost complex structure $J$ also descends to an almost complex structure $J_G$ on $TX_G$, and $h^L, h^E, g^{TX}$ descend to $h^{L_G}$, $h^{E_G}, g^{TX_G}$ respectively.

One can construct the corresponding spin$^c$ Dirac operator $D_{G,p}$ on $X_G$.

Assume for simplicity that $G$ acts freely on $P$.

The geometric quantization conjecture of Guillemin-Sternberg [**21**] can be stated as follows: for any $p > 0$,

$$\dim\left(\mathrm{Ind}(D_p^+)^G\right) = \dim\Big(\mathrm{Ind}(D_{G,p}^+)\Big), \tag{0.5}$$

holds when $E$ is the trivial bundle $\mathbb{C}$ on $X$.

When $G$ is abelian, this conjecture was proved by Meinrenken [**34**] and Vergne [**45**]. The remaining nonabelian case was proved by Meinrenken [**35**] using the symplectic cut techniques of Lerman, and by Tian and Zhang [**44**] using analytic localization techniques.

More generally, by a result of Tian and Zhang [**44**, Theorem 0.2], for any general vector bundle $E$ as above, there exists $p_0 > 0$ such that for any $p \geqslant p_0$, (0.5) still holds.

On the other hand, by [**27**, Theorem 2.5] (cf. (2.15)), which is a direct consequence of the Lichnerowicz formula for $D_p$, for $p$ large enough, both Coker $D_p^+$ and Coker $D_{G,p}^+$ are null (cf. also [**10**], [**13**]). Thus there exists $p_0 > 0$ such that for any $p \geqslant p_0$,

$$\begin{aligned}\dim(\mathrm{Ker}\, D_p)^G &= \dim(\mathrm{Ker}\, D_{G,p}) = \dim\Big(\mathrm{Ind}(D_{G,p}^+)\Big)\\ &= \int_{X_G} \mathrm{Td}(TX_G)\,\mathrm{ch}(L_G^p\otimes E_G)\\ &= \mathrm{rk}(E)\int_{X_G} \frac{(p\,c_1(L_G))^{n-n_0}}{(n-n_0)!}\\ &\quad + \int_{X_G}\Big(c_1(E_G) + \frac{\mathrm{rk}(E)}{2}c_1(TX_G)\Big)\frac{(p\,c_1(L_G))^{n-n_0-1}}{(n-n_0-1)!} + \mathscr{O}(p^{n-n_0-2}),\end{aligned} \tag{0.6}$$

where $\mathrm{ch}(\cdot), c_1(\cdot), \mathrm{Td}(\cdot)$ are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles ($TX_G$ is a complex vector bundle with complex structure $J_G$).

Set $E_p := \Lambda(T^{*(0,1)}X)\otimes L^p\otimes E$. Let $\langle\ \ \rangle$ be the $L^2$-scalar product on $\Omega^{0,\bullet}(X, L^p\otimes E) = \mathscr{C}^\infty(X, E_p)$ induced by $g^{TX}, h^L, h^E$ as in (1.19).

Let $P_p^G$ be the orthogonal projection from $(\Omega^{0,\bullet}(X, L^p\otimes E), \langle\ \ \rangle)$ on $(\mathrm{Ker}\, D_p)^G$. The $G$-invariant Bergman kernel is $P_p^G(x,x')$ $(x,x'\in X)$, the smooth kernel of $P_p^G$ with respect to the Riemannian volume form $dv_X(x')$.

Let $\mathrm{pr}_1$ and $\mathrm{pr}_2$ be the projections from $X\times X$ onto the first and the second factor $X$ respectively. Then $P_p^G(x,x')$ is a smooth section of $\mathrm{pr}_1^*(E_p)\otimes \mathrm{pr}_2^*(E_p^*)$ on $X\times X$. In particular, $P_p^G(x,x)\in \mathrm{End}(E_p)_x = \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_x$.

The $G$-invariant Bergman kernel $P_p^G(x,x')$ is an analytic version of $(\mathrm{Ker}\, D_p)^G$. In view of (0.6), it is natural to expect that the kernel $P_p^G(x,x')$ should be closely related to the corresponding Bergman kernel on the symplectic reduction $X_G$. The purpose of this paper is to study the asymptotic expansion of the $G$-invariant Bergman kernel

$P_p^G(x,x')$ as $p\to\infty$, and we will relate it to the asymptotic expansion of the Bergman kernel on the symplectic reduction $X_G$.

Let $d^X(x,x')$ be the Riemannian distance between $x$, $x'\in X$.

In Section 2.4, we prove the following result which allows us to reduce our problem as a problem near $P=\mu^{-1}(0)$, it works without the assumption on the freeness of the action of $G$ on $P$.

***Theorem 0.1.*** — *For any open $G$-neighborhood $U$ of $P$ in $X$, $\varepsilon_0>0$, $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ (depending on $U$, $\varepsilon_0$) such that for $p\geqslant 1$, $x,x'\in X, d^X(Gx,x')\geqslant\varepsilon_0$ or $x,x'\in X\setminus U$,*

$$|P_p^G(x,x')|_{\mathscr{C}^m}\leqslant C_{l,m}p^{-l}, \tag{0.7}$$

*where $\mathscr{C}^m$ is the $\mathscr{C}^m$-norm induced by $\nabla^L,\nabla^E$, $\nabla^{TX}$, $h^L,h^E$ and $g^{TX}$.*

Let $U$ be an open $G$-neighborhood of $\mu^{-1}(0)$ such that $G$ acts freely on $U$.

For any $G$-equivariant vector bundle $(F,\nabla^F)$ on $U$, we denote by $F_B$ the bundle on $U/G=B$ induced naturally by $G$-invariant sections of $F$ on $U$. The connection $\nabla^F$ induces canonically a connection $\nabla^{F_B}$ on $F_B$. Let $R^{F_B}$ be its curvature. Let

$$\mu^F(K)=\nabla^F_{K^X}-L_K\in \operatorname{End}(F) \tag{0.8}$$

for $K\in\mathfrak{g}$ and $K^X$ the corresponding vector field on $U$.

Note that $P_p^G\in(\mathscr{C}^\infty(U\times U,\mathrm{pr}_1^*E_p\otimes\mathrm{pr}_2^*E_p^*))^{G\times G}$, thus we can view $P_p^G(x,x')$ $(x,\,x'\in U)$ as a smooth section of $\mathrm{pr}_1^*(E_p)_B\otimes\mathrm{pr}_2^*(E_p^*)_B$ on $B\times B$.

Let $g^{TB}$ be the Riemannian metric on $U/G=B$ induced by $g^{TX}$. Let $\nabla^{TB}$ be the Levi-Civita connection on $(TB,g^{TB})$ with curvature $R^{TB}$. Let $N_G$ be the normal bundle to $X_G$ in $B$. We identify $N_G$ with the orthogonal complement of $TX_G$ in $(TB|_{X_G},g^{TB})$.

Let $g^{TX_G}$, $g^{N_G}$ be the metrics on $TX_G$, $N_G$ induced by $g^{TB}$ respectively.

Let $P^{TX_G}$, $P^{N_G}$ be the orthogonal projections from $TB|_{X_G}$ on $TX_G$, $N_G$ respectively. Set

$$\begin{aligned}&\nabla^{N_G}=P^{N_G}(\nabla^{TB}|_{X_G})P^{N_G},\quad \nabla^{TX_G}=P^{TX_G}(\nabla^{TB}|_{X_G})P^{TX_G},\\ &{}^0\nabla^{TB}=\nabla^{TX_G}\oplus\nabla^{N_G},\quad A=\nabla^{TB}|_{X_G}-{}^0\nabla^{TB}.\end{aligned} \tag{0.9}$$

Then $\nabla^{N_G}$, ${}^0\nabla^{TB}$ are Euclidean connections on $N_G$, $TB|_{X_G}$ respectively, $\nabla^{TX_G}$ is the Levi-Civita connection on $(TX_G,g^{TX_G})$, and $A$ is the associated second fundamental form.

Denote by $\mathrm{vol}(Gx)$ $(x\in U)$ the volume of the orbit $Gx$ equipped with the metric induced by $g^{TX}$. Following [**44**, (3.10)], let $h(x)$ be the function on $U$ defined by

$$h(x)=(\mathrm{vol}(Gx))^{1/2}. \tag{0.10}$$

Then $h$ reduces to a function on $B$.

Denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X)\otimes E$ onto $\mathbb{C}\otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X)\otimes E = \mathbb{C}\otimes E\oplus\Lambda^{>0}(T^{*(0,1)}X)\otimes E$, and $I_{\mathbb{C}\otimes E_B}$ the corresponding projection on $B$.

In the whole paper, for any $x_0 \in X_G$, $Z \in T_{x_0}B$, we write $Z = Z^0 + Z^\perp$, with $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$.

Let $\tau_{Z^0}Z^\perp \in N_{G,\exp^{X_G}_{x_0}(Z^0)}$ be the parallel transport of $Z^\perp$ with respect to the connection $\nabla^{N_G}$ along the geodesic in $X_G$, $[0,1]\ni t \to \exp^{X_G}_{x_0}(tZ^0)$.

For $\varepsilon_0 > 0$ small enough, we identify $Z \in T_{x_0}B$, $|Z| < \varepsilon_0$ with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp) \in B$. Then for $x_0 \in X_G$, $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| < \varepsilon_0$, the map $\Psi : TB|_{X_G} \times TB|_{X_G} \to B \times B$,

$$\Psi(Z, Z') = (\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp), \exp^B_{\exp^{X_G}_{x_0}(Z'^0)}(\tau_{Z'^0}Z'^\perp))$$

is well defined.

We identify $(E_p)_{B,Z}$ to $(E_p)_{B,x_0}$ by using parallel transport with respect to $\nabla^{(E_p)_B}$ along $[0,1]\ni u \to uZ$.

Let $\pi_B : TB|_{X_G} \times TB|_{X_G} \to X_G$ be the natural projection from the fiberwise product of $TB|_{X_G}$ on $X_G$ onto $X_G$.

From Theorem 0.1, we only need to understand $P^G_p\circ\Psi$, and under our identification, $P^G_p \circ \Psi(Z, Z')$ is a smooth section of

$$\pi^*_B(\mathrm{End}(E_p)_B) = \pi^*_B(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$$

on $TB|_{X_G} \times TB|_{X_G}$.

Let $|\ \ |_{\mathscr{C}^{m'}(X_G)}$ be the $\mathscr{C}^{m'}$-norm on $\mathscr{C}^\infty(X_G, \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$ induced by $\nabla^{\mathrm{Cliff}_B}$, $\nabla^{E_B}$, $h^E$ and $g^{TX}$. The norm $|\ \ |_{\mathscr{C}^{m'}(X_G)}$ induces naturally a $\mathscr{C}^{m'}$-norm along $X_G$ on $\mathscr{C}^\infty(TB|_{X_G} \times TB|_{X_G}, \pi^*_B(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B))$, we still denote it by $|\ \ |_{\mathscr{C}^{m'}(X_G)}$.

Let $dv_B, dv_{X_G}, dv_{N_G}$ be the Riemannian volume forms on $(B, g^{TB})$, $(X_G, g^{TX_G})$, $(N_G, g^{N_G})$ respectively. Let $\kappa \in \mathscr{C}^\infty(TB|_{X_G}, \mathbb{R})$, with $\kappa = 1$ on $X_G$, be defined by that for $Z \in T_{x_0}B$, $x_0 \in X_G$,

$$dv_B(x_0, Z) = \kappa(x_0, Z)dv_{T_{x_0}B}(Z) = \kappa(x_0, Z)dv_{X_G}(x_0)dv_{N_{G,x_0}}. \tag{0.11}$$

The following result is one of the main results of this paper.

***Theorem 0.2.*** — *Assume that $G$ acts freely on $\mu^{-1}(0)$ and $\mathbf{J} = J$ on $\mu^{-1}(0)$. Then there exist $\mathscr{Q}_r(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}$ ($x_0 \in X_G, r \in \mathbb{N}$), polynomials in $Z, Z'$ with the same parity as $r$, whose coefficients are polynomials in $A$, $R^{TB}$, $R^{\mathrm{Cliff}_B}$, $R^{E_B}$, $\mu^E$, $\mu^{\mathrm{Cliff}}$ (resp. $r^X$, $R^{\det}$, $R^E$; resp. $h$, $R^L$, $R^{L_B}$; resp. $\mu$) and their derivatives at $x_0$ to order $r-1$ (resp. $r-2$; resp. $r$; resp. $r+1$), such that if we denote by*

$$P^{(r)}_{x_0}(Z, Z') = \mathscr{Q}_r(Z, Z')P(Z, Z'), \quad \mathscr{Q}_0(Z, Z') = I_{\mathbb{C}\otimes E_B}, \tag{0.12}$$

*with*

$$P(Z,Z')=\exp\Big(-\frac{\pi}{2}|Z^0-Z'^0|^2-\pi\sqrt{-1}\left\langle J_{x_0}Z^0,Z'^0\right\rangle\Big)\times 2^{\frac{n_0}{2}}\exp\Big(-\pi\big(|Z^\perp|^2+|Z'^\perp|^2\big)\Big), \tag{0.13}$$

*then there exists $C''>0$ such that for any $k,m,m',m''\in\mathbb{N}$, there exists $C>0$ such that for $x_0\in X_G$, $Z,Z'\in T_{x_0}B$, $|Z|,|Z'|\leqslant\varepsilon_0$,* [1]

$$(1+\sqrt{p}|Z^\perp|+\sqrt{p}|Z'^\perp|)^{m''}\sup_{|\alpha|+|\alpha'|\leqslant m}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\left(p^{-n+\frac{n_0}{2}}(h\kappa^{\frac12})(Z)(h\kappa^{\frac12})(Z')P_p^G\circ\Psi(Z,Z')-\sum_{r=0}^k P_{x_0}^{(r)}(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}\right)\right|_{\mathscr{C}^{m'}(X_G)}$$
$$\leqslant Cp^{-(k+1-m)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m}\exp(-\sqrt{C''}\sqrt{p}|Z-Z'|)+\mathscr{O}(p^{-\infty}). \tag{0.14}$$

*Furthermore, the expansion is uniform in the following sense: for any fixed $k,m,m',m''\in\mathbb{N}$, assume that the derivatives of $g^{TX}$, $h^L$, $\nabla^L$, $h^E$, $\nabla^E$,and $J$ with order $\leqslant 2n+k+m+m'+5$ run over a set bounded in the $\mathscr{C}^{m'}$–norm taken with respect to the parameters and, moreover, $g^{TX}$ runs over a set bounded below, then the constant $C$ is independent of $g^{TX}$; and the $\mathscr{C}^{m'}$-norm in* (0.14) *includes also the derivatives on the parameters.*

In (0.14), the term $\mathscr{O}(p^{-\infty})$ means that for any $l,l_1\in\mathbb{N}$, there exists $C_{l,l_1}>0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1}p^{-l}$.

It is interesting to see that the kernel $P(Z,Z')$ is the product of two kernels : along $T_{x_0}X_G$, it is the classical Bergman kernel on $T_{x_0}X_G$ with complex structure $J_{x_0}$, while along $N_G$, it is the kernel of a harmonic oscillator on $N_{G,x_0}$.

***Remark 0.3***. — i) Theorem 0.2 is a special case of Theorem 2.23 where we do not assume $\mathbf{J}=J$ on $P=\mu^{-1}(0)$. In Theorem 3.2, we get explicit informations on $P^{(r)}$ when $\mathbf{J}$ verifies (3.2).

ii) If $G$ does not act freely on $P$, then $X_G$ is an orbifold. In Section 4.1, we explain how to modify our arguments to get the asymptotic expansion, Theorem 4.1. Analogous to the usual orbifold case [**17**, (5.27)], $P_p^G(x,x)(x\in P)$ does not have a uniform asymptotic expansion if the singular set of $X_G$ is not empty.

iii) Let $\mathcal{V}$ be an irreducible representation of $G$, let $P_p^{\mathcal{V}}$ be the orthogonal projection from $\Omega^{0,\bullet}(X,L^p\otimes E)$ on $\mathrm{Hom}_G(\mathcal{V},\mathrm{Ker}\,D_p)\otimes\mathcal{V}\subset\mathrm{Ker}\,D_p$. In Section 4.2, we get the asymptotic expansion of the kernel $P_p^{\mathcal{V}}(x,x')$ from Theorems 0.1, 0.2.

iv) When $G=\{1\}$, Theorem 0.2 is [**17**, Theorem 4.18′].

[1] In the exponential factor of [**32**, (7)], we missed $m'$ as in the last line of (0.14) here.

v) If we take $Z = Z' = 0$ in (0.14), then we get for $x_0 \in X_G$,

$$P^{(0)}_{x_0}(0,0) = 2^{\frac{n_0}{2}} I_{\mathbb{C}\otimes E_B}, \tag{0.15}$$

and

$$\Big| p^{-n+\frac{n_0}{2}} h^2(x_0) P^G_p(x_0,x_0) - \sum_{r=0}^{k} P^{(2r)}_{x_0}(0,0)p^{-r} \Big|_{\mathscr{C}^{m'}(X_G)} \leqslant Cp^{-k-1}. \tag{0.16}$$

In Section 4.3, we show that (0.15) and (0.16) are direct consequences of the full off-diagonal asymptotic expansion of the Bergman kernel [**17**, Theorem 4.18′]. In fact, one possible way to get Theorem 0.2 is to average the full off-diagonal asymptotic expansion of the Bergman kernel on $X$ [**17**, Theorem 4.18′] with respect to a Haar measure on $G$. However, we do not know how to get the full off-diagonal expansion, especially the fast decay along $N_G$ in (0.14) in this way.

In this paper we will apply the analytic localization techniques to prove Theorem 0.2, and this method also gives us an effective way to compute the coefficients in the asymptotic expansion (cf. §3.2). The key observation is that the $G$-invariant Bergman kernel is exactly the kernel of the orthogonal projection to the zero space of a deformation of $D^2_p$ by the Casimir operator (i.e., to consider $D^2_p - p\mathrm{Cas}$). This plays an essential role in proving Theorems 0.1, 0.2.

Let $\mathscr{I}_p$ be a section of $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B$ on $X_G$ defined by

$$\mathscr{I}_p(x_0) = \int_{Z\in N_G, |Z|\leqslant\varepsilon_0} h^2(x_0,Z) P^G_p \circ \Psi((x_0,Z),(x_0,Z))\kappa(x_0,Z) dv_{N_G}(Z). \tag{0.17}$$

By Theorem 0.1, modulo $\mathscr{O}(p^{-\infty})$, $\mathscr{I}_p(x_0)$ does not depend on $\varepsilon_0$, and

$$\begin{aligned} \dim(\mathrm{Ker}\, D_p)^G &= \int_X \mathrm{Tr}[P^G_p(y,y)]dv_X(y) \\ &= \int_U \mathrm{Tr}[P^G_p(y,y)]dv_X(y) + \mathscr{O}(p^{-\infty}) \\ &= \int_B h^2(y)\,\mathrm{Tr}[P^G_p(y,y)]dv_B(y) + \mathscr{O}(p^{-\infty}) \\ &= \int_{X_G} \mathrm{Tr}[\mathscr{I}_p(x_0)]dv_{X_G}(x_0) + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{0.18}$$

A direct consequence of Theorem 0.2 is the following corollary.

***Corollary 0.4***. — *Taking $Z = Z' \in N_{G,x_0}$, $m = 0$ in* (0.14)*, we get*

$$\begin{aligned} \Big| p^{-n+\frac{n_0}{2}} (h^2\kappa)(Z) P^G_p(Z,Z) - \sum_{r=0}^{k} P^{(r)}_{x_0}(\sqrt{p}Z,\sqrt{p}Z)p^{-r/2} \Big|_{\mathscr{C}^{m'}(X_G)} \\ \leqslant Cp^{-(k+1)/2}(1+\sqrt{p}|Z|)^{-m''} + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{0.19}$$

*In particular, there exist $\Phi_r \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}$ ($r\in\mathbb{N}$) which are polynomials in $A$, $R^{TB}$, $R^{\mathrm{Cliff}_B}$, $R^{E_B}$, $\mu^E$, $\mu^{\mathrm{Cliff}}$, (resp. $r^X$, $R^{\det}$, $R^E$; resp. $h$, $R^{L_B}$, $R^L$;*

*resp.* $\mu$*), and their derivatives at* $x_0$ *up to order* $2r-1$ *(resp.* $2r-2$*; resp.* $2r$*; resp.* $2r+1$*), and* $\Phi_0 = I_{\mathbb{C}\otimes E_B}$*, such that for any* $k, m' \in \mathbb{N}$*, there exists* $C_{k,m'} > 0$ *such that for any* $x_0 \in X_G$*,* $p \in \mathbb{N}$*,*

$$\left| p^{-n+n_0} \mathscr{I}_p(x_0) - \sum_{r=0}^{k} \Phi_r(x_0) p^{-r} \right|_{\mathscr{C}^{m'}} \leqslant C_{k,m'} p^{-k-1}. \tag{0.20}$$

In the rest of Introduction, we will specify our results in the Kähler case.

We suppose now that $(X, \omega, J)$ is a compact Kähler manifold and $\mathbf{J} = J$ on $X$. Assume also that $(L, h^L, \nabla^L)$, $(E, h^E, \nabla^E)$ are holomorphic Hermitian vector bundles with holomorphic Hermitian connections, and the action of $G$ on $X, L, E$ is holomorphic.

Let $H^j(X, L^p \otimes E)$ $(0 \leqslant j \leqslant n)$ be the Dolbeault cohomology of the Dolbeault complex $(\Omega^{0,\bullet}(X, L^p \otimes E), \overline{\partial}^{L^p\otimes E})$ of $X$ with values in $L^p \otimes E$. Espeically, $H^0(X, L^p \otimes E)$ is the space of the holomorphic sections of $L^p \otimes E$ on $X$.

Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$, then

$$D_p = \sqrt{2}(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}), \tag{0.21}$$

and

$$D_p^2 = 2\left(\overline{\partial}^{L^p\otimes E}\overline{\partial}^{L^p\otimes E,*} + \overline{\partial}^{L^p\otimes E,*}\overline{\partial}^{L^p\otimes E}\right) \tag{0.22}$$

preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p \otimes E)$.

By the Kodaira vanishing theorem, for $p$ large enough,

$$(\operatorname{Ker} D_p)^G = H^0(X, L^p \otimes E)^G. \tag{0.23}$$

Thus for $p$ large enough, $P_p^G(x, x') \in (L^p \otimes E)_x \otimes (L^p \otimes E)^*_{x'}$ and so $P_p^G(x, x) \in \operatorname{End}(E_x)$, $\mathscr{I}_p(x_0) \in \operatorname{End}(E_{x_0})$. In particular, in (0.15),

$$P_{x_0}^{(0)}(0, 0) = 2^{\frac{n_0}{2}} \operatorname{Id}_{E_G}. \tag{0.24}$$

***Remark 0.5***. — In the special case of $E = \mathbb{C}$, $P_p^G(x_0, x_0)$ is a non-negative function on $X_G$, and (0.16) has been proved in [**36**, Theorem 1] (without obtaining the informations on $P_{x_0}^{(2r)}(0,0)$), while in [**37**, Theorem 1], it was claimed that $P_{x_0}^{(0)}(0,0) = 1$. In [**36**, Prop. 1], Paoletti showed that for any $l \in \mathbb{N}$, there is $C > 0$ such that for any $p$, $|P_p^G(x, x)| \leqslant Cp^{-l}$ uniformly on any compact subset of $X \setminus (\mu^{-1}(0) \cup R)$, with $R$ the subset of unstable points of the action of $G$. In [**37**], some Toeplitz operator type properties on $X_G$ were also claimed to follow from the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [**11**], Boutet de Monvel-Sjöstrand [**12**] and Shiffman-Zelditch [**40**]. If we suppose moreover that $G$ is a torus, Charles [**15**] has also a different version on the Toeplitz operator type properties on $X_G$.

In Section 4.5, we will show that Theorem 0.2 implies properties of Toeplitz operators on $X_G$ (which also hold in the symplectic case). In particular, we recover the results on Toeplitz operators from [**15**], [**37**].

Let $\widetilde{h}$ denote the restriction to $X_G$ of the function $h$ defined in (0.10).

The second main result of this paper is that we can in fact obtain the scalar curvature $r^{X_G}$ on the symplectic reduction $X_G$ from $\mathscr{I}_p$.

We will use the following notation: when a subscript index appears two times in a formula, we sum up with this index.

***Theorem 0.6***. — *If $(X,\omega)$ is a compact Kähler manifold and $L, E$ are holomorphic vector bundles with holomorphic Hermitian connections $\nabla^L, \nabla^E$, $\mathbf{J} = J$, and $G$ acts freely on $\mu^{-1}(0)$, then for $p$ large enough, $\mathscr{I}_p(x_0) \in \mathrm{End}(E_G)_{x_0}$, and in* (0.20), *$\Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$ are polynomials in $A$, $R^{TB}$, $R^{E_B}$, $\mu^E$, $R^E$ (resp. $h$, $R^{L_B}$; resp. $\mu$) and their derivatives at $x_0$ to order $2r-1$ (resp. $2r$; resp. $2r+1$), and $\Phi_0 = \mathrm{Id}_{E_G}$. Moreover*

$$\Phi_1(x_0) = \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{3}{4\pi}\Delta_{X_G} \log \widetilde{h} + \frac{1}{2\pi} R^{E_G}_{x_0}(w^0_j, \overline{w}^0_j). \tag{0.25}$$

*Here $r^{X_G}$ is the Riemannian scalar curvature of $(TX_G, g^{TX_G})$, $\Delta_{X_G}$ is the Bochner-Laplacian on $X_G$ (cf.* (1.21)*), and $\{w^0_j\}$ is an orthonormal basis of $T^{(1,0)}X_G$.*

Since the non-equivariant version of this result has already played a crucial role in the work of Donaldson mentioned before, we have reason to believe that Theorem 0.6 might also play a role in the study of stability properties of projective manifolds. Indeed, as Donaldson usually interprets his results in the framework of geometric quantization, this seems likely to be so.

We recover (0.6) from (0.25) after taking the trace, and then the integration on $X_G$. Thus (0.25) is a local version of (0.6) in the spirit of the Local Index Theory. The appearance of the term $\frac{3}{4\pi}\Delta_{X_G} \log \widetilde{h}$ is unexpected.

Let $T$ be the torsion of the connection ${}^0\nabla^{TX}$ in (1.2) on $U$. The curvature $\Theta$ of the principal bundle $U \to B$ relates to the torsion $T$ by (1.6).

Following (3.6) and (5.21), we choose $\{e^\perp_j\}$ to be an orthonormal basis of $N_{G,x_0}$ and $\{\frac{\partial}{\partial z^0_j}\} \in T^{(1,0)}_{x_0} X_G$ to be the holomorphic basis of the normal coordinate on $X_G$, and define $\mathcal{T}_{klm}, \widetilde{\mathcal{T}}_{jkl}$ as in (5.14). In particular, by Remark 5.3, $\widetilde{\mathcal{T}}_{jkl} = 0$ if $G$ is abelian.

The $G$-invariant section $\widetilde{\mu}^E$ of $TY \otimes \mathrm{End}(E)$ on $U$ is defined by (1.13) and (1.14).

If there is no other specific notification in the next formula (0.26), when we meet the operation $|\quad|^2$, we will first do this operation, then take the sum of the indices.

***Theorem 0.7.*** — *Under the assumption of Theorem 0.6, for $p>0$ large enough, $P^G_p(x,x)\in \operatorname{End}(E_x)$ and $P^{(r)}_{x_0}(0,0)\in \operatorname{End}(E_{x_0})$. Moreover,*

$$
\begin{aligned}
(0.26)\quad P^{(2)}_{x_0}(0,0) = 2^{\frac{n_0}{2}} \Big\{ & \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{1}{\pi} R^{E_G}\big(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial \overline{z}^0_j}\big) + \frac{1}{\pi}\Delta_{X_G}\log \widetilde{h} \\
& - \frac{3}{8\pi}\nabla_{e^\perp_k}\nabla_{e^\perp_k}\log h - \frac{2}{\pi}\sqrt{-1}\nabla_{JT(\frac{\partial}{\partial z^0_j},\frac{\partial}{\partial \overline{z}^0_l})}\log h - \frac{3}{\pi}\Big|\nabla_{\frac{\partial}{\partial \overline{z}^0_j}}\log h\Big|^2 \\
& - \frac{5}{4\pi}\Big|\nabla_{e^\perp_j}\log h\Big|^2 + \frac{1}{2\pi}\big|T(e^\perp_k,\tfrac{\partial}{\partial \overline{z}^0_j})\big|^2 + \frac{1}{2\pi}\Big|T(\tfrac{\partial}{\partial z^0_i},\tfrac{\partial}{\partial \overline{z}^0_j})\Big|^2 \\
& - \frac{1}{2\pi}\Big|\sum_j T(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial \overline{z}^0_j})\Big|^2 + \frac{1}{24\pi}\mathcal{T}^2_{klm} + \frac{1}{2^6\pi}\widetilde{\mathcal{T}}_{ijk}(-\widetilde{\mathcal{T}}_{kji}+3\widetilde{\mathcal{T}}_{ijk}) \\
& + \frac{1}{2\pi}\left\langle \widetilde{\mu}^E_{x_0},\widetilde{\mu}^E_{x_0}\right\rangle_{g^{TY}} + \frac{1}{\pi}\Big\langle \widetilde{\mu}^E, T(\tfrac{\partial}{\partial z^0_l},\tfrac{\partial}{\partial \overline{z}^0_l})\Big\rangle \\
& + \frac{3\sqrt{-1}}{2\pi}\left\langle \widetilde{\mu}^E, Je^\perp_k\right\rangle \nabla_{e^\perp_k}\log h + \frac{\sqrt{-1}}{4\pi}\Big\langle Je^\perp_k, \nabla^{TY}_{e^\perp_k}\widetilde{\mu}^E\Big\rangle \Big\}.
\end{aligned}
$$

***Remark 0.8.*** — Certainly, if we only assume that $\mathbf{J}=J$ on a neighborhood $U$ of $P=\mu^{-1}(0)$, then we still have $\Phi_r(x_0)\in \operatorname{End}(E_G)_{x_0}$, as we work on the kernel of the Dirac operator $D_p$. Set $\mathscr{I}_{p,0}=I_{\mathbb{C}\otimes E_G}\mathscr{I}_p I_{\mathbb{C}\otimes E_G}$, the component of $\mathscr{I}_p$ on $\mathbb{C}\otimes E_G$. As the computation is local, we still have Theorem 0.6 with $\mathscr{I}_p$ replaced by $\mathscr{I}_{p,0}$ and $\mathscr{I}_p-\mathscr{I}_{p,0}=\mathscr{O}(p^{-\infty})$ (cf. (5.19)). If we only work on the $\overline{\partial}$-operator, i.e. the holomorphic sections, in Section 5.5, we explain how to reduce the case of general $\mathbf{J}$ to the case $\mathbf{J}=J$. Same remark holds for $P^G_p(x_0,x_0)$.

Let $i: P\hookrightarrow X$ be the natural injection.

Let $\pi_G:\mathscr{C}^\infty(P,L^p\otimes E)^G\to \mathscr{C}^\infty(X_G,L^p_G\otimes E_G)$ be the natural identification.

By a result of Zhang [**48**, Theorem 1.1 and Proposition 1.2], one sees that for $p$ large enough, the map

$$\pi_G\circ i^*:\mathscr{C}^\infty(X,L^p\otimes E)^G\to \mathscr{C}^\infty(X_G,L^p_G\otimes E_G)$$

induces a natural isomorphism

$$\sigma_p=\pi_G\circ i^*: H^0(X,L^p\otimes E)^G\to H^0(X_G,L^p_G\otimes E_G). \tag{0.27}$$

(When $E=\mathbb{C}$, this result was first proved in [**21**, Theorem 3.8].)

The following result is a symplectic version of the above isomorphism which is proved in Corollary 4.13, as a simple application of the Toeplitz operator type properties proved in that subsection. It might be regarded as an "asymptotic symplectic quantization identification", generalizing the corresponding holomorphic identification (0.27).

***Theorem 0.9.*** — *If $X$ is a compact symplectic manifold and $\mathbf{J}=J$, then the natural map $\sigma_p:(\operatorname{Ker}D_p)^G\to \operatorname{Ker}D_{G,p}$ defined in (4.88) is an isomorphism for $p$ large enough.*

Now we go back to the holomorphic situation.

Let $\langle\,,\,\rangle_{L^p_G\otimes E_G}$ be the metric on $L^p_G\otimes E_G$ induced by $h^{L_G}$ and $h^{E_G}$.

In view of [**44**, (3.54)], the natural Hermitian product on $\mathscr{C}^\infty(X_G, L^p_G\otimes E_G)$ is the following weighted Hermitian product $\langle\,,\,\rangle_{\widetilde{h}}$:

$$\langle s_1, s_2\rangle_{\widetilde{h}} = \int_{X_G} \langle s_1, s_2\rangle_{L^p_G\otimes E_G}(x_0)\widetilde{h}^2(x_0)\, dv_{X_G}(x_0). \tag{0.28}$$

In fact, $\pi_G : (\mathscr{C}^\infty(P, L^p\otimes E)^G, \langle\,,\,\rangle) \to (\mathscr{C}^\infty(X_G, L^p_G\otimes E_G), \langle\,,\,\rangle_{\widetilde{h}})$ is an isometry.

We still denote by $\langle\quad\rangle$ the scalar product on $H^0(X, L^p\otimes E)^G$ induced by (0.23).

***Theorem 0.10***. — *The isomorphism $(2p)^{-\frac{n_0}{4}}\sigma_p$ is an asymptotic isometry from $(H^0(X, L^p\otimes E)^G, \langle\,,\,\rangle)$ onto $(H^0(X_G, L^p_G\otimes E_G), \langle\,,\,\rangle_{\widetilde{h}})$, i.e., if $\{s_i^p\}_{i=1}^{d_p}$ is an orthonormal basis of $(H^0(X, L^p\otimes E)^G, \langle\,,\,\rangle)$, then*

$$(2p)^{-\frac{n_0}{2}}\langle \sigma_p s_i^p, \sigma_p s_j^p\rangle_{\widetilde{h}} = \delta_{ij} + \mathscr{O}\left(\frac{1}{p}\right). \tag{0.29}$$

From the explicit formula (0.26), one can also get the coefficient of $p^{-1}$ in the expansion (0.29) (cf. [**31**, Problem 7.2]). We leave it to the interested readers.

***Remark 0.11***. — Theorem 0.10 also admits a natural symplectic extension corresponding to the asymptotic identification result in Theorem 0.9 (cf. Chapter 7).

Let $\widetilde{P}^{X_G}_p$ denote the orthogonal projection from $(\mathscr{C}^\infty(X_G, L^p_G\otimes E_G), \langle\,,\,\rangle_{\widetilde{h}})$ onto $H^0(X, L^p_G\otimes E_G)$. Let $\widetilde{P}^{X_G}_p(x_0, x_0')$ $(x_0,\, x_0'\in X_G)$ be the smooth kernel of the operator $\widetilde{P}^{X_G}_p$ with respect to $\widetilde{h}^2(x_0')dv_{X_G}(x_0')$.

The following result is an easy consequence of [**17**, Theorem 1.3].

***Theorem 0.12***. — *Under the assumption of Theorem 0.6, there exist smooth coefficients $\widetilde{\Phi}_r(x_0)\in \operatorname{End}(E_G)_{x_0}$ which are polynomials in $R^{TX_G}$, $R^{E_G}$ (resp. $\widetilde{h}$), and their derivatives at $x_0$ to order $2r-1$ (resp. $2r$), and $\widetilde{\Phi}_0 = \operatorname{Id}_{E_G}$, such that for any $k, l\in\mathbb{N}$, there exists $C_{k,l}>0$ such that for any $x_0\in X_G$, $p\in\mathbb{N}$,*

$$\left| p^{-n+n_0}\widetilde{h}^2(x_0)\widetilde{P}^{X_G}_p(x_0,x_0) - \sum_{r=0}^{k}\widetilde{\Phi}_r(x_0)p^{-r}\right|_{\mathscr{C}^l} \leqslant C_{k,l}p^{-k-1}. \tag{0.30}$$

*Moreover, the following identity holds,*

$$\widetilde{\Phi}_1(x_0) = \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{1}{2\pi}\Delta_{X_G}\log\widetilde{h} + \frac{1}{2\pi}R^{E_G}_{x_0}(w_j^0, \overline{w}_j^0). \tag{0.31}$$

***Remark 0.13***. — From (0.25) and (0.31), one sees that in general $\Phi_1\neq\widetilde{\Phi}_1$, if $\widetilde{h}$ is not constant on $X_G$. This reflects a subtle defect between the Bergman kernel and the geometric quantization.

From the works [**17**], [**28**] and the present paper, we see clearly that the asymptotic expansion of Bergman kernel is parallel to the small time asymptotic expansion of the heat kernel. To localize the problem, the spectral gap property (2.15) and the finite propagation speed of solutions of hyperbolic equations play essential roles.

Let $U$ be a $G$-neighborhood of $\mu^{-1}(0)$ as in Theorem 0.2, in this paper, we will then work on $U/G$.

Indeed, after doing suitable rescaling on the coordinates, we get the limit operator $\mathscr{L}_2^0$ (cf. (3.13)) which is the sum of two terms, one along $T_{x_0}X_G$, whose kernel is infinite dimensional and gives us the classical Bergman kernel as in $\mathbb{C}^{n-n_0}$, the other along $N_G$, which is a harmonic oscillator and its kernel is one dimensional. This explains well why we can expect to get the fast decay estimate along $N_G$ in (0.14).

This paper is organized as follows. In Chapter 1, we study connections and Laplacians associated to a principal bundle. In Chapter 2, we localize the problem by using the spectral gap property and finite propagation speed, then we use the rescaling technique in local index theory to prove Theorem 2.23 which is a version of Theorem 0.2 without assumption on $\mathbf{J}$. We assume $G$ acts freely on $P = \mu^{-1}(0)$ in Sections 2.5-2.8, and in Section 4.1 we explain Theorem 4.1, the version of Theorem 0.2 where we only assume that $\mu$ is regular at 0. In Chapter 3, we get explicit informations on the coefficients $P^{(r)}$ when $\mathbf{J}$ verifies (3.2), thus we get an effective way to compute its first coefficients of the asymptotic expansion (0.14). Especially, we establish (0.12) and (0.13). In Chapter 4, we explain various applications of our Theorem 0.2, including Toeplitz operator properties, etc. In Chapter 5, we compute the coefficient $\Phi_1$ in Theorem 0.6 and in the general case: $\mathbf{J} \neq J$. In Chapter 6, we compute the coefficient $P^{(2)}_{x_0}(0,0)$ in Theorem 0.7. In Chapter 7, we prove Theorems 0.10, 0.12.

Some results of this paper have been announced in [**32, 33**].

**Notations.** — We denote by $\mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}$ the complex, natural, rational, real, integer numbers, and $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$, $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$, $\mathbb{R}_+ = [0,\infty[$, $\mathbb{R}_+^* = ]0,\infty[$. For $u \in \mathbb{R}$, we denote by $\lfloor u \rfloor$ the integer part of $u$.

For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$, $B = (B_1, \cdots, B_m) \in \mathbb{C}^m$, we denote by

$$|\alpha| = \sum_{j=1}^m \alpha_j, \quad \alpha! = \prod_j (\alpha_j!), \quad B^\alpha = \prod_j B_j^{\alpha_j}.$$

We denote by dim or $\dim_{\mathbb{C}}$ the complex dimension of a complex (vector) space. We denote also by $\dim_{\mathbb{R}}$ the real dimension of a space.

For a complex vector bundle $E$ on a manifold $X$, $\operatorname{rank}(E)$ denotes its rank, and $\mathrm{Id}_E$ the identity endomorphism. Also, $\det(E) := \Lambda^{\operatorname{rank}(E)}(E)$ is the determinant line bundle of $E$, $E^*$ is the dual bundle of $E$ and $\operatorname{End}(E) := E \otimes E^*$. The space of smooth sections of $E$ over $X$ is denoted by $\mathscr{C}^\infty(X, E)$.

If $Q$ is an operator, we denote by $\operatorname{Ker}(Q)$ its kernel, $\operatorname{Im}(Q)$ its image set.

If $V$ is a representation of the group $G$, then we denote its $G$-invariant sub-space by $V^G$.

In the whole paper, if there is no other specific notification, when an index variable appears twice in a singe term, it implies that we are summing over all its possible values.

**Acknowledgments.** — We thank Professor Jean-Michel Bismut for useful conversations. The work of the second author was partially supported by MOEC, MOSTC and NNSFC. Part of work was done while the second author was visiting IHES during February and March, 2005. He would like to thank Professor Jean-Pierre Bourguignon for the hospitality. The final version of this paper was prepared while the second author was visiting Ecole Polytechnique. He would like to thank Ecole Polytechnique for financial support and Université Paris VII for the hospitality.

# CHAPTER 1

# CONNECTIONS AND LAPLACIANS ASSOCIATED TO A PRINCIPAL BUNDLE

In this Chapter, for a $G$-principal bundle $\pi : X \to B = X/G$, we will study the associated connections and Bochner-Laplacians. The results in this chapter extend the corresponding ones in [**2**, §1d)] and [**1**, §5.1, 5.2] where the metric along the fiber is parallel along the horizontal direction. These results will be used in Proposition 2.7 and in Sections 3.3, 5.

If $G$ acts only infinitesimal freely on $X$, then $B = X/G$ is an orbifold. The results in this chapter can be extended easily to this situation, as will be explained in Section 4.1.

This Chapter is organized as follows. In Section 1.1, we study the Levi-Civita connection for a principal bundle which extends the results of [**2**, §1d)]. In Section 1.2, we study the relation of the Laplacians on the total and base manifolds.

## 1.1. Connections associated to a principal bundle

Let a compact connected Lie group $G$ act smoothly on the left on a smooth manifold $X$ and $\dim_{\mathbb{R}} G = n_0$. We suppose temporary that $G$ acts freely on $X$. Then

$$\pi : X \to B = X/G$$

is a $G$-principal bundle. We denote by $TY$ the relative tangent bundle for the fibration $\pi : X \to B$.

Let $g^{TX}$ be a $G$-invariant metric on $TX$. Let $\nabla^{TX}$ be the Levi-Civita connection on $TX$. By the explicit equation for $\left\langle \nabla^{TX}_{\cdot} \cdot, \cdot \right\rangle$ in [**1**, (1.18)], for $W, Z, Z'$ vector fields on $X$,

$$\begin{aligned} (1.1) \quad 2\left\langle \nabla^{TX}_W Z, Z' \right\rangle = {} & W \langle Z, Z' \rangle + Z \langle W, Z' \rangle - Z' \langle W, Z \rangle \\ & - \langle W, [Z, Z'] \rangle - \langle Z, [W, Z'] \rangle + \langle Z', [W, Z] \rangle . \end{aligned}$$

Let $T^H X$ be the orthogonal complement of $TY$ in $TX$.

For $U \in TB$, let $U^H \in T^H X$ be the lift of $U$ such that $\pi_* U^H = U$. Let $L_{U^H}$ be the corresponding Lie derivative.

Let $g^{TY}, g^{T^H X}$ be $G$-invariant metrics on $TY, T^H X$ induced by $g^{TX}$. Let $P^{TY}$, $P^{T^H X}$ be the orthogonal projections from $TX$ onto $TY$, $T^H X$.

Let $g^{TB}$ be the metric on $TB$ induced by $g^{T^H X}$. Let $\nabla^{TB}$ be the Levi-Civita connection on $(TB, g^{TB})$ with curvature $R^{TB}$. Set

$$\nabla^{T^H X} = \pi^* \nabla^{TB}, \quad \nabla^{TY} = P^{TY} \nabla^{TX} P^{TY}, \quad {}^0\nabla^{TX} = \nabla^{TY} \oplus \nabla^{T^H X}. \tag{1.2}$$

Then $\nabla^{T^H X}$, ${}^0\nabla^{TX}$ define Euclidean connections on $T^H X$, $TX$, and $\nabla^{TY}$ is the connection on $TY$ induced by $\nabla^{TX}$ (cf. [**2**, Def. 1.6]).

Let $T$ be the torsion of ${}^0\nabla^{TX}$, and let $S \in T^*X \otimes \mathrm{End}(TX)$, $\dot{g}^{TY}_{\cdot} \in T^*B \otimes \mathrm{End}(TY)$ be defined by

$$S = \nabla^{TX} - {}^0\nabla^{TX}, \qquad \dot{g}^{TY}_U = (g^{TY})^{-1}(L_{U^H} g^{TY}) \quad \text{for } U \in TB. \tag{1.3}$$

Then $S$ is a 1-form on $X$ taking values in skew-adjoint endomorphisms of $TX$.

By [**6**, Theorem 1.2] (cf. [**5**, Theorems 1.1 and 1.2]) the proof of which can also be found in [**1**, Prop. 10.2] where one applies directly (1.1), we know that $\nabla^{TY}$ is the Levi-Civita connection on $TY$ along the fiber $Y$, and for $U \in TB$,

$$\nabla^{TY}_{U^H} = L_{U^H} + \frac{1}{2}(g^{TY})^{-1}(L_{U^H} g^{TY}) = L_{U^H} + \frac{1}{2}\dot{g}^{TY}_U. \tag{1.4}$$

Let $\mathfrak{g}$ be the Lie algebra of $G$. For $K \in \mathfrak{g}$, we denote by $K^X_x = \frac{\partial}{\partial t} e^{-tK} x|_{t=0}$ the corresponding vector field on $X$, then $gK^X_x = (\mathrm{Ad}_g(K))^X_{gx}$. Thus we can identify the trivial bundle $X \times \mathfrak{g}$ with Ad-action of $G$ on $\mathfrak{g}$ to the $G$-equivariant bundle $TY$ by the map $K \to K^X$.

Let $\theta : TX \to \mathfrak{g}$ be the connection form of the principal bundle $\pi : X \to B$ such that $T^H X = \mathrm{Ker}\,\theta$, and $\Theta$ its curvature.

For $K_1, K_2 \in \mathfrak{g}$, $U, V \in TB$, as $U^H$ is $G$-invariant, we have

$$L_{U^H} K^X_1 = -[K^X_1, U^H] = 0. \tag{1.5}$$

By (1.4), (1.5), we get $T \in \Lambda^2(T^*X) \otimes TY$ and

$$\begin{aligned} &T(U^H, V^H) = \Theta(U^H, V^H) = -P^{TY}[U^H, V^H], \quad T(K^X_1, K^X_2) = 0, \\ &T(U^H, K^X_1) = \frac{1}{2}(g^{TY})^{-1}(L_{U^H} g^{TY}) K^X_1 = \frac{1}{2}\dot{g}^{TY}_U K^X_1. \end{aligned} \tag{1.6}$$

And by (1.1), (1.4), (1.5) and (1.6), for $W \in TX$, we have (cf. also [**2**, (1.28)], [**1**, Prop. 10.6]),

$$\begin{aligned} &S(W)(TY) \subset T^H X, \quad S(U^H)V^H \in TY, \\ &2\left\langle S(U^H)K^X_1, V^H\right\rangle = 2\left\langle S(K^X_1)U^H, V^H\right\rangle = \left\langle T(U^H, V^H), K^X_1\right\rangle, \\ &\left\langle S(K^X_2)U^H, K^X_1\right\rangle = -\left\langle S(K^X_2)K^X_1, U^H\right\rangle \\ &\qquad\qquad = \frac{1}{2} U^H \left\langle K^X_2, K^X_1\right\rangle = \left\langle T(U^H, K^X_1), K^X_2\right\rangle. \end{aligned} \tag{1.7}$$

Let $\{e_i\}$ be an orthonormal basis of $TB$. By (1.3) and (1.7), for $Y$ a section of $TY$,

$$\nabla^{TX}_{U^H} Y = \nabla^{TY}_{U^H} Y + \frac{1}{2}\left\langle T(U^H, e_i^H), Y\right\rangle e_i^H. \tag{1.8}$$

***Proposition 1.1.*** — *Let $\{f_l\}_{l=1}^{n_0}$ be a $G$-invariant orthonormal frame of $TY$, then*

$$\sum_{l=1}^{n_0} \nabla^{TY}_{f_l} f_l = 0. \tag{1.9}$$

*Proof.* — (1.9) is analogous to the fact that any left invariant volume form on $G$ is also right invariant. We only need to work on a fiber $Y_b$, $b \in B$.

Let $dv_Y$ be the Riemannian volume form on $Y_b$.

By using $L_{f_k} f_l = \nabla^{TY}_{f_k} f_l - \nabla^{TY}_{f_l} f_k$ and $dv_Y$ is preserved by $\nabla^{TY}$ on $Y_b$, we get

$$L_{f_k} dv_Y = \sum_{l=1}^{n_0} \left\langle \nabla^{TY}_{f_l} f_k, f_l\right\rangle dv_Y. \tag{1.10}$$

Now from $L_{f_k} = i_{f_k} d^Y + d^Y i_{f_k}$ and $\left\langle \nabla^{TY}_{f_l} f_k, f_l\right\rangle$ is $G$-invariant and (1.10), we get

$$0 = \int_{Y_b} L_{f_k} dv_Y = \sum_{l=1}^{n_0} \left\langle \nabla^{TY}_{f_l} f_k, f_l\right\rangle \int_{Y_b} dv_Y. \tag{1.11}$$

From (1.11), we get (1.9). □

***Remark 1.2.*** — If $g^{TY}$ is induced by a family of $\mathrm{Ad}_G$-invariant metric on $\mathfrak{g}$ under the isomorphism from $X \times \mathfrak{g}$ to $TY$ defined by $K \to K^X$, then (1.9) is trivial. In this case, as in [**19**, Theorem 11.3], for $Y_1, Y_2$ two $G$-invariant sections of $TY$, by (1.1), we have

$$\nabla^{TY}_{Y_1} Y_2 = \frac{1}{2}[Y_1, Y_2]. \tag{1.12}$$

## 1.2. Curvatures and Laplacians associated to a principal bundle

Let $(F, h^F)$ be a $G$-equivariant Hermitian vector bundle on $X$ with a $G$-invariant Hermitian connection $\nabla^F$ on $X$. For any $K \in \mathfrak{g}$, denote by $L_K$ the infinitesimal action induced by $K$ on the corresponding vector bundles.

Let $\mu^F$ be the section of $\mathfrak{g}^* \otimes \mathrm{End}(F)$ on $X$ defined by,

$$\mu^F(K) = \nabla^F_{K^X} - L_K \quad \text{for } K \in \mathfrak{g}. \tag{1.13}$$

By using the identification $X \times \mathfrak{g} \to TY$, $\mu^F$ defines a $G$-invariant section $\widetilde{\mu}^F$ of $TY \otimes \mathrm{End}(F)$ on $X$ such that

$$\left\langle \widetilde{\mu}^F, K^X\right\rangle = \mu^F(K). \tag{1.14}$$

The curvature $R^F_\mu$ of the Hermitian connection $\nabla^F - \mu^F(\theta)$ on $F$ is $G$-invariant. Moreover as $\nabla^F$ is $G$-invariant, by (1.13),

$$R^F_\mu(K^X, v) = [L_K, \nabla^F - \mu^F(\theta)](v) = 0 \tag{1.15}$$

for $K \in \mathfrak{g}$, $v \in TX$, and

$$R^F_\mu = R^F - \nabla^F(\mu^F(\theta)) + \mu^F(\theta) \wedge \mu^F(\theta). \tag{1.16}$$

The Hermitian vector bundle $(F, h^F)$ induces a Hermitian vector bundle $(F_B, h^{F_B})$ on $B$ by identifying $G$-invariant sections of $F$ on $X$.

For $s \in \mathscr{C}^\infty(B, F_B) \simeq \mathscr{C}^\infty(X, F)^G$, we define

$$\nabla^{F_B}_U s = \nabla^F_{U^H} s. \tag{1.17}$$

Then $\nabla^{F_B}$ is a Hermitian connection on $F_B$ with curvature $R^{F_B}$.

Observe that $\nabla^{F_B}$ is the restriction of the connection $\nabla^F - \mu^F(\theta)$ to $\mathscr{C}^\infty(X, F)^G$, and $R^{F_B}$ is the section induced by $R^F_\mu$. From (1.16), for $U_1, U_2 \in TB$, we get

$$R^{F_B}(U_1, U_2) = R^F(U_1^H, U_2^H) - \mu^F(\Theta)(U_1, U_2). \tag{1.18}$$

Let $dv_X$ be the Riemannian volume form on $(X, g^{TX})$. We define a scalar product on $\mathscr{C}^\infty(X, F)$ by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, s_2 \rangle_F(x)\, dv_X(x)\,. \tag{1.19}$$

As in (1.19), $h^{F_B}$, $g^{TB}$ induce a natural scalar product $\langle \quad \rangle$ on $\mathscr{C}^\infty(B, F_B)$.

Denote by $\mathrm{vol}(Gx)$ $(x \in X)$ the volume of the orbit $Gx$ equipped with the metric induced by $g^{TX}$. The function

$$h(x) = \sqrt{\mathrm{vol}(Gx)}, \qquad x \in X,$$

as in (0.10) is $G$-invariant and defines a function on $B$.

Denote by $\pi_G : \mathscr{C}^\infty(X, F)^G \to \mathscr{C}^\infty(B, F_B)$ the natural identification. Then the map

$$\Phi = h\pi_G : (\mathscr{C}^\infty(X, F)^G, \langle\, , \rangle) \to (\mathscr{C}^\infty(B, F_B), \langle\, , \rangle) \tag{1.20}$$

is an isometry.

Let $\{e_a\}_{a=1}^m$ be an orthonormal frame of $TX$.

Let $(E, h^E)$ be a Hermitian vector bundle on $X$ and let $\nabla^E$ be a Hermitian connection on $E$. The usual Bochner Laplacians $\Delta^E, \Delta_X$ are defined by

$$\Delta^E := -\sum_{a=1}^m \left( (\nabla^E_{e_a})^2 - \nabla^E_{\nabla^{TX}_{e_a} e_a} \right), \quad \Delta_X = \Delta^{\mathbb{C}}. \tag{1.21}$$

Let $\{f_l\}_{l=1}^{n_0}$ be a $G$-invariant orthonormal frame of $TY$, and $\{f^l\}$ its dual frame, and let $\{e_i\}$ be an orthonormal frame of $TB$, then $\{e_i^H, f_l\}$ is an orthonormal frame of $TX$.

To simplify the notation, for $\sigma_1, \sigma_2 \in TY \otimes \mathrm{End}(F)$, we denote by $\langle \sigma_1, \sigma_2 \rangle_{g^{TY}} \in \mathrm{End}(F)$ the contraction of $\sigma_1 \otimes \sigma_2$ on the part of $TY$ by $g^{TY}$. In particular,

$$\langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} = \sum_{l=1}^{n_0} \langle \widetilde{\mu}^F, f_l \rangle^2 \in \mathrm{End}(F). \tag{1.22}$$

The following result extends [**1**, Prop. 5.6, 5.10] where $F = X \times_G V$ for a $G$-representation $V$, and where $g^{TY}$ is induced by a fixed $\mathrm{Ad}_G$-invariant metric on $\mathfrak{g}$ under the isomorphism from $X \times \mathfrak{g}$ to $TY$ defined by $K \to K^X$ (Thus $h$ is constant on $B$).

***Theorem 1.3***. — *As an operator on $\mathscr{C}^\infty(B, F_B)$, we have*

$$\Phi \Delta^F \Phi^{-1} = \Delta^{F_B} - \langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} - \frac{1}{h} \Delta_B h. \tag{1.23}$$

*Proof*. — At first by (1.6) and (1.7),

$$\begin{aligned} \frac{1}{h}(e_i h) &= \frac{1}{2}(L_{e_i^H} dv_Y)/dv_Y = \frac{1}{2}\left\langle L_{e_i^H} f^l, f^l \right\rangle = -\frac{1}{2}\left\langle L_{e_i^H} f_l, f_l \right\rangle \\ &= \frac{1}{4}(L_{e_i^H} g^{TY})(f_l, f_l) = \frac{1}{2}\left\langle T(e_i^H, f_l), f_l \right\rangle = -\frac{1}{2}\left\langle S(f_l) f_l, e_i^H \right\rangle. \end{aligned} \tag{1.24}$$

As $\widetilde{\mu}^F$ is $G$-invariant, then $\langle \widetilde{\mu}^F, f_l \rangle$ is also a $G$-invariant section of $\mathrm{End}(F)$.

By (1.13), $\nabla^F_{f_l} = \langle \widetilde{\mu}^F, f_l \rangle$ on $\mathscr{C}^\infty(X, F)^G$, and by (1.3), $\nabla^{TX}_{f_l} f_l = \nabla^{TY}_{f_l} f_l + S(f_l) f_l$, thus by (1.20), we get for $1 \leqslant l \leqslant n_0$,

$$\Phi[(\nabla^F_{f_l})^2 - \nabla^F_{\nabla^{TX}_{f_l} f_l}]\Phi^{-1} = \langle \widetilde{\mu}^F, f_l \rangle^2 - \langle \widetilde{\mu}^F, \nabla^{TY}_{f_l} f_l \rangle - h \nabla^{F_B}_{S(f_l) f_l} h^{-1}. \tag{1.25}$$

From (1.7), (1.9), (1.21), (1.22), (1.24) and (1.25), we have

$$\begin{aligned} \Phi \Delta^F \Phi^{-1} &= -\sum_{i=1}^{m-n_0} \Phi \left[ (\nabla^F_{e_i^H})^2 - \nabla^F_{\nabla^{TX}_{e_i^H} e_i^H} \right] \Phi^{-1} - \sum_{l=1}^{n_0} \Phi \left[ (\nabla^F_{f_l})^2 - \nabla^F_{\nabla^{TX}_{f_l} f_l} \right] \Phi^{-1} \\ &= h \Delta^{F_B} h^{-1} - \sum_{l=1}^{n_0} \langle \widetilde{\mu}^F, f_l \rangle^2 - 2(e_i h) \nabla^{F_B}_{e_i} h^{-1} = \Delta^{F_B} - \langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} - \frac{1}{h} \Delta_B h. \end{aligned} \tag{1.26}$$

□

# CHAPTER 2

# $G$-INVARIANT BERGMAN KERNELS

In this Chapter, we study the uniform estimate with its derivatives on $t = \frac{1}{\sqrt{p}}$ of the $G$-invariant Bergman kernel $P_p^G(x, x')$ of $D_p^2$ as $p \to \infty$.

The first main difficulty is to localize the problem to arbitrary small neighborhoods of $P = \mu^{-1}(0)$, so that one can study the $G$-invariant Bergman kernel in the spirit of [**17**]. Our observation here is that the $G$-invariant Bergman kernel is exactly the kernel of the orthogonal projection on the zero space of an operator $\mathcal{L}_p$, which is a deformation of $D_p^2$ by the Casimir operator. Moreover, $\mathcal{L}_p$ has a spectral gap property (cf. (2.24), (2.25)). In the spirit of [**17**, §4], this allows us to localize the problem to a problem near a $G$-neighborhood of $Gx$. By combining with the Lichnerowicz formula, we get Theorem 0.1 in Section 2.4.

After localizing the problem to a problem near $P$, we first replace $X$ by $G \times \mathbb{R}^{2n-n_0}$, then we reduce it to a problem on $\mathbb{R}^{2n-n_0}$. On $\mathbb{R}^{2n-n_0}$, the problem in Section 2.7 is similar to a problem on $\mathbb{R}^{2n}$ considered in [**17**, §4.3].

Comparing with the operator in [**17**, §4.3], we have an extra quadratic term along the normal direction of $X_G$. This allows us to improve the estimate in the normal direction. After suitable rescaling, we will introduce a family of Sobolev norms defined by the rescaled connection on $L^p$ and the rescaled moment map in this situation, then we can extend the functional analysis techniques developed in [**17**, §4.3] and [**7**, §11].

This Chapter is organized as follows. In Section 2.1, we recall a basic property on the Casimir operator of a compact connected Lie group. In Section 2.2, we recall the definition of spin$^c$ Dirac operators for an almost complex manifold. In Section 2.3, we introduce the operator $\mathcal{L}_p$ to study the $G$-invariant Bergman kernel $P_p^G$ of $D_p^2$. In Section 2.4, we explain that the asymptotic expansion of $P_p^G(x, x')$ is localized on a $G$-neighborhood of $Gx$, and we establish Theorem 0.1. In Section 2.5, we show that our problem near $P$ is equivalent to a problem on $U/G$ for any open $G$-neighborhood $U$ of $P$. In Section 2.6, we derive an asymptotic expansion of $\Phi\mathcal{L}_p\Phi^{-1}$ in coordinates of $U/G$. In Section 2.7, we study the uniform estimate, with its derivatives on $t$, of the Bergman kernel associated to the rescaled operator $\mathscr{L}_2^t$ from $\Phi\mathcal{L}_p\Phi^{-1}$, using the heat kernel. In Theorem 2.21, we estimate uniformly the remaining term of the Taylor expansion of $e^{-u\mathscr{L}_2^t}$ for $u \geqslant u_0 > 0,\ 0 < t \leqslant t_0 \leqslant 1$. In Section 2.8, we identify $J_{r,u}$, the coefficient of the Taylor expansion of $e^{-u\mathscr{L}_2^t}$, with the Volterra expansion of the heat kernel, thus giving a way to compute the coefficient $P_{x_0}^{(r)}$ in Theorem 0.2. In Section 2.9, we prove Theorem 0.2 except (0.12) and (0.13).

We use the notation in Chapter 1. In Sections 2.5-2.9, we assume $G$ acts freely on $P = \mu^{-1}(0)$.

## 2.1. Casimir operator

Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\dim_{\mathbb{R}} G = n_0$. We choose an $\mathrm{Ad}_G$-invariant metric on $\mathfrak{g}$ such that it is the minus Killing form on the semi-simple part of $\mathfrak{g}$.

Let $\{K_j\}_{j=1}^{n_0}$ be an orthogonal basis of $\mathfrak{g}$ and $\{K^j\}$ be its dual basis of $\mathfrak{g}^*$.

The Casimir operator Cas of $\mathfrak{g}$ is defined as the following element of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$,

$$\mathrm{Cas} := \sum_{j=1}^{n_0} K_j K_j. \tag{2.1}$$

Then Cas is independent of the choice of $\{K_j\}$ and belongs to the center of $U(\mathfrak{g})$.

Let $\mathfrak{t}$ be the Lie algebra of a maximum torus $T$ of $G$, and $\mathfrak{t}^*$ its dual. Let $|\ \ |$ denote the norm on $\mathfrak{t}^*$ induced by the $\mathrm{Ad}_G$-invariant metric on $\mathfrak{g}$.

Let $\mathcal{W} \subset \mathfrak{t}^*$ be the fundamental Weyl chamber associated to the set of positive roots $\Delta^+$ of $G$, and its closure $\overline{\mathcal{W}} \subset \mathfrak{t}^*$.

Let $I = \{K \in \mathfrak{t}; \exp(2\pi K) = 1 \in T\}$ be the integer lattice such that $T = \mathfrak{t}/2\pi I$, and $P = \{\alpha \in \mathfrak{t}^*; \alpha(I) \subset \mathbb{Z}\}$ the lattice of integral forms.

Let $\varrho_G$ be the half sum of the positive roots of $G$.

By the Weyl character formula [**19**, Theorem 8.21], the irreducible representations of $G$ correspond one to one to $\vartheta \in \overline{\mathcal{W}} \cap P$, the highest weight of the representation.

Moreover, for any irreducible representation $\rho : G \to \mathrm{End}(V)$ with highest weight $\vartheta \in \overline{\mathcal{W}} \cap P$, classically, the action of Cas on $V$ is given by (cf. [**19**, Theorem 10.6]),

$$\rho(\mathrm{Cas}) = -(|\vartheta + \varrho_G|^2 - |\varrho_G|^2)\,\mathrm{Id}_V\,. \tag{2.2}$$

Set

$$\nu_1 := \inf_{0 \neq \vartheta \in \overline{\mathcal{W}} \cap P} (|\vartheta + \varrho_G|^2 - |\varrho_G|^2) > 0. \tag{2.3}$$

By (2.2), for any representation $\rho : G \to \mathrm{End}(V)$, if the $G$-invariant subspace $V^G$ of $V$ is zero, then

$$-\rho(\mathrm{Cas}) \geqslant \nu_1\,\mathrm{Id}_V\,. \tag{2.4}$$

## 2.2. Spin$^c$ Dirac operator

Let $(X, \omega)$ be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle $L$ over $X$ endowed with a Hermitian connection $\nabla^L$ with the property that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega, \tag{2.5}$$

where $R^L = (\nabla^L)^2$ is the curvature of $(L, \nabla^L)$.

Let $(E, h^E)$ be a Hermitian vector bundle on $X$ with Hermitian connection $\nabla^E$ and its curvature $R^E$.

Let $g^{TX}$ be a Riemannian metric on $X$.

Let $\mathbf{J} : TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u, v) = g^{TX}(\mathbf{J}u, v) \tag{2.6}$$

for $u, v \in TX$.

Let $J$ be an almost complex structure such that

$$g^{TX}(Ju, Jv) = g^{TX}(u, v), \qquad \omega(Ju, Jv) = \omega(u, v), \tag{2.7}$$

and that $\omega(\cdot, J\cdot)$ defines a metric on $TX$. Then $J$ commutes with $\mathbf{J}$ and

$$-\langle J\mathbf{J}\cdot, \cdot\rangle = \omega(\cdot, J\cdot)$$

is positive by our assumption. Thus $-J\mathbf{J} \in \mathrm{End}(TX)$ is symmetric and positive, and one verifies easily that

$$-J\mathbf{J} = (-\mathbf{J}^2)^{1/2}, \quad J = \mathbf{J}(-\mathbf{J}^2)^{-1/2}. \tag{2.8}$$

The almost complex structure $J$ induces a splitting

$$TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X,$$

where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles.

For any $v \in TX \otimes_{\mathbb{R}} \mathbb{C}$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0} \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then

$$c(v) := \sqrt{2}(\overline{v}^*_{1,0} \wedge -i_{v_{0,1}}) \tag{2.9}$$

defines the Clifford action of $v$ on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and $i$ denote the exterior and interior multiplications respectively.

Set

$$\nu_0 := \inf_{u \in T_x^{(1,0)}X,\, x \in X} R^L_x(u, \overline{u})/|u|^2_{g^{TX}} > 0. \tag{2.10}$$

Let $\nabla^{TX}$ be the Levi-Civita connection of the metric $g^{TX}$ with curvature $R^{TX}$. We denote by $P^{T^{(1,0)}X}$ the projection from $TX \otimes_{\mathbb{R}} \mathbb{C}$ to $T^{(1,0)}X$.

Let $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X}\nabla^{TX}P^{T^{(1,0)}X}$ be the Hermitian connection on $T^{(1,0)}X$ induced by $\nabla^{TX}$ with curvature $R^{T^{(1,0)}X}$. Let $\nabla^{\det}$ be the connection on $\det(T^{(1,0)}X)$ induced by $\nabla^{T^{(1,0)}X}$.

Formally,

$$\Lambda(T^{*(0,1)}X) = S(TX)\otimes(\det(T^{(1,0)}X))^{1/2}, \tag{2.11}$$

here $S(TX)$ is the possible (non-existent) spinors bundle associated to $(X, g^{TX})$, and $(\det(T^{(1,0)}X))^{1/2}$ is the possible (non-existent) square root of $\det(T^{(1,0)}X)$. By [**24**, pp.397–398], [**31**, §1.3], $\nabla^{TX}$ induces canonically a Clifford connection $\nabla^{\mathrm{Cliff}}$ on $\Lambda(T^{*(0,1)}X)$ and its curvature $R^{\mathrm{Cliff}}$ (cf. also [**27**, §2]).

Let $\{e_a\}_a$ be an orthonormal basis of $TX$. Then

$$R^{\mathrm{Cliff}} = \frac{1}{4}\sum_{ab}\langle R^{TX}e_a, e_b\rangle c(e_a)c(e_b) + \frac{1}{2}\operatorname{Tr}\left[R^{T^{(1,0)}X}\right]. \tag{2.12}$$

For $p\in\mathbb{N}$, we denote by $L^p := L^{\otimes p}$. Let $\nabla^{E_p}$ be the connection on

$$E_p := \Lambda(T^{*(0,1)}X)\otimes L^p\otimes E \tag{2.13}$$

induced by $\nabla^{\mathrm{Cliff}}$, $\nabla^L$ and $\nabla^E$.

Let $\langle\ \ \rangle_{E_p}$ be the metric on $E_p$ induced by $g^{TX}$, $h^L$ and $h^E$.

The $L^2$–scalar product $\langle\ \ \rangle$ on $\Omega^{0,\bullet}(X, L^p\otimes E)$, the space of smooth sections of $E_p$, is given by (1.19). We denote the corresponding norm by $\|\cdot\|_{L^2}$.

***Definition 2.1***. — The spin$^c$ Dirac operator $D_p$ is defined by

$$D_p := \sum_{a=1}^{2n} c(e_a)\nabla^{E_p}_{e_a} : \Omega^{0,\bullet}(X, L^p\otimes E)\longrightarrow \Omega^{0,\bullet}(X, L^p\otimes E)\,. \tag{2.14}$$

Clearly, $D_p$ is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, L^p\otimes E)$, which interchanges $\Omega^{0,\mathrm{even}}(X, L^p\otimes E)$ and $\Omega^{0,\mathrm{odd}}(X, L^p\otimes E)$.

If $A$ is any operator, we denote by Spec(A) the spectrum of $A$.

The following result was proved in [**27**, Theorems 1.1, 2.5] by applying directly the Lichnerowicz formula (cf. also [**8**, Theorem 1] in the holomorphic case).

***Theorem 2.2***. — *There exists* $C_L > 0$ *such that for any* $p\in\mathbb{N}$ *and any* $s\in \Omega^{>0}(X, L^p\otimes E) = \bigoplus_{q\geqslant 1}\Omega^{0,q}(X, L^p\otimes E)$,

$$\|D_p s\|^2_{L^2} \geqslant (2p\nu_0 - C_L)\|s\|^2_{L^2}\,. \tag{2.15}$$

*Moreover* $\operatorname{Spec}(D_p^2)\subset\{0\}\cup[2p\nu_0 - C_L, +\infty[$.

## 2.3. $G$-invariant Bergman kernel

Suppose that the compact connected Lie group $G$ acts on the left of $X$, and the action of $G$ lifts on $L, E$ and preserves the metrics and connections, $\omega$ and the almost complex structure $J$.

Let $\mu : X \to \mathfrak{g}^*$ be defined by

$$2\pi\sqrt{-1}\mu(K) := \mu^L(K) = \nabla^L_{K^X} - L_K, \ K \in \mathfrak{g}. \tag{2.16}$$

Then $\mu$ is the corresponding moment map (cf. [**1**, Example 7.9]), i.e. for any $K \in \mathfrak{g}$,

$$d\mu(K) = i_{K^X}\omega. \tag{2.17}$$

For $V$ a subspace of $\Omega^{0,\bullet}(X, L^p \otimes E)$, we denote by $V^\perp$ the orthogonal complement of $V$ in $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle \quad \rangle)$.

Let $\Omega^{0,\bullet}(X, L^p \otimes E)^G$, $(\operatorname{Ker} D_p)^G$ be the $G$-invariant subspaces of $\Omega^{0,\bullet}(X, L^p \otimes E)$, $\operatorname{Ker} D_p$. Let $P^G_p$ be the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ on $(\operatorname{Ker} D_p)^G$.

***Definition 2.3.*** — The $G$-invariant Bergman kernel $P^G_p(x, x')$ $(x, x' \in X)$ of $D_p$ is the smooth kernel of $P^G_p$ with respect to the Riemannian volume form $dv_X(x')$.

Let $\{S^p_i\}_{i=1}^{d_p}$ $(d_p = \dim(\operatorname{Ker} D_p)^G)$ be any orthonormal basis of $(\operatorname{Ker} D_p)^G$ with respect to the norm $\| \ \|_{L^2}$, then

$$P^G_p(x, x') = \sum_{i=1}^{d_p} S^p_i(x) \otimes (S^p_i(x'))^* \in (E_p)_x \otimes (E^*_p)_{x'}. \tag{2.18}$$

Especially, $P^G_p(x, x) \in \operatorname{End}(E_p)_x \simeq \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$.

We use the notation $\mu^F$ in (1.13) now.

Observe that the Lie derivative $L_K$ on $TX$ is given by

$$L_K V = \nabla^{TX}_{K^X} V - \nabla^{TX}_V K^X. \tag{2.19}$$

Thus

$$\mu^{TX}(K) = \nabla^{TX}_{\cdot} K^X \in \operatorname{End}(TX). \tag{2.20}$$

By (2.11), the action on $\Lambda(T^{*(0,1)}X)$ induced by $\mu^{TX}(K)$ is given by

$$\mu^{\mathrm{Cliff}}(K) = \frac{1}{4}\sum_{a=1}^{2n} c(e_a)c(\nabla^{TX}_{e_a} K^X) + \frac{1}{2}\operatorname{Tr}[P^{T^{(1,0)}X}\nabla^{TX}_{\cdot} K^X]. \tag{2.21}$$

Thus the action $L_K$ of $K$ on smooth sections of $\Lambda(T^{*(0,1)}X)$ is given by (cf. also [**44**, (1.24)])

$$L_K = \nabla^{\mathrm{Cliff}}_{K^X} - \mu^{\mathrm{Cliff}}(K). \tag{2.22}$$

By (2.16) and (2.22), the action $L_K$ of $K$ on $\Omega^{0,\bullet}(X, L^p \otimes E)$ is $\nabla^{E_p}_{K^X} - \mu^{E_p}(K)$ with

$$\mu^{E_p}(K) = 2\pi\sqrt{-1}p\mu(K) + \mu^E(K) + \mu^{\mathrm{Cliff}}(K). \tag{2.23}$$

***Definition 2.4***. — The (formally) self-adjoint operator $\mathcal{L}_p$ acting on $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle\, ,\, \rangle)$ is defined by,

$$\mathcal{L}_p = D_p^2 - p\sum_{i=1}^{n_0} L_{K_i} L_{K_i}. \tag{2.24}$$

The following result will play a crucial role in the whole paper.

***Theorem 2.5***. — *The projection $P_p^G$ is the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto* $\operatorname{Ker}(\mathcal{L}_p)$. *Moreover, there exist $\nu$, $C_L > 0$ such that for any $p \in \mathbb{N}$,*

$$\begin{aligned} &\operatorname{Ker}(\mathcal{L}_p) = (\operatorname{Ker} D_p)^G, \\ &\operatorname{Spec}(\mathcal{L}_{\rm p}) \subset \{0\} \cup [2p\nu - C_L, +\infty[. \end{aligned} \tag{2.25}$$

*Proof.* — By (2.24), for any $s \in \Omega^{0,\bullet}(X, L^p \otimes E)$,

$$\langle \mathcal{L}_p s, s\rangle = \|D_p s\|_{L^2}^2 + p\sum_{i=1}^{n_0} \|L_{K_i} s\|_{L^2}^2. \tag{2.26}$$

Thus $\mathcal{L}_p s = 0$ is equivalent to

$$D_p s = L_{K_i} s = 0. \tag{2.27}$$

This means $s$ is fixed by the $G$-action. Thus we get the first equation of (2.25).

For $s \in (\operatorname{Ker} \mathcal{L}_p)^\perp$, there exist $s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G \cap (\operatorname{Ker} D_p)^\perp$, $s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp$, such that $s = s_1 + s_2$. Clearly,

$$D_p s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G, \quad D_p s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp.$$

By Theorem 2.2 and (2.4),

$$\begin{aligned} \langle \mathcal{L}_p s, s\rangle &= -p\langle \rho(\mathrm{Cas}) s_2, s_2\rangle + \|D_p s_2\|_{L^2}^2 + \|D_p s_1\|_{L^2}^2 \\ &\geqslant p\nu_1 \|s_2\|_{L^2}^2 + (2p\nu_0 - C_L)\|s_1\|_{L^2}^2, \end{aligned} \tag{2.28}$$

from which we get (2.25). □

We assume that $0 \in \mathfrak{g}^*$ is a regular value of $\mu$. Then $X_G = \mu^{-1}(0)/G$ is an orbifold ($X_G$ is smooth if $G$ acts freely on $P = \mu^{-1}(0)$). Furthermore, $\omega$ descends to a symplectic form $\omega_G$ on $X_G$. Thus one gets the Marsden-Weinstein symplectic reduction $(X_G, \omega_G)$.

Moreover, $(L, \nabla^L)$, $(E, \nabla^E)$ descend to $(L_G, \nabla^{L_G})$, $(E_G, \nabla^{E_G})$ over $X_G$ so that the corresponding curvature condition holds [**21**] :

$$\frac{\sqrt{-1}}{2\pi} R^{L_G} = \omega_G. \tag{2.29}$$

The $G$-invariant almost complex structure $J$ also descends to an almost complex structure $J_G$ on $TX_G$, and $h^L, h^E, g^{TX}$ descend to $h^{L_G}$, $h^{E_G}, g^{TX_G}$.

We can construct the corresponding spin$^c$ Dirac operator $D_{G,p}$ on $X_G$.

Let $P_{G,p}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$ on $\operatorname{Ker} D_{G,p}$, and let $P_{G,p}(x,x')$ be the smooth kernel of $P_{G,p}$ with respect to the Riemannian volume form $dv_{X_G}(x')$.

The purpose of this paper is to study the asymptotic expansion of $P_p^G(x,x')$ when $p \to \infty$, and we will relate it to the asymptotic expansion of the Bergman kernel $P_{G,p}$ on $X_G$.

## 2.4. Localization of the problem and proof of Theorem 0.1

Let $a^X$ be the injectivity radius of $(X, g^{TX})$, and $\varepsilon \in ]0, a^X/4[$. If $x \in X$, $Z \in T_xX$, let $\mathbb{R} \ni u \to x_u = \exp_x^X(uZ) \in X$ be the geodesic in $(X, g^{TX})$, such that $x_0 = x$, $\frac{dx_u}{du}|_{u=0} = Z$.

For $x \in X$, we denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in $X$ and $T_xX$ with center $x$ and radius $\varepsilon$, respectively. The map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\varepsilon)$ on $B^X(x,\varepsilon)$ for $\varepsilon \leqslant a^X$.

From now on, we identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ for $\varepsilon \leqslant a^X/4$.

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$f(v) = \begin{cases} 1 & \text{for} \quad |v| \leqslant \varepsilon/2, \\ 0 & \text{for} \quad |v| \geqslant \varepsilon. \end{cases} \tag{2.30}$$

Set

$$F(a) = \Big( \int_{-\infty}^{+\infty} f(v)dv \Big)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v)dv. \tag{2.31}$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$.

Let $\widetilde{F}$ be the holomorphic function on $\mathbb{C}$ such that $\widetilde{F}(a^2) = F(a)$. The restriction of $\widetilde{F}$ to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Let $\widetilde{F}(\mathcal{L}_p)(x,x')$ be the smooth kernel of $\widetilde{F}(\mathcal{L}_p)$ with respect to the volume form $dv_X(x')$.

***Proposition 2.6.*** — *For any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $p \geqslant C_L/\nu$,*

$$|\widetilde{F}(\mathcal{L}_p)(x,x') - P_p^G(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m} p^{-l}. \tag{2.32}$$

*Here the $\mathscr{C}^m$ norm is induced by $\nabla^L, \nabla^E$, $\nabla^{\text{Cliff}}$, $h^L, h^E$ and $g^{TX}$.*

*Proof.* — For $a \in \mathbb{R}$, set

$$\phi_p(a) = 1_{[p\nu,+\infty[}(a)\widetilde{F}(a). \tag{2.33}$$

Then by Theorem 2.5, for $p > C_L/\nu$,

$$\widetilde{F}(\mathcal{L}_p) - P_p^G = \phi_p(\mathcal{L}_p). \tag{2.34}$$

By (2.31), for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$\sup_{a\in\mathbb{R}} |a|^m |\widetilde{F}(a)| \leqslant C_m. \tag{2.35}$$

As $X$ is compact, there exist $\{x_i\}_{i=1}^r \subset X$ such that $\{U_i = B^X(x_i,\varepsilon)\}_{i=1}^r$ is a covering of $X$. We identify $B^{T_{x_i}X}(0,\varepsilon)$ with $B^X(x_i,\varepsilon)$ by geodesics as above.

We identify $(E_p)_Z$ for $Z \in B^{T_{x_i}X}(0,\varepsilon)$ to $(E_p)_{x_i}$ by parallel transport with respect to the connection $\nabla^{E_p}$ along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_i}(uZ)$.

Let $\{e_a\}_{a=1}^{2n}$ be an orthonormal basis of $T_{x_i}X$. Let $\widetilde{e}_a(Z)$ be the parallel transport of $e_a$ with respect to $\nabla^{TX}$ along the above curve.

Let $\Gamma^E, \Gamma^L, \Gamma^{\mathrm{Cliff}}$ be the corresponding connection forms of $\nabla^E$, $\nabla^L$ and $\nabla^{\mathrm{Cliff}}$ with respect to any fixed frame for $E, L$, $\Lambda(T^{*(0,1)}X)$ which is parallel along the curve $\gamma_Z$ under the trivialization on $U_i$. Then $\Gamma^L$ is a usual 1-form.

Denote by $\nabla_U$ the ordinary differentiation operator on $T_{x_i}X$ in the direction $U$. Then

$$\nabla^{E_p} = \nabla + p\,\Gamma^L + \Gamma^{\mathrm{Cliff}} + \Gamma^E, \quad D_p = c(\widetilde{e}_j)\nabla^{E_p}_{\widetilde{e}_j}. \tag{2.36}$$

Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$.

For $l \in \mathbb{N}$, we define a Sobolev norm on the $l$-th Sobolev space $\boldsymbol{H}^l(X,E_p)$ by

$$\|s\|^2_{\boldsymbol{H}^l_p} = \sum_i \sum_{k=0}^{l} \sum_{i_1,\cdots,i_k=1}^{2n} \|\nabla_{e_{i_1}}\cdots\nabla_{e_{i_k}}(\varphi_i s)\|^2_{L^2} \tag{2.37}$$

Then by (2.36), there exist $C, C', C'' > 0$ such that for $p \geqslant 1$, $s \in \boldsymbol{H}^2(X,E_p)$,

$$C'\|D_p^2 s\|_{L^2} - C''p^2\|s\|_{L^2} \leq \|s\|_{\boldsymbol{H}^2_p} \leqslant C(\|D_p^2 s\|_{L^2} + p^2\|s\|_{L^2}). \tag{2.38}$$

Observe that $D_p$ commutes with the $G$-action, thus

$$[D_p, L_{K_j}] = 0. \tag{2.39}$$

By (2.24), (2.39), and the facts that $D_p$ is self-adjoint and $L_{K_j}$ is skew-adjoint, we know

$$\begin{aligned} \|\mathcal{L}_p s\|^2_{L^2} &= \|D_p^2 s\|^2_{L^2} + p^2\|\sum_j L_{K_j}L_{K_j}s\|^2_{L^2} - 2p\,\mathrm{Re}\sum_j\langle D_p^2 s, L_{K_j}L_{K_j}s\rangle \\ &= \|D_p^2 s\|^2_{L^2} + p^2\|\sum_j L_{K_j}L_{K_j}s\|^2_{L^2} + 2p\sum_j\|L_{K_j}D_p s\|^2_{L^2}. \end{aligned} \tag{2.40}$$

From (2.38) and (2.40), there exists $C > 0$ such that

$$\|s\|_{\boldsymbol{H}^2_p} \leqslant C(\|\mathcal{L}_p s\|_{L^2} + p^2\|s\|_{L^2}). \tag{2.41}$$

Let $Q$ be a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in $U_i$, then

$$[\mathcal{L}_p, Q] = [D_p^2, Q] - p\sum_j [L_{K_j}L_{K_j}, Q] \tag{2.42}$$

is a differential operator of order $m+1$. Moreover, by (2.23), (2.36), the leading term of order $m-1$ differential operator in $[L_{K_j}L_{K_j}, Q]$ is $p^2[((\Gamma^L - 2\pi\sqrt{-1}\mu)(K_j))^2, Q]$.

Thus by (2.41) and (2.42),

$$\begin{aligned}\|Qs\|_{\boldsymbol{H}_p^2} &\leqslant C(\|\mathcal{L}_p Qs\|_{L^2} + p^2\|Qs\|_{L^2})\\ &\leqslant C(\|Q\mathcal{L}_p s\|_{L^2} + p\|s\|_{\boldsymbol{H}_p^{m+1}} + p^2\|s\|_{\boldsymbol{H}_p^m} + p^3\|s\|_{\boldsymbol{H}_p^{m-1}}).\end{aligned} \tag{2.43}$$

This means

$$\|s\|_{\boldsymbol{H}_p^{2m+2}} \leqslant C_m p^{2m+2} \sum_{j=0}^{m+1} \|\mathcal{L}_p^j s\|_{L^2}. \tag{2.44}$$

Moreover, from

$$\langle \mathcal{L}_p^{m'} \phi_p(\mathcal{L}_p)Qs, s'\rangle = \langle s, Q^*\phi_p(\mathcal{L}_p)\mathcal{L}_p^{m'} s'\rangle,$$

(2.33) and (2.35), we know that for any $l, m' \in \mathbb{N}$, there exists $C_{l,m'} > 0$ such that for $p \geqslant 1$,

$$\|\mathcal{L}_p^{m'} \phi_p(\mathcal{L}_p)Qs\|_{L^2} \leqslant C_{l,m'} p^{-l+m}\|s\|_{L^2}. \tag{2.45}$$

We deduce from (2.44) and (2.45) that if $Q_1, Q_2$ are differential operators of order $m, m'$ with compact support in $U_i$, $U_j$ respectively, then for any $l > 0$, there exists $C_l > 0$ such that for $p \geqslant 1$,

$$\|Q_1\phi_p(\mathcal{L}_p)Q_2 s\|_{L^2} \leqslant C_l p^{-l}\|s\|_{L^2}. \tag{2.46}$$

On $U_i \times U_j$, by using Sobolev inequality and (2.34), we get Proposition 2.6. □

Observe that $K_j^X$ are vector fields along the orbits of the $G$-action, thus the contribution of $pL_{K_j}L_{K_j}$ in the wave operator $\cos(t\sqrt{\mathcal{L}_p})$ will propagate along the $G$-orbits, and the principal symbol of $\mathcal{L}_p$ is given by

$$\sigma(\mathcal{L}_p)(\xi) = |\xi|^2 + p\sum_j \langle K_j^X, \xi\rangle^2 \quad \text{for } \xi \in T^*X.$$

By the finite propagation speed for solutions of hyperbolic equations [**16**, §7.8], [**41**, §4.4], [**42**, I. §2.6, §2.8], [**31**, Append. D.2], $\widetilde{F}(\mathcal{L}_p)(x, x')$ only depends on the restriction of $\mathcal{L}_p$ to $G \cdot B^X(x, \varepsilon)$ and

$$\widetilde{F}(\mathcal{L}_p)(x, x') = 0, \quad \text{if } d^X(Gx, x') \geqslant \varepsilon. \tag{2.47}$$

(When we apply the proof of [**42**, §2.6, §2.8], [**31**, Append. D.2], we need to suppose that $\Sigma_1, \Sigma_2$ therein are $G$-space-like surfaces for the operator $\frac{\partial^2}{\partial t^2} + D_p^2$).

Combining with Proposition 2.6, we know that the asymptotic of $P_p^G(x, x')$ as $p \to \infty$ is localized on a neighborhood of $Gx$.

*Proof of Theorem 0.1.* — From Proposition 2.6 and (2.47), we get (0.7) for any $x, x' \in X$, $d^X(Gx, x') \geqslant \varepsilon_0$. Now we will establish (0.7) for $x, x' \in X \setminus U$.

Recall that $U$ is a $G$-open neighborhood of $P = \mu^{-1}(0)$.

As 0 is a regular value of $\mu$, there exists $\epsilon_0 > 0$ such that $\mu : X_{2\epsilon_0} = \mu^{-1}(B^{\mathfrak{g}^*}(0, 2\epsilon_0)) \to B^{\mathfrak{g}^*}(0, 2\epsilon_0)$ is a submersion, $X_{2\epsilon_0}$ is a $G$-open subset of $X$.

Fix $\varepsilon, \epsilon_0 > 0$ small enough such that $X_{2\epsilon_0} \subset U$, and $d^X(x,y) > 4\varepsilon$ for any $x \in X_{\epsilon_0}$, $y \in X \setminus U$. Then $V_{\epsilon_0} = X \setminus X_{\epsilon_0}$ is a smooth $G$-manifold with boundary $\partial V_{\epsilon_0}$.

Consider the operator $\mathcal{L}_p$ on $V_{\epsilon_0}$ with the Dirichlet boundary condition. We denote it by $\mathcal{L}_{p,D}$. Note that $\mathcal{L}_{p,D}$ is self-adjoint.

Still from [**42**, §2.6, §2.8], [**31**, Append. D.2], the wave operator $\cos(t\sqrt{\mathcal{L}_{p,D}})$ is well defined and $\cos(t\sqrt{\mathcal{L}_{p,D}})(x,x')$ only depends on the restriction of $\mathcal{L}_p$ to $G \cdot B^X(x,t) \cap V_{\epsilon_0}$, and is zero if $d^X(Gx,x') \geqslant t$. Thus, by (2.31),

$$\widetilde{F}(\mathcal{L}_p)(x,x') = \widetilde{F}(\mathcal{L}_{p,D})(x,x'), \quad \text{if } x, x' \in X \setminus U. \tag{2.48}$$

Now for $s \in \mathscr{C}_0^\infty(V_{\epsilon_0}, E_p)$, after taking an integration over $G$, we can get the decomposition $s = s_1 + s_2$ with $s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G$, $s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp$ and $\operatorname{supp}(s_i) \subset V_{\epsilon_0} \setminus \partial V_{\epsilon_0}$.

Since $\sum_{i=1}^{\dim G} L_{K_i} L_{K_i}$ commutes with the $G$-action, $\mathcal{L}_p s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G$, $\mathcal{L}_p s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp$ and, by (2.24), (2.28),

$$\begin{aligned} \langle \mathcal{L}_p s, s\rangle &= \langle \mathcal{L}_p s_1, s_1\rangle + \langle \mathcal{L}_p s_2, s_2\rangle \\ &= \|D_p s_2\|_{L^2}^2 - p\langle \rho(\mathrm{Cas}) s_2, s_2\rangle + \langle D_p^2 s_1, s_1\rangle \\ &\geqslant p\nu_1 \|s_2\|_{L^2}^2 + \langle D_p^2 s_1, s_1\rangle. \end{aligned} \tag{2.49}$$

To estimate the term $\langle D_p^2 s_1, s_1\rangle$, we will use the Lichnerowicz formula.

Recall that the Bochner-Laplacian $\Delta^{E_p}$ on $E_p$ is defined by (1.21).

Let $r^X$ be the Riemannian scalar curvature of $(TX, g^{TX})$.

Let $\{w_a\}$ be an orthonormal frame of $(T^{(1,0)}X, g^{TX})$. Set

$$\begin{aligned} &\omega_d = -\sum_{a,b} R^L(w_a, \overline{w}_b)\, \overline{w}^b \wedge i_{\overline{w}_a}\,, \\ &\tau(x) = \sum_a R^L(w_a, \overline{w}_a)\,, \qquad R^E_\tau = \sum_a R^E(w_a, \overline{w}_a)\,, \\ &\mathbf{c}(R) = \sum_{a<b} \Big(R^E + \tfrac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]\Big)(e_a, e_b)\, c(e_a)\, c(e_b)\,. \end{aligned} \tag{2.50}$$

The Lichnerowicz formula [**1**, Theorem 3.52] (cf. [**27**, Theorem 2.2]) for $D_p^2$ is

$$D_p^2 = \Delta^{E_p} - 2p\omega_d - p\tau + \tfrac{1}{4} r^X + \mathbf{c}(R). \tag{2.51}$$

Especially, as $\operatorname{supp}(s_i) \subset V_{\epsilon_0} \setminus \partial V_{\epsilon_0}$, from (2.51), we get

$$\langle D_p^2 s_1, s_1\rangle = \|\nabla^{E_p} s_1\|_{L^2}^2 - p\langle (2\omega_d + \tau) s_1, s_1\rangle + \langle (\tfrac{1}{4} r^X + \mathbf{c}(R)) s_1, s_1\rangle. \tag{2.52}$$

Since $s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G$, from (1.13), for any $K \in \mathfrak{g}$,

$$\nabla^{E_p}_{K^X} s_1 = (L_K + \mu^{E_p}(K)) s_1 = \mu^{E_p}(K) s_1. \tag{2.53}$$

From (2.23) and (2.53), there exist $C, C' > 0$ such that

$$\|\nabla^{E_p} s_1\|^2_{L^2} \geqslant C \sum_j \|\nabla^{E_p}_{K_j^X} s_1\|^2_{L^2} = C \sum_j \|\mu^{E_p}(K_j) s_1\|^2_{L^2} \geqslant Cp^2 \||\mu| s_1\|^2_{L^2} - C'\|s_1\|^2_{L^2} \geqslant C\epsilon_0^2 p^2 \|s_1\|^2_{L^2} - C'\|s_1\|^2_{L^2}. \tag{2.54}$$

From (2.49)-(2.54), for $p$ large enough,

$$\langle \mathcal{L}_p s, s\rangle \geqslant p\nu_1 \|s_2\|^2_{L^2} + Cp^2\|s_1\|^2_{L^2}. \tag{2.55}$$

Thus there are $C, C' > 0$ such that for $p \geqslant 1$,

$$\mathrm{Spec}(\mathcal{L}_{p,D}) \subset [Cp - C', \infty[. \tag{2.56}$$

Now as $K_j^X|_{\partial V_{\epsilon_0}} \in T\partial V_{\epsilon_0}$ for any $j$, thus $L_{K_j}$ preserves the Dirichlet boundary condition. We get for $l \in \mathbb{N}$,

$$L_{K_j}\phi_p(\mathcal{L}_{p,D}) = \phi_p(\mathcal{L}_{p,D}) L_{K_j}, \quad (\mathcal{L}_{p,D})^l \phi_p(\mathcal{L}_{p,D}) = \phi_p(\mathcal{L}_{p,D})(\mathcal{L}_{p,D})^l. \tag{2.57}$$

Thus from (2.24), (2.39) and (2.57),

$$D^2_{p,D} \leqslant \mathcal{L}_{p,D}, \tag{2.58}$$

and for $l \in \mathbb{N}$, $(D^2_{p,D})^l$ commutes with the operator $\phi_p(\mathcal{L}_{p,D})$.

Let $\phi_p(\mathcal{L}_{p,D})(x,x')$ be the smooth kernel of $\phi_p(\mathcal{L}_{p,D})$ with respect to $dv_X(x')$.

Then from the above argument we get that $(D^2_{p,x})^l (D^2_{p,x'})^k \phi_p(\mathcal{L}_{p,D})(x,x')$ verifies the Dirichlet boundary condition for $x, x'$ respectively for any $l, k \in \mathbb{N}$.

By (2.36) and the elliptic estimate for Laplacian with Dirichlet boundary condition [**42**, Theorem 5.1.3], there exists $C > 0$ such that for $s \in \boldsymbol{H}^{2m+2}(X, E_p) \cap \boldsymbol{H}^1_0(X, E_p)$, $p \in \mathbb{N}$,we have

$$\|s\|_{\boldsymbol{H}^{2m+2}_p} \leqslant C(\|D^2_p s\|_{\boldsymbol{H}^{2m}_p} + p^2\|s\|_{\boldsymbol{H}^{2m+1}_p}). \tag{2.59}$$

Thus if $Q_1, Q_2$ are differential operators of order $2m, 2m'$ with compact support in $U_i$, $U_j$ respectively, by (2.59) and (2.58), as in (2.44), we get for $s \in \mathscr{C}^\infty_0(V_{\epsilon_0}, E_p)$,

$$\|Q_1\phi_p(\mathcal{L}_{p,D})Q_2 s\|_{L^2} \leqslant Cp^{4m+4m'} \sum_{j_1=0}^{m}\sum_{j_2=0}^{m'} \|(D^2_{p,D})^{j_1}\phi_p(\mathcal{L}_{p,D})(D^2_{p,D})^{j_2} s\|_{L^2} \leqslant Cp^{4m+4m'} \sum_{j_1=0}^{m}\sum_{j_2=0}^{m'} \|(\mathcal{L}_{p,D})^{j_1}\phi_p(\mathcal{L}_{p,D})(\mathcal{L}_{p,D})^{j_2} s\|_{L^2}. \tag{2.60}$$

From (2.56), (2.60), as in (2.46), we get

$$\|Q_1\phi_p(\mathcal{L}_{p,D})Q_2 s\|_{L^2} \leqslant C_l p^{-l}\|s\|_{L^2}. \tag{2.61}$$

By using Sobolev inequality as in the proof of Proposition 2.6, from (2.32), (2.48) and (2.61), we get Theorem 0.1. □

## 2.5. Induced operator on $U/G$

Let $U$ be a $G$-neighborhood of $P = \mu^{-1}(0)$ in $X$ such that $G$ acts freely on $\overline{U}$, the closure of $U$. We will use the notation as in Introduction and Sections 1.1, 1.2 with $X$ therein replaced by $U$, especially $B = U/G$.

Let $\pi : U \to B$ be the natural projection with fiber $Y$. Let $TY$ be the sub-bundle of $TU$ generated by the $G$-action, let $g^{TY}$, $g^{TP}$ be the metrics on $TY$, $TP$ induced by $g^{TX}$.

Let $T^HU$, $T^HP$ be the orthogonal complements of $TY$ in $TU$, $(TP, g^{TP})$. Let $g^{T^HU}$ be the metric on $T^HU$ induced by $g^{TX}$, and it induces naturally a Riemannian metric $g^{TB}$ on $B$.

Let $dv_B$ be the Riemannian volume form on $(B, g^{TB})$.

Recall that in (1.20), we defined the isometry

$$\Phi = h\pi_G : (\mathscr{C}^\infty(U, E_p)^G, \langle\ ,\ \rangle) \to (\mathscr{C}^\infty(B, E_{p,B}), \langle\ ,\ \rangle).$$

By (1.14), $\mu^{E_p}$ defines a $G$-invariant section $\widetilde{\mu}^{E_p}$ of $TY \otimes \mathrm{End}(E_p)$ on $U$.

Remark that $\omega_d$, $\tau$, $\mathbf{c}(R)$ in (2.50) are $G$-invariant. We still denote by $\omega_d$, $\tau$, $\mathbf{c}(R)$ the induced sections on $B$.

As a direct corollary of Theorem 1.3 and (2.51), we get the following result,

***Proposition 2.7***. — *As an operator on* $\mathscr{C}^\infty(B, E_{p,B})$,

$$\begin{aligned}(2.62)\quad \Phi\mathcal{L}_p\Phi^{-1} &= \Phi D_p^2\Phi^{-1}\\ &= \Delta^{E_{p,B}} - \langle\widetilde{\mu}^{E_p}, \widetilde{\mu}^{E_p}\rangle_{g^{TY}} - \frac{1}{h}\Delta_B h - 2p\omega_d - p\tau + \tfrac{1}{4}r^X + \mathbf{c}(R).\end{aligned}$$

From Theorem 0.1, Prop. 2.6 and (2.47), modulo $\mathscr{O}(p^{-\infty})$, $P_p^G(x, x')$ depends only the restriction of $\mathcal{L}_p$ on $U$.

To get a complete picture on $P_p^G(x, x')$, we explain now that modulo $\mathscr{O}(p^{-\infty})$, $P_p^G(x, x')$ depends only on the restriction of $\Phi\mathcal{L}_p\Phi^{-1}$ on any neighborhood of $X_G$ in $B$.

As in the proof of Theorem 0.1, we will fix $\epsilon_0 > 0$ small enough such that $X_{2\epsilon_0} = \mu^{-1}(B^{\mathfrak{g}^*}(0, 2\epsilon_0)) \subset U$, and the constant $\varepsilon > 0$ verifying that $d^X(x, y) > 4\varepsilon$ for any $x \in X_{\epsilon_0}$, $y \in X \setminus U$. Set $B_{\epsilon_0} = \pi(X_{\epsilon_0})$.

Let $\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(x, x')$ $(x, x' \in B_{\epsilon_0})$ be the smooth kernel of $\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})$ with respect to $dv_B(x')$. We will also view $\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})$ as a $G \times G$-invariant section of $\mathrm{pr}_1^*E_p \otimes \mathrm{pr}_2^*E_p^*$ on $X_{\epsilon_0} \times X_{\epsilon_0}$.

***Theorem 2.8***. — *For any* $l, m \in \mathbb{N}$, *there exists* $C_{l,m} > 0$ *such that for* $p \geqslant 1$, $x, x' \in X_{\epsilon_0}$,

$$(2.63)\qquad |h(x)h(x')P_p^G(x, x') - \widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(\pi(x), \pi(x'))|_{\mathscr{C}^m(X_{\epsilon_0}\times X_{\epsilon_0})} \leqslant C_{l,m}p^{-l}.$$

*Proof.* — Let $Q : \mathscr{C}^\infty(X, E_p) \to \mathscr{C}^\infty(X, E_p)^G$ be the orthogonal projection and $Q^\perp = \mathrm{Id} - Q$. Then $D_p$, $\mathcal{L}_p$ commute with $Q, Q^\perp$, thus

$$\widetilde{F}(\mathcal{L}_p) = \widetilde{F}(\mathcal{L}_p)Q + \widetilde{F}(\mathcal{L}_p)Q^\perp. \tag{2.64}$$

Let $(\widetilde{F}(\mathcal{L}_p)Q)(x,x')$, $(\widetilde{F}(\mathcal{L}_p)Q^\perp)(x,x')$ be the Schwartz kernel of the operators $\widetilde{F}(\mathcal{L}_p)Q$, $\widetilde{F}(\mathcal{L}_p)Q^\perp$ with respect to $dv_X(x')$.

Now, by (2.4), (2.24), on $\mathrm{Im}(Q^\perp)$, $\mathrm{Spec}(\mathcal{L}_\mathrm{p}) \subset [\mathrm{p}\nu_1, +\infty[$. As $\mathcal{L}_p$ commutes with $Q^\perp$, by the same argument as in (2.32), (2.46), we get for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $p \geqslant 1$,

$$|(\widetilde{F}(\mathcal{L}_p)Q^\perp)(x,x')|_{\mathscr{C}^m(X_{\epsilon_0}\times X_{\epsilon_0})} \leqslant C_{l,m}p^{-l}. \tag{2.65}$$

Let $d^B(\cdot,\cdot)$ be the Riemannian distance on $B$.

By (2.62) and the finite propagation speed for solutions of hyperbolic equations [**16**, §7.8], [**41**, §4.4] (cf. [**31**, Append. D]), $\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(x,x')$ only depends on the restriction of $\Phi\mathcal{L}_p\Phi^{-1}$ to $B^B(x,\varepsilon)$ and

$$\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(x,x') = 0, \quad \text{if } d^B(x,x') \geqslant \varepsilon. \tag{2.66}$$

Now by (2.47), (2.66) and the isometry $\Phi$ in (1.20), we get

$$\Phi(\widetilde{F}(\mathcal{L}_p)Q)\Phi^{-1} = \widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1}). \tag{2.67}$$

From (2.67), for $x, x' \in X_{\epsilon_0}$, we have

$$h(x)h(x')(\widetilde{F}(\mathcal{L}_p)Q)(x,x') = \widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(\pi(x),\pi(x')). \tag{2.68}$$

In fact, by (0.10) and (2.67), for any $s \in \mathscr{C}_0^\infty(B_{\epsilon_0}, E_{p,G})$,

$$\begin{aligned}(\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})s)(\pi(x)) &= (\Phi(\widetilde{F}(\mathcal{L}_p)Q)\Phi^{-1}s)(\pi(x))\\ &= h(x)\int_{X_{\epsilon_0}} (\widetilde{F}(\mathcal{L}_p)Q)(x,x')h^{-1}(x')s(x')dv_X(x')\\ &= h(x)\int_{B_{\epsilon_0}} (\widetilde{F}(\mathcal{L}_p)Q)(x,y')h(y')s(y')dv_B(y').\end{aligned} \tag{2.69}$$

From (2.32), (2.64), (2.65) and (2.68), we get (2.63). □

Theorem 2.8 and (2.66) help us to understand that the asymptotic behavior of $P_p^G(x,x')$ is local near $X_G$. In the rest, we will not use directly Theorem 2.8.

## 2.6. Rescaling and a Taylor expansion of the operator $\Phi\mathcal{L}_p\Phi^{-1}$

Recall that $N_G$ is the normal bundle of $X_G$ in $B$, and we identify $N_G$ as the orthogonal complement of $TX_G$ in $(TB|_{X_G}, g^{TB})$.

Let $P^{TX_G}, P^{N_G}$ be the orthogonal projection from $TB|_{X_G}$ on $TX_G$, $N_G$.

Recall that $\nabla^{N_G}, {}^0\nabla^{TB}$ are connections on $N_G$, $TB$ on $X_G$, and $A$ is the associated second fundamental form defined in (0.9).

We fix $x_0 \in X_G$.

If $W \in T_{x_0}X_G$, let $\mathbb{R} \ni t \to x_t = \exp_{x_0}^{X_G}(tW) \in X_G$ be the geodesic in $X_G$ such that $x_t|_{t=0} = x_0$, $\frac{dx}{dt}|_{t=0} = W$.

If $W \in T_{x_0}X_G$, $|W| \leqslant \varepsilon$, $V \in N_{G,x_0}$, let $\tau_W V \in N_{G,\exp_{x_0}^{X_G}(W)}$ be the natural parallel transport of $V$ with respect to the connection $\nabla^{N_G}$ along the curve $[0,1] \ni t \to \exp_{x_0}^{X_G}(tW)$.

If $Z \in T_{x_0}B$, $Z = Z^0 + Z^\perp$, $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$, $|Z^0|, |Z^\perp| \leqslant \varepsilon$, we identify $Z$ with $\exp^B_{\exp_{x_0}^{X_G}(Z^0)}(\tau_{Z^0}Z^\perp)$. This identification is a diffeomorphism from $B_{x_0}^{TX_G}(0,\varepsilon) \times B_{x_0}^{N_G}(0,\varepsilon)$ into an open neighborhood $\mathscr{U}(x_0)$ of $x_0$ in $B$. We denote it by $\Psi$, and $\mathscr{U}(x_0) \cap X_G = B_{x_0}^{TX_G}(0,\varepsilon) \times \{0\}$.

From now on, we use indifferently the notation $B_{x_0}^{TX_G}(0,\varepsilon) \times B_{x_0}^{N_G}(0,\varepsilon)$ or $\mathscr{U}(x_0)$, $x_0$ or $0, \cdots$.

We identify $(L_B)_Z, (E_B)_Z$ and $(E_{p,B})_Z$ to $(L_B)_{x_0}, (E_B)_{x_0}$ and $(E_{p,B})_{x_0}$ by using parallel transport with respect to $\nabla^{L_B}, \nabla^{E_B}$ and $\nabla^{E_{p,B}}$ along the curve $\gamma_u : [0,1] \ni u \to uZ$.

Recall that $T^HU \subset TX$ is the horizontal bundle for $\pi : U \to B$ defined in Section 2.5.

Let $P^{T^HU}$ be the orthogonal projection from $TX$ onto $T^HU$.

For $W \in TB$, let $W^H \in T^HU$ be the horizontal lift of $W$.

For $y_0 \in \pi^{-1}(x_0)$, we define the curve $\widetilde{\gamma}_u : [0,1] \to X$ to be the lift of the curve $\gamma_u$ with $\widetilde{\gamma}_0 = y_0$ and $\frac{\partial \widetilde{\gamma}_u}{\partial u} \in T^HU$. Then on $\pi^{-1}(B^{TB}(0,\varepsilon))$, we use the parallel transport with respect to $\nabla^L, \nabla^E$ and $\nabla^{E_p}$ along the curve $\widetilde{\gamma}_u$ to trivialize the corresponding bundles. By (1.17), the previous trivialization is naturally induced by this one.

This also gives a trivialization of $\pi^{-1}(B^{TB}(0,\varepsilon))$ as $G \times B^{TB}(0,\varepsilon)$, and the $G$-action on $G \times B^{TB}(0,\varepsilon)$ induced from its action on $\pi^{-1}(B^{TB}(0,\varepsilon))$ is

$$g(1,Z) = (g,Z). \tag{2.70}$$

Let $\{e_i^0\}$, $\{e_j^\perp\}$ be orthonormal basis of $T_{x_0}X_G$, $N_{G,x_0}$, then $\{e_i\} = \{e_i^0, e_j^\perp\}$ is an orthonormal basis of $T_{x_0}B$. Let $\{e^i\}$ be its dual basis. We will also denote $\Psi_*(e_i^0), \Psi_*(e_j^\perp)$ by $e_i^0, e_j^\perp$. Thus in our coordinates,

$$\frac{\partial}{\partial Z_i^0} = e_i^0, \quad \frac{\partial}{\partial Z_j^\perp} = e_j^\perp. \tag{2.71}$$

In what follows, for $\varepsilon > 0$ small enough, we will extend the geometric objects on $B^{TB}(x_0,\varepsilon)$ to $\mathbb{R}^{2n-n_0} \simeq T_{x_0}B$ (here we identify $(Z_1, \cdots, Z_{2n-n_0}) \in \mathbb{R}^{2n-n_0}$ to $\sum_i Z_i e_i \in T_{x_0}B$) such that $D_p$ will become the restriction of a spin$^c$ Dirac operator on $G \times \mathbb{R}^{2n-n_0}$ associated to a Hermitian line bundle with positive curvature. In this way, we can replace $X$ by $G \times \mathbb{R}^{2n-n_0}$.

First of all, we denote by $L_0$, $E_0$ the trivial bundles $L|_{Gy_0}, E|_{Gy_0}$, lifted on $X_0 = G \times \mathbb{R}^{2n-n_0}$, and we still denote by $\nabla^L, \nabla^E$, $h^L$ etc. the connections and metrics on $L_0$, $E_0$ on $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$ induced by the above identification. Then $h^L$, $h^E$ is identified with the constant metrics $h^{L_0} = h^{L_{y_0}}$, $h^{E_0} = h^{E_{y_0}}$.

Set

$$\mathcal{R}^{\perp} = \sum_j Z_j^{\perp} e_j^{\perp} = Z^{\perp}, \quad \mathcal{R}^0 = \sum_i Z_i^0 e_i^0 = Z^0, \quad \mathcal{R} = \mathcal{R}^{\perp} + \mathcal{R}^0 = Z. \tag{2.72}$$

Then $\mathcal{R}$ is the radial vector field on $\mathbb{R}^{2n-n_0}$.

Let $\varepsilon > 0$ with $\varepsilon < \varepsilon_0/2$. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\varphi(v) = 1 \ \text{ if } \ |v| < 2; \quad \varphi(v) = 0 \ \text{ if } |v| > 4. \tag{2.73}$$

Let $\varphi_\varepsilon : X_0 \to X_0$ be the map defined by $\varphi_\varepsilon(g, Z) = (g, \varphi(|Z|/\varepsilon)Z)$ for $(g, Z) \in G \times \mathbb{R}^{2n-n_0}$.

Let $g^{TX_0}(g, Z) = g^{TX}(\varphi_\varepsilon(g, Z))$, $J_0(g, Z) = J(\varphi_\varepsilon(g, Z))$ be the metric and almost-complex structure on $X_0$.

Let $\nabla^{E_0} = \varphi_\varepsilon^* \nabla^E$, then $\nabla^{E_0}$ is the extension of $\nabla^E$ on $\pi^{-1}(B^{T_{x_0}B}(0, \varepsilon))$.

Let $\nabla^{L_0}$ be the Hermitian connection on $(L_0, h^{L_0})$ on $G \times \mathbb{R}^{2n-n_0}$ defined by that for $Z \in \mathbb{R}^{2n-n_0}$,

$$\nabla^{L_0} = \varphi_\varepsilon^* \nabla^L + \Big(1 - \varphi(\tfrac{|Z|}{\varepsilon})\Big) R^L_{y_0}(\mathcal{R}^H, P^{TY}_{y_0} \cdot) + \frac{1}{2}\Big(1 - \varphi^2(\tfrac{|Z|}{\varepsilon})\Big) R^L_{y_0}(\mathcal{R}^H, P^{T^HU}_{y_0} \cdot). \tag{2.74}$$

We calculate directly that its curvature $R^{L_0} = (\nabla^{L_0})^2$ is

$$\begin{aligned} R^{L_0}_Z = \psi_\varepsilon^* R^L + d\Big((1 - \varphi(\tfrac{|Z|}{\varepsilon}))R^L_{y_0}(Z, P^{TY}_{y_0} \cdot) + \frac{1}{2}(1 - \varphi^2(\tfrac{|Z|}{\varepsilon}))R^L_{y_0}(Z, P^{T^HU}_{y_0} \cdot)\Big) \\ = R^L_{\psi_\varepsilon(Z)}(P^{TY}_{y_0} \cdot, P^{TY}_{y_0} \cdot) + R^L_{y_0}(P^{T^HU}_{y_0} \cdot, \cdot) \\ + \varphi^2(\tfrac{|Z|}{\varepsilon})(R^L_{\psi_\varepsilon(Z)} - R^L_{y_0})(P^{T^HU}_{y_0} \cdot, P^{T^HU}_{y_0} \cdot) \\ + \varphi(\tfrac{|Z|}{\varepsilon})(R^L_{\psi_\varepsilon(Z)} - R^L_{y_0})(P^{T^HU}_{y_0} \cdot, P^{TY}_{y_0} \cdot) \\ - \varphi'(\tfrac{|Z|}{\varepsilon})\frac{Z^*}{\varepsilon|Z|} \wedge [R^L_{y_0}(Z, P^{TY}_{y_0} \cdot) - R^L_{\psi_\varepsilon(Z)}(Z, P^{TY}_{y_0} \cdot)] \\ - (\varphi\varphi')(\tfrac{|Z|}{\varepsilon})\frac{Z^*}{\varepsilon|Z|} \wedge [R^L_{y_0}(Z, P^{T^HU}_{y_0} \cdot) - R^L_{\psi_\varepsilon(Z)}(Z, P^{T^HU}_{y_0} \cdot)]. \end{aligned} \tag{2.75}$$

Here $Z^* \in T^*_{x_0}B$ is the dual of $Z \in T_{x_0}B$ with respect to the metric $g^{T_{x_0}B}$.

From (2.75), one deduces that $R^{L_0}$ is positive in the sense of (2.10) for $\varepsilon$ small enough, and the corresponding constant $\nu_0$ for $R^{L_0}$ is bigger than $\frac{4}{5}\nu_0$ uniformly for $y_0 \in P$.

From now on, we fix $\varepsilon$ as above.

Now $G$ acts naturally on $X_0$ by (2.70), and under our identification, the $G$-action on $L, E$ on $G \times B^{T_{x_0}B}(0, \varepsilon)$ is exactly the $G$-action on $L|_{Gy_0}, E|_{Gy_0}$.

We define a $G$-action on $L_0, E_0$ by its $G$-action on $Gy_0$, then it extends the $G$-action on $L, E$ on $G \times B^{T_{x_0}B}(0, \varepsilon)$ to $X_0$.

By (2.17), for any $K \in \mathfrak{g}$, $W \in TP$ on $P = \mu^{-1}(0)$, we have

$$\begin{aligned} R^L(W, K^X) &= -2\pi\sqrt{-1}\omega(W, K^X) = 2\pi\sqrt{-1}W(\mu(K)) = 0, \\ R^L_{(1,Z^0)}(\mathcal{R}^H, K^X) &= R^L_{(1,Z^0)}((\mathcal{R}^\perp)^H, K^X). \end{aligned} \tag{2.76}$$

Observe that for $(1, Z) \in G \times \mathbb{R}^{2n-n_0}$, by (2.70), $\varphi_{\varepsilon *} K^{X_0}_{(1,Z)} = K^X_{y_0}$ for $K \in \mathfrak{g}$, by (2.16), the moment map $\mu_{X_0} : X_0 \to \mathfrak{g}^*$ of the $G$-action on $X_0$ is given by

$$2\pi\sqrt{-1}\mu_{X_0}(K)_{(1,Z)} = (1 - \varphi(\tfrac{|Z|}{\varepsilon}))R^L_{y_0}(\mathcal{R}^H, K^X_{y_0}) + 2\pi\sqrt{-1}\mu(K)_{\varphi_\varepsilon(1,Z)}. \tag{2.77}$$

Now from the choice of our coordinate, we know that $\mu_{X_0} = 0$ on $G \times \mathbb{R}^{2n-2n_0} \times \{0\}$. Moreover,

$$2\pi\sqrt{-1}\mu(K)_{\varphi_\varepsilon(1,Z)} = R^L_{(1,Z)}(\varphi(\tfrac{|Z|}{\varepsilon})(\mathcal{R}^\perp)^H, K^X) + \mathscr{O}(\varphi(\tfrac{|Z|}{\varepsilon})|Z||Z^\perp|). \tag{2.78}$$

From our construction, (2.77) and (2.78), we know that

$$\mu^{-1}_{X_0}(0) = G \times \mathbb{R}^{2n-2n_0} \times \{0\}. \tag{2.79}$$

By (2.76) and (2.77), for $Z \in T_{x_0}B$, $|Z| \geqslant 4\varepsilon$,

$$2\pi\sqrt{-1}\mu_{X_0}(K)_{(1,Z)} = R^L_{y_0}((\mathcal{R}^\perp)^H, K^X_{y_0}). \tag{2.80}$$

Let $D^{X_0}_p$ be the Dirac operator on $X_0$ associated to the above data by the construction in Section 2.2. By the argument in [**27**, p. 656-657] and the proof of Theorem 2.5, we know the analogue of Theorems 2.2, 2.5 still holds for $D^{X_0}_p$. Let $\mathcal{L}^{X_0}_p$ be the operator on $X_0$ defined as in (2.24). Then there exists $C > 0$ such that for $p \geqslant 1$,

$$\operatorname{Spec}\left(\mathcal{L}^{X_0}_p\right) \subset \{0\} \cup [p\nu - C, +\infty[. \tag{2.81}$$

Set

$$E_{0,p} = \Lambda(T^{*(0,1)}X_0) \otimes L^p_0 \otimes E_0. \tag{2.82}$$

Let $g^{TB_0}$ be the metric on $B_0 = \mathbb{R}^{2n-n_0}$ induced by $g^{TX_0}$, and let $dv_{B_0}$ be the Riemannian volume form on $(B_0, g^{TB_0})$.

The operator $\Phi\mathcal{L}^{X_0}_p\Phi^{-1}$ is also well-defined on $T_{x_0}B \simeq \mathbb{R}^{2n-n_0}$.

Let $P_{x_0,p}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0}, (E_{0,p})_{B_0})$ onto $\operatorname{Ker}(\Phi\mathcal{L}^{X_0}_p\Phi^{-1})$ on $\mathbb{R}^{2n-n_0}$. Let $P_{x_0,p}(Z, Z')$ $(Z, Z' \in \mathbb{R}^{2n-n_0})$ be the smooth kernel of $P_{x_0,p}$ with respect to $dv_{B_0}(Z')$. As before, we view $P_{x_0,p}$ as a $G \times G$-invariant section of $\mathrm{pr}_1^*(E_{0,p}) \otimes \mathrm{pr}_2^*(E_{0,p})^*$ on $X_0 \times X_0$.

Let $P^G_{0,p}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_0, L^p_0 \otimes E_0)$ onto $(\operatorname{Ker} D^{X_0}_p)^G$, and let $P^G_{0,p}(x, x')$ be the smooth kernel of $P^G_{0,p}$ with respect to the volume form $dv_{X_0}(x')$.

Note that $\Phi$ in (1.20) defines an isometry from $(\operatorname{Ker} D^{X_0}_p)^G = \operatorname{Ker} \mathcal{L}^{X_0}_p$ onto $\operatorname{Ker}(\Phi\mathcal{L}^{X_0}_p\Phi^{-1})$, as in (2.68), we get

$$h(x)h(x')P^G_{0,p}(x, x') = P_{x_0,p}(\pi(x), \pi(x')). \tag{2.83}$$

***Proposition 2.9***. — *For any $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ such that for $x,x'\in G\times B^{T_{x_0}B}(0,\varepsilon)$,*

$$\Big|(P^G_{0,p}-P^G_p)(x,x')\Big|_{\mathscr{C}^m}\leqslant C_{l,m}p^{-l}. \tag{2.84}$$

*Proof*. — By the analogue of Theorems 2.2, 2.5, we know that for $x,x'\in G\times B^{T_{x_0}B}(0,\varepsilon)$, $P^G_{0,p}-\widetilde{F}(\mathcal{L}_p^{X_0})$ verifies also (2.32), and for $x,x'\in G\times B^{T_{x_0}B}(0,\varepsilon)$,

$$\widetilde{F}(\mathcal{L}_p^{X_0})(x,x')=\widetilde{F}(\mathcal{L}_p)(x,x')$$

by finite propagation speed. Thus we get (2.84). □

Let $T^{*(0,1)}X_0$ be the anti-holomorphic cotangent bundle of $(X_0,J_0)$. Since $J_0(g,Z)=J(\varphi_\varepsilon(g,Z))$, $T^{*(0,1)}_{Z,J_0}X_0$ is naturally identified with $T^{*(0,1)}_{\varphi_\varepsilon(g,Z),J}X_0$.

Let $\nabla^{\mathrm{Cliff}_0}$ be the Clifford connection on $\Lambda(T^{*(0,1)}X_0)$ induced by the Levi-Civita connection $\nabla^{TX_0}$ on $(X_0,g^{TX_0})$. Let $R^{E_0},R^{TX_0}$, $R^{\mathrm{Cliff}_0}$ be the corresponding curvatures on $E_0,TX_0$ and $\Lambda(T^{*(0,1)}X_0)$ (cf. (2.12)).

We identify $\Lambda(T^{*(0,1)}X_0)_{(g,Z)}$ with $\Lambda(T^{*(0,1)}_{(g,0)}X)$ by identifying first $\Lambda(T^{*(0,1)}X_0)_{(g,Z)}$ with $\Lambda(T^{*(0,1)}_{\varphi_\varepsilon(g,Z),J}X_0)$, which in turn is identified with $\Lambda(T^{*(0,1)}_{gy_0}X)$ by using parallel transport along $u\to u\varphi_\varepsilon(g,Z)$ with respect to $\nabla^{\mathrm{Cliff}_0}$. We also trivialize $\Lambda(T^{*(0,1)}X_0)$ in this way.

Let $S_L$ be a $G$-invariant unit section of $L|_{Gy_0}$. Using $S_L$ and the above discussion, we get an isometry

$$\Lambda(T^{*(0,1)}X_0)\otimes L_0^p\otimes E_0\simeq(\Lambda(T^{*(0,1)}X)\otimes E)|_{\pi^{-1}(x_0)}=:\mathbf{E}|_{\pi^{-1}(x_0)}.$$

For any $1\leqslant i\leqslant 2n-n_0$, let $\widetilde{e}_i(Z)$ be the parallel transport of $e_i$ with respect to the connection ${}^0\nabla^{TB}$ along $[0,1]\ni u\to uZ^0$, and with respect to the connection $\nabla^{TB}$ along $[1,2]\ni u\to Z^0+(u-1)Z^\perp$.

Recall that $A$, $\mathcal{R}^\perp$ have been defined in (0.9), (2.72).

The following Lemma extends [**1**, Prop. 1.28] (cf. also [**17**, Lemma 4.5]).

***Lemma 2.10***. — *The Taylor expansion of $\widetilde{e}_i(Z)$ with respect to the basis $\{e_i\}$ to order $r$ is a polynomial of the Taylor expansion of the curvature coefficients of $R^{TB}$ to order $r-2$ and $A$ to order $r-1$.*

*Proof*. — Let $\partial_i=\nabla_{e_i}$ be the partial derivatives along $e_i$.

Let $\Gamma^{TB}$ be the connection form of $\nabla^{TB}$ with respect to the frame $\{\widetilde{e}_i\}$ of $TB$. By the definition of our fixed frame, we have $i_{\mathcal{R}^\perp}\Gamma^{TB}=0$. As in [**1**, (1.12)],

$$L_{\mathcal{R}^\perp}\Gamma^{TB}=[i_{\mathcal{R}^\perp},d]\Gamma^{TB}=i_{\mathcal{R}^\perp}(d\Gamma^{TB}+\Gamma^{TB}\wedge\Gamma^{TB})=i_{\mathcal{R}^\perp}R^{TB}. \tag{2.85}$$

Let $\Theta(Z)=(\theta^i_j(Z))_{i,j=1}^{2n-n_0}$ be the $(2n-n_0)\times(2n-n_0)$-matrix such that

$$e_i=\sum_j\theta^j_i(Z)\widetilde{e}_j(Z),\quad \widetilde{e}_j(Z)=(\Theta(Z)^{-1})^k_je_k. \tag{2.86}$$

Set $\theta^j(Z) = \sum_i \theta^j_i(Z) e^i$ and

$$\theta = \sum_j e^j \otimes e_j = \sum_j \theta^j \widetilde{e}_j \in T^*B \otimes TB. \tag{2.87}$$

As $\nabla^{TB}$ is torsion free, $\nabla^{TB}\theta = 0$. Thus the $\mathbb{R}^{2n-n_0}$-valued one-form $\theta = (\theta^j(Z))$ satisfies the structure equation,

$$d\theta + \Gamma^{TB} \wedge \theta = 0. \tag{2.88}$$

By the same proof of [**1**, Prop. 1.27], we have

$$\mathcal{R}^\perp = \sum_j Z^\perp_j \widetilde{e}^\perp_j(Z), \quad i_{\mathcal{R}^\perp}\theta = \sum_j Z^\perp_j e^\perp_j = Z^\perp. \tag{2.89}$$

Here under our trivialization by $\{\widetilde{e}_i\}$, we consider $Z^\perp = (0, Z^\perp_1, \cdots, Z^\perp_{n_0})$ as a $\mathbb{R}^{2n-n_0}$-valued function.

Substituting (2.89) and $(L_{\mathcal{R}^\perp} - 1)Z^\perp = 0$ into the identity $i_{\mathcal{R}^\perp}(d\theta + \Gamma^{TB} \wedge \theta) = 0$, we obtain

$$(L_{\mathcal{R}^\perp} - 1)L_{\mathcal{R}^\perp}\theta = (L_{\mathcal{R}^\perp} - 1)(dZ^\perp + \Gamma^{TB} Z^\perp) = (L_{\mathcal{R}^\perp}\Gamma^{TB})Z^\perp = (i_{\mathcal{R}^\perp} R^{TB})Z^\perp. \tag{2.90}$$

Here we consider $R^{TB}$ as a matrix of 2-forms, so that $R^{TB}Z^\perp$ is a vector of 2-forms, and $\theta$ is a $\mathbb{R}^{2n-n_0}$-valued 1-form.

By (2.89) and (2.90), we get

$$i_{e_j}(L_{\mathcal{R}^\perp} - 1)L_{\mathcal{R}^\perp}\theta^i(Z) = \left\langle R^{TB}(\mathcal{R}^\perp, e_j)\mathcal{R}^\perp, \widetilde{e}_i \right\rangle (Z). \tag{2.91}$$

We will denote by $\partial^\perp$, $\partial^0$ the partial derivatives along $N_G$, $TX_G$ respectively. Then we have the following Taylor expansions of (2.91): for $j \in \{2(n-n_0)+1, \cdots, 2n-n_0\}$, i.e. $e_j \in N_G$, by $L_{\mathcal{R}^\perp} e^j = e^j$, we have

$$\sum_{|\alpha^\perp| \geqslant 1} (|\alpha^\perp|^2 + |\alpha^\perp|)((\partial^\perp)^{\alpha^\perp}\theta^i_j)(Z^0)\frac{(Z^\perp)^{\alpha^\perp}}{\alpha^\perp!} = \left\langle R^{TB}(\mathcal{R}^\perp, e_j)\mathcal{R}^\perp, \widetilde{e}_i \right\rangle (Z). \tag{2.92}$$

and for $j \in \{1, \cdots, 2(n-n_0)\}$, i.e. $e_j \in TX_G$, by $L_{\mathcal{R}^\perp} e^j = 0$, we have

$$\sum_{|\alpha^\perp| \geqslant 1} (|\alpha^\perp|^2 - |\alpha^\perp|)((\partial^\perp)^{\alpha^\perp}\theta^i_j)(Z^0)\frac{(Z^\perp)^{\alpha^\perp}}{\alpha^\perp!} = \left\langle R^{TB}(\mathcal{R}^\perp, e_j)\mathcal{R}^\perp, \widetilde{e}_i \right\rangle (Z). \tag{2.93}$$

From (2.92), (2.93), we still need to obtain the Taylor expansions for $\theta^i_j(Z^0)$, $(1 \leqslant i, j \leqslant 2n - n_0)$ and $(\partial^\perp_k \theta^i_j)(Z^0)$, $(1 \leqslant j \leqslant 2(n-n_0))$.

By our construction, we know that for $i$ or $j \in \{2(n-n_0)+1, \cdots, 2n-n_0\}$,

$$\widetilde{e}^\perp_k(Z^0) = e^\perp_k(Z^0), \quad \theta^i_j(Z^0) = \delta_{i,j}. \tag{2.94}$$

By [**1**, (1.21)] (cf. [**17**, (4.35)]), we know that on $\mathbb{R}^{2n-2n_0}\times\{0\}$, for $i,j\in\{1,\cdots,2(n-n_0)\}$,

$$\begin{aligned}&\theta^i_j(0)=\delta_{i,j},\\ &\sum_{|\alpha^0|\geqslant 1}(|\alpha^0|^2+|\alpha^0|)((\partial^0)^{\alpha^0}\theta^i_j)(0)\frac{(Z^0)^{\alpha^0}}{\alpha^0!}=\left\langle R^{TX_G}(\mathcal{R}^0,e_j)\mathcal{R}^0,\widetilde{e}_i\right\rangle(Z^0),\end{aligned}\tag{2.95}$$

while by (0.9), (2.86), and $[e^\perp_i,e^\perp_j]=0$ (cf. (2.71)), we get

$$\begin{aligned}(\partial^\perp_k\theta^i_j)(Z^0)&=e^\perp_k\langle e^0_j,\widetilde{e}^0_i\rangle(Z^0)=\langle\nabla^{TB}_{e^\perp_k}e^0_j,\widetilde{e}^0_i\rangle(Z^0)\\ &=\langle\nabla^{TB}_{e^0_j}e^\perp_k,\widetilde{e}^0_i\rangle(Z^0)=-\langle\nabla^{TB}_{e^0_j}\widetilde{e}^0_i,e^\perp_k\rangle(Z^0)=-\langle A(e^0_j)\widetilde{e}^0_i,e^\perp_k\rangle(Z^0).\end{aligned}\tag{2.96}$$

Let $R^{TX_G}$, $R^{N_G}$ be the curvatures of $\nabla^{TX_G},\nabla^{N_G}$. By (0.9),

$$R^{TX_G}+R^{N_G}+A^2+{}^0\nabla^{TB}_{\cdot}A=R^{TB}|_{X_G}\in\Lambda^2(TX_G)\otimes\mathrm{End}(TB).\tag{2.97}$$

For $1\leqslant j\leqslant 2(n-n_0)$, $2(n-n_0)+1\leqslant i\leqslant 2n-n_0$, $i'=i-2(n-n_0)$, by $[e^\perp_k,e^0_j]=0$, as in (2.96), we get

$$(\partial^\perp_k\theta^i_j)(Z^0)=e^\perp_k\langle e^0_j,\widetilde{e}^\perp_{i'}\rangle(Z^0)=\langle\nabla^{TB}_{e^0_j}e^\perp_k,\widetilde{e}^\perp_{i'}\rangle(Z^0)=\langle\nabla^{N_G}_{e^0_j}e^\perp_k,e^\perp_{i'}\rangle(Z^0).\tag{2.98}$$

By [**1**, Prop. 1.18] (cf. (2.103)) and (2.98), the Taylor expansion of $(\partial^\perp_k\theta^i_j)(Z^0)$ at 0 to order $r$ only determines by those of $R^{N_G}$ to order $r-1$.

Now by (2.86), (2.92)-(2.98) determine the Taylor expansion of $\theta^i_j(Z)$ to order $m$ in terms of the Taylor expansion of the curvature coefficients of $R^{TB}$ to order $m-2$ and $A$ to order $m-1$.

By (2.86), we get Lemma 2.10. □

Let $dv_{TB}$ be the Riemannian volume form on $(T_{x_0}B,g^{TB})$.

Let $\kappa(Z)$ $(Z\in\mathbb{R}^{2n-n_0})$ be the smooth positive function defined by the equation

$$dv_{B_0}(Z)=\kappa(Z)dv_{TB}(Z),\tag{2.99}$$

with $\kappa(0)=1$.

For $s\in\mathscr{C}^\infty(\mathbb{R}^{2n-n_0},\mathbf{E}_{x_0})$ and $Z\in\mathbb{R}^{2n-n_0}$, for $t=\frac{1}{\sqrt{p}}$, set

$$\begin{aligned}&(S_ts)(Z):=s(Z/t),\quad \nabla_t:=S_t^{-1}t\kappa^{\frac12}\nabla^{E_p,B_0}\kappa^{-\frac12}S_t,\\ &\mathscr{L}^t_2:=S_t^{-1}t^2\kappa^{\frac12}\Phi D_p^{X_0,2}\Phi^{-1}\kappa^{-\frac12}S_t.\end{aligned}\tag{2.100}$$

As in (1.18), we denote by $R^{L_B},R^{E_B},R^{\mathrm{Cliff}_B}$ the curvatures on $L_B,E_B$, $\Lambda(T^{*(0,1)}X)_B$ induced by $\nabla^L,\nabla^E$,$\nabla^{\mathrm{Cliff}}$ on $X$.

As in (1.14), $\widetilde{\mu}\in TY$, $\widetilde{\mu}^E\in TY\otimes\mathrm{End}(E)$, $\widetilde{\mu}^{\mathrm{Cliff}}\in TY\otimes\mathrm{End}(\Lambda(T^{*(0,1)}X))$ are sections induced by $\mu,\mu^E,\mu^{\mathrm{Cliff}}$ in (2.17), (2.23).

Denote by $\nabla_V$ the ordinary differentiation operator on $T_{x_0}B$ in the direction $V$.

Denote by $(\partial^\alpha R^{L_B})_{x_0}$ the tensor $(\partial^\alpha R^{L_B})_{x_0}(e_i,e_j):=\partial^\alpha(R^{L_B}(e_i,e_j))_{x_0}$.

***Theorem 2.11***. — *There exist* $\mathcal{A}_{i,j,r}$ *(resp.* $\mathcal{B}_{i,r}$, $\mathcal{C}_r$*)* $(r\in\mathbb{N}, i,j\in\{1,\cdots,2n-n_0\})$ *polynomials in* $Z$*, and* $\mathcal{A}_{i,j,r}$ *is a homogeneous polynomial in* $Z$ *with degree* $r$*, the degree on* $Z$ *of* $\mathcal{B}_{i,r}$ *is* $\leqslant r+1$ *(resp.* $\mathcal{C}_r$ *is* $\leqslant r+2$*), and has the same parity with* $r-1$ *(resp.* $r$*), with the following properties:*

*– the coefficients of* $\mathcal{A}_{i,j,r}$ *are polynomials in* $R^{TB}$ *(resp.* $A$*) and their derivatives at* $x_0$ *to order* $r-2$ *(resp.* $r-1$*);*

*– the coefficients of* $\mathcal{B}_{i,r}$ *are polynomials in* $R^{TB}$, $R^{\mathrm{Cliff}_B}$, $R^{E_B}$, *(resp.* $A$, $R^{L_B}$*) and their derivatives at* $x_0$ *to order* $r-2$ *(resp.* $r-1$, $r$*);*

*– the coefficients of* $\mathcal{C}_r$ *are polynomials in* $R^{TB}$, $R^{\mathrm{Cliff}_B}$, $R^{E_B}$, $r^X$, $\mathrm{Tr}[R^{T^{(1,0)}X}]$, $R^E$ *(resp.* $A$, $\widetilde{\mu}^E$, $\widetilde{\mu}^{\mathrm{Cliff}}$*; resp.* $h$, $R^L$, $R^{L_B}$*; resp.* $\mu$*) and their derivatives at* $x_0$ *to order* $r-2$ *(resp.* $r-1$*; resp.* $r$*; resp.* $r+1$*).*

*– if we denote by*

$$\mathcal{O}_r=\mathcal{A}_{i,j,r}\nabla_{e_i}\nabla_{e_j}+\mathcal{B}_{i,r}\nabla_{e_i}+\mathcal{C}_r,$$

$$\mathscr{L}_2^0=-\sum_{j=1}^{2n-n_0}\Big(\nabla_{e_j}+\frac{1}{2}R^{L_B}_{x_0}(\mathcal{R},e_j)\Big)^2-2\omega_{d,x_0}-\tau_{x_0}+4\pi^2|P^{TY}\mathbf{J}_{x_0}\mathcal{R}|^2, \tag{2.101}$$

*then*

$$\mathscr{L}_2^t=\mathscr{L}_2^0+\sum_{r=1}^{m}t^r\mathcal{O}_r+\mathscr{O}(t^{m+1}). \tag{2.102}$$

*Moreover, there exists* $m'\in\mathbb{N}$ *such that for any* $k\in\mathbb{N}$, $t\leqslant 1$, $|tZ|\leqslant\varepsilon$*, the derivatives of order* $\leqslant k$ *of the coefficients of the operator* $\mathscr{O}(t^{m+1})$ *are dominated by* $Ct^{m+1}(1+|Z|)^{m'}$*.*

*Proof*. — Let $\Gamma^{E_B}$, $\Gamma^{L_B}$ and $\Gamma^{\mathrm{Cliff}_B}$ be the connection forms of $\nabla^{E_B}$, $\nabla^{L_B}$ and $\nabla^{\mathrm{Cliff}_B}$ with respect to any fixed frames for $E_B$, $L_B$ and $\Lambda(T^{*(1,0)}X)_B$ which are parallel along the curve $\gamma_u:[0,1]\ni u\to uZ$ under our trivialization on $B^{T_{x_0}B}(0,\varepsilon)$. Then $\Gamma^{E_B}$ is a $\mathrm{End}(\mathbb{C}^{\dim E})$-valued 1-form on $\mathbb{R}^{2n-n_0}$ and $\Gamma^{L_B}$ is a 1-form on $\mathbb{R}^{2n-n_0}$.

Now for $\Gamma^\bullet=\Gamma^{E_B},\Gamma^{L_B}$ or $\Gamma^{\mathrm{Cliff}_B}$ and $R^\bullet=R^{E_B},R^{L_B}$ or $R^{\mathrm{Cliff}_B}$ respectively, by the definition of our fixed frame and [**1**, Proposition 1.18] (cf. also [**31**, Prop. 1.2.4]), the Taylor coefficients of $\Gamma^\bullet(e_j)(Z)$ at $x_0$ to order $r$ only determines by those of $R^\bullet$ to order $r-1$, and

$$\sum_{|\alpha|=r}(\partial^\alpha\Gamma^\bullet)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!}=\frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^\bullet)_{x_0}(\mathcal{R},e_j)\frac{Z^\alpha}{\alpha!}. \tag{2.103}$$

Especially,

$$\Gamma^\bullet_Z(e_j)=\frac{1}{2}R^\bullet_{x_0}(\mathcal{R},e_j)+\mathscr{O}(|Z|^2). \tag{2.104}$$

By (2.100), for $t=1/\sqrt{p}$, if $|Z|\leqslant\sqrt{p}\varepsilon$, then

$$\nabla_t=\kappa^{\frac12}(tZ)\Big(\nabla+\Big(t\Gamma^{\mathrm{Cliff}_B}+t\Gamma^{E_B}+\frac{1}{t}\Gamma^{L_B}\Big)(tZ)\Big)\kappa^{-\frac12}(tZ). \tag{2.105}$$

Moreover, set

$$(\nabla^{TB}_{e_i} e_j)(Z) = \Gamma^k_{ij}(Z) e_k, \quad g_{ij}(Z) = g^{TB}(e_i, e_j)(Z) = \theta^k_i \theta^k_j(Z), \tag{2.106}$$

then $\Gamma^k_{ij}$ is the connection form of $\nabla^{TB}$ with respect to the frame $\{e_i\}$.

Let $(g^{ij})$ be the inverse matrix of $(g_{ij})$, then

$$\Delta^{E_{p,B}} = -\sum_{i,j} g^{ij}\Big(\nabla^{E_{p,B}}_{e_i}\nabla^{E_{p,B}}_{e_j} - \Gamma^k_{ij}\nabla^{E_{p,B}}_{e_k}\Big), \tag{2.107}$$

and by (1.1), (2.99),

$$\begin{aligned} &\kappa(Z) = (\det g_{ij})^{1/2}(Z),\\ &\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \end{aligned} \tag{2.108}$$

By (2.62), (2.100) and (2.107),

$$\begin{aligned} \mathscr{L}^t_2(Z) = -g^{ij}(tZ)(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_k}) - \langle t\widetilde{\mu}^{E_p}, t\widetilde{\mu}^{E_p}\rangle_{g^{TY}}(tZ)\\ - 2\omega_d(tZ) - \tau(tZ) + t^2\Big(\frac{1}{4}r^X + \mathbf{c}(R) - \frac{1}{h}\Delta_{B_0} h\Big)(tZ). \end{aligned} \tag{2.109}$$

By (2.23),

$$\langle t\widetilde{\mu}^{E_p}, t\widetilde{\mu}^{E_p}\rangle_{g^{TY}} = -4\pi^2|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}} + \langle 4\pi\sqrt{-1}\widetilde{\mu} + t^2(\widetilde{\mu}^{\text{Cliff}} + \widetilde{\mu}^E), \widetilde{\mu}^{\text{Cliff}} + \widetilde{\mu}^E\rangle_{g^{TY}}. \tag{2.110}$$

By (2.6), (2.17), and $\widetilde{\mu}_{y_0} = 0$, for $y_0 \in P, \pi(y_0) = x_0$, we get for $K \in \mathfrak{g}$,

$$-\langle \mathbf{J}e^H_i, K^X\rangle_{y_0} = \omega(K^X, e^H_i) = \nabla_{e^H_i}(\mu(K)) = \langle \nabla^{TY}_{e^H_i}\widetilde{\mu}, K^X\rangle_{y_0}, \tag{2.111}$$

thus

$$|\widetilde{\mu}|^2_{g^{TY}}(Z) = |\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}|^2_{g^{TY}} + \mathscr{O}(|Z|^3) = |P^{TY}\mathbf{J}_{x_0}\mathcal{R}|^2 + \mathscr{O}(|Z|^3). \tag{2.112}$$

By Lemma 2.10, (2.103), (2.105), (2.109) and (2.112), we know that $\mathscr{L}^t_2$ has the expansion (2.102), in particular, we get the formula $\mathscr{L}^0_2$ in (2.101).

By (2.97), (2.103) and (2.109), we get the properties on $\mathcal{A}_{i,j,r}$, $\mathcal{B}_{i,r}$.

By (2.97), (2.109) and (2.110), we get the properties on $\mathcal{C}_r$.

The proof of Theorem 2.11 is complete. □

## 2.7. Uniform estimate on the $G$-invariant Bergman kernel

Recall that the operators $\mathscr{L}^t_2$, $\nabla_t$ were defined in (2.100), and $\mathbf{E}_0 = \Lambda(T^{*(0,1)}X_0)\otimes E_0$. We have trivialized the bundle $\mathbf{E}_{0,B_0}$ to $\mathbf{E}_{B,x_0}$ in Section 2.6. We still denote by $h^{\mathbf{E}_{0,B_0}}$ the metric on the trivial bundle $\mathbf{E}_{B,x_0}$ on $\mathbb{R}^{2n-n_0}$ induced by the corresponding metric on $\mathbf{E}_{0,B_0}$. By our trivialization, $(E_{0,B}, h^{\mathbf{E}_{0,B_0}})$ is identified to the trivial Hermitian vector bundle $(E_{B,x_0}, h^{\mathbf{E}_{B,x_0}})$.

We also denote by $\langle\ ,\ \rangle_{0,L^2}$ and $\|\ \|_{0,L^2}$ the scalar product and the $L^2$ norm on $\mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ induced by $g^{T_{x_0}B}, h^{\mathbf{E}_{0,B_0}}$ as in (1.19).

Let $\widetilde{\mu}_{X_0}, \widetilde{\mu}^{E_{0,p}}$ be the $G$-invariant sections of $TY$, $TY \otimes \mathrm{End}(E_{0,p})$ on $X_0$ induced by $\mu_{X_0}$, $\mu^{E_{0,p}}$ as in (1.14).

Let $\{f_l\}$ be a $G$-invariant orthonormal frame of $TY$ on $\pi^{-1}(B^B(x_0,\varepsilon))$, then $(f_{0,l})_Z = (f_l)_{\varphi_\varepsilon(Z)}$ is a $G$-invariant orthonormal frame of $TY_0$ on $X_0$.

***Definition 2.12.*** — Set

$$\mathcal{D}_t = \{\nabla_{t,e_i}, 1 \leqslant i \leqslant 2n-n_0; \frac{1}{t}\langle \widetilde{\mu}_{X_0}, f_{0,l}\rangle(tZ), 1 \leqslant l \leqslant n_0\}. \tag{2.113}$$

For $k \in \mathbb{N}^*$, let $\mathcal{D}_t^k$ be the family of operators acting on $\mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ which can be written in the form $Q = Q_1 \cdots Q_k$, $Q_i \in \mathcal{D}_t$.

For $s \in \mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$, $k \geqslant 1$, set

$$\begin{aligned} \|s\|_{t,0}^2 &= \int_{\mathbb{R}^{2n-n_0}} |s(Z)|^2_{h^{\mathbf{E}_{B,x_0}}} dv_{T_{x_0}B}(Z), \\ \|s\|_{t,k}^2 &= \|s\|_{t,0}^2 + \sum_{l=1}^{k} \sum_{Q\in\mathcal{D}_t^l} \|Qs\|_{t,0}^2. \end{aligned} \tag{2.114}$$

We denote by $\langle s', s\rangle_{t,0}$ the inner product on $\mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ corresponding to $\| \ \|_{t,0}^2$.

Let $\boldsymbol{H}_t^m$ be the Sobolev space of order $m$ with norm $\| \ \|_{t,m}$. Let $\boldsymbol{H}_t^{-1}$ be the Sobolev space of order $-1$ and let $\| \ \|_{t,-1}$ be the norm on $\boldsymbol{H}_t^{-1}$ defined by $\|s\|_{t,-1} = \sup_{0\neq s'\in \boldsymbol{H}_t^1} |\langle s, s'\rangle_{t,0}|/\|s'\|_{t,1}$.

If $A \in \mathscr{L}(\boldsymbol{H}_t^m, \boldsymbol{H}_t^{m'})$ $(m, m' \in \mathbb{Z})$, we denote by $\|A\|_t^{m,m'}$ the norm of $A$ with respect to the norms $\| \ \|_{t,m}$ and $\| \ \|_{t,m'}$.

Then $\mathscr{L}_2^t$ is a formally self-adjoint elliptic operator with respect to $\| \ \|_{t,0}^2$, and is a smooth family of operators with respect to the parameter $x_0 \in X_G$.

***Theorem 2.13.*** — *There exist constants $C_1, C_2, C_3 > 0$ such that for $t \in ]0,1]$ and any $s, s' \in C_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$,*

$$\begin{aligned} \left\langle \mathscr{L}_2^t s, s\right\rangle_{t,0} &\geqslant C_1\|s\|_{t,1}^2 - C_2\|s\|_{t,0}^2, \\ |\left\langle \mathscr{L}_2^t s, s'\right\rangle_{t,0}| &\leqslant C_3\|s\|_{t,1}\|s'\|_{t,1}. \end{aligned} \tag{2.115}$$

*Proof.* — By (2.80) and our construction for $L_0, E_0$ on $X_0$, we know for $Z \in T_{x_0}B$, $|Z| > 4\varepsilon$,

$$\mu^{E_{0,p}}(K)_{(1,Z)} = p\, R^L_{y_0}((\mathcal{R}^\perp)^H, K^X_{y_0}). \tag{2.116}$$

Thus from (2.109) and (2.114),

$$\begin{aligned} \left\langle \mathscr{L}_2^t s, s\right\rangle_{t,0} &= \|\nabla_t s\|_{t,0}^2 - t^2 \left\langle \langle \widetilde{\mu}^{E_{0,p}}, \widetilde{\mu}^{E_{0,p}}\rangle_{g^{TY}}(tZ) s, s\right\rangle_{t,0} \\ &\quad + \left\langle \left(-2S_t^{-1}\omega_d - S_t^{-1}\tau + t^2 S_t^{-1}(\tfrac{1}{4}r^X + \mathbf{c}(R) - \frac{1}{h}\Delta_{B_0}h)\right) s, s\right\rangle_{t,0}. \end{aligned} \tag{2.117}$$

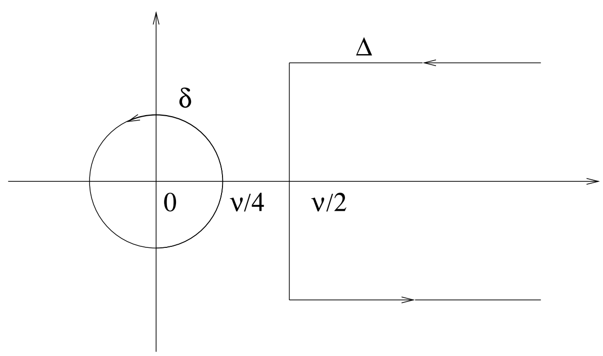
From (2.77), (2.110), (2.116), and our construction on $\nabla^{E_0}$,

(2.118)
$$-t^2 \left\langle \langle \widetilde{\mu}^{E_{0,p}}, \widetilde{\mu}^{E_{0,p}} \rangle_{g^{TY}}(tZ)s, s \right\rangle_{t,0} \geqslant 2\pi^2 \sum_{l=1}^{n_0} \left\| \frac{1}{t} \langle \widetilde{\mu}_{X_0}, f_{0,l} \rangle (tZ)s \right\|_{t,0}^2 - Ct\|s\|_{t,0}^2.$$

From (2.117) and (2.118), we get (2.115). □

Recall that $\nu$ is the constant in (2.25).

Let $\delta$ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\nu/4$, and let $\Delta$ be the oriented path in $\mathbb{C}$ which goes parallel to the real axis from $+\infty + i$ to $\frac{\nu}{2} + i$ then parallel to the imaginary axis to $\frac{\nu}{2} - i$ and the parallel to the real axis to $+\infty - i$.




Theorems 2.14–2.16 are the analogues of [**17**, Theorems 4.8-4.10] (cf. also [**31**, Theorems 4.1.10-4.1.12]). Especially, the proofs of Theorems 2.14, 2.16 are exactly the same as the proof of [**17**, Theorems 4.8, 4.10], we include the proofs for the sake of completeness.

***Theorem 2.14.*** — *There exist $t_0 > 0$, $C > 0$ such that for $t \in ]0, t_0]$, $\lambda \in \delta \cup \Delta$ and $x_0 \in X_G$, $(\lambda - \mathscr{L}_2^t)^{-1}$ exists and*

(2.119)
$$\|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{0,0} \leqslant C,$$
$$\|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{-1,1} \leqslant C(1 + |\lambda|^2).$$

*Proof.* — By (2.25), (2.62) for $D_p^{X_0}$, and (2.100), there exists $t_0 > 0$ such that for $t \in ]0, t_0]$,

(2.120)
$$\operatorname{Spec}\left(\mathscr{L}_2^t\right) \subset \{0\} \cup [\nu, +\infty[\,.$$

Thus $(\lambda - \mathscr{L}_2^t)^{-1}$ exists for $\lambda \in \delta \cup \Delta$.

The first inequality of (2.119) is from (2.120).

By (2.115), for $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leqslant -2C_2$, $(\lambda_0 - \mathscr{L}_2^t)^{-1}$ exists, and we have $\|(\lambda_0 - \mathscr{L}_2^t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{C_1}$. Now,

$$(\lambda - \mathscr{L}_2^t)^{-1} = (\lambda_0 - \mathscr{L}_2^t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_2^t)^{-1}(\lambda_0 - \mathscr{L}_2^t)^{-1}. \tag{2.121}$$

Thus for $\lambda \in \delta \cup \Delta$, from (2.121), we get

$$\|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{-1,0} \leq \frac{1}{C_1}\Big(1 + \frac{4}{\nu}|\lambda - \lambda_0|\Big). \tag{2.122}$$

Now we change the last two factors in (2.121), and apply (2.122), we get

$$\begin{aligned}\|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{-1,1} &\leqslant \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{{C_1}^2}\Big(1 + \frac{4}{\nu}|\lambda - \lambda_0|\Big) \\ &\leqslant C(1 + |\lambda|^2).\end{aligned} \tag{2.123}$$

The proof of our Theorem is complete. □

***Proposition 2.15***. — *Take $m \in \mathbb{N}^*$. There exists $C_m > 0$ such that for $t \in ]0,1]$, $Q_1, \cdots, Q_m \in \mathcal{D}_t \cup \{Z_i\}_{i=1}^{2n-n_0}$ and $s, s' \in \mathscr{C}_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$,*

$$\Big|\big\langle [Q_1, [Q_2, \ldots, [Q_m, \mathscr{L}_2^t] \ldots]] s, s'\big\rangle_{t,0}\Big| \leqslant C_m \|s\|_{t,1} \|s'\|_{t,1}. \tag{2.124}$$

*Proof*. — Note that $[\nabla_{t,e_i}, Z_j] = \delta_{ij}$. By (2.109), we know that $[Z_j, \mathscr{L}_2^t]$ verifies (2.124).

Recall that by (2.77) and (2.80), $(\nabla_{e_i}\langle\widetilde{\mu}_{X_0}, f_{0,l}\rangle)(tZ)$ is uniformly bounded with its derivatives for $t \in [0,1]$ and

$$\nabla_{e_i}\langle\widetilde{\mu}_{X_0}, f_{0,l}\rangle = (e_i\langle\widetilde{\mu}_{X_0}, f_{0,l}\rangle)_{x_0} = \omega(f_{0,l}, e_i)_{x_0} \tag{2.125}$$

for $|Z| \geqslant 4\varepsilon$. Thus $[\frac{1}{t}\langle\widetilde{\mu}_{X_0}, f_{0,l}\rangle(tZ), \mathscr{L}_2^t]$ also verifies (2.124).

Note that by (2.100),

$$[\nabla_{t,e_i}, \nabla_{t,e_j}] = \big(R^{L_{0,B_0}}(tZ) + t^2 R^{\mathbf{E}_{0,B_0}}(tZ)\big)(e_i, e_j). \tag{2.126}$$

Thus from (2.109), (2.125) and (2.126), we know that $[\nabla_{t,e_k}, \mathscr{L}_2^t]$ has the same structure as $\mathscr{L}_2^t$ for $t \in ]0,1]$, i.e. $[\nabla_{t,e_k}, \mathscr{L}_2^t]$ has the type as

$$\begin{aligned}&\sum_{ij} a_{ij}(t,tZ)\nabla_{t,e_i}\nabla_{t,e_j} + \sum_i c_i(t,tZ)\nabla_{t,e_i} \\ &\qquad + \sum_l \Big[c_l'(t,tZ)\frac{1}{t}\langle\widetilde{\mu}_{X_0}, f_{0,l}\rangle(tZ) + d\big|\frac{1}{t}\widetilde{\mu}_{X_0}\big|^2_{g^{TY}}(tZ)\Big] + c(t,tZ),\end{aligned} \tag{2.127}$$

where $d \in \mathbb{C}$; $a_{ij}(t,Z), c_i(t,Z), c_j'(t,Z), c(t,Z)$ and their derivatives on $Z$ are uniformly bounded for $Z \in \mathbb{R}^{2n-n_0}, t \in [0,1]$; moreover, they are polynomials in $t$. In fact, for $[\nabla_{t,e_k}, \mathscr{L}_2^t]$, $d = 0$ in (2.127).

Let $(\nabla_{t,e_i})^*$ be the adjoint of $\nabla_{t,e_i}$ with respect to $\langle\ ,\ \rangle_{t,0}$, then by (2.114),

$$(\nabla_{t,e_i})^* = -\nabla_{t,e_i} - t(k^{-1}\nabla_{e_i}k)(tZ), \tag{2.128}$$

the last term of (2.128) and its derivatives in $Z$ are uniformly bounded in $Z \in \mathbb{R}^{2n-n_0}, t \in [0,1]$.

By (2.127) and (2.128), (2.124) is verified for $m = 1$.

By iteration, we know that $[Q_1, [Q_2, \dots, [Q_m, \mathscr{L}_2^t] \dots]]$ has the same structure (2.127) as $\mathscr{L}_2^t$. By (2.128), we get Proposition 2.15. □

***Theorem 2.16***. — *For any $t \in ]0, t_0]$, $\lambda \in \delta \cup \Delta$, $m \in \mathbb{N}$, the resolvent $(\lambda - \mathscr{L}_2^t)^{-1}$ maps $\boldsymbol{H}_t^m$ into $\boldsymbol{H}_t^{m+1}$. Moreover for any $\alpha \in \mathbb{N}^{2n-n_0}$, there exist $N \in \mathbb{N}$, $C_{\alpha,m} > 0$ such that for $t \in ]0, t_0]$, $\lambda \in \delta \cup \Delta$, $s \in \mathscr{C}_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$,*

$$\|Z^\alpha (\lambda - \mathscr{L}_2^t)^{-1} s\|_{t,m+1} \leqslant C_{\alpha,m} (1 + |\lambda|^2)^N \sum_{\alpha' \leqslant \alpha} \|Z^{\alpha'} s\|_{t,m}. \tag{2.129}$$

*Proof.* — For $Q_1, \cdots, Q_m \in \mathcal{D}_t$, $Q_{m+1}, \cdots, Q_{m+|\alpha|} \in \{Z_i\}_{i=1}^{2n-n_0}$, we can express $Q_1 \cdots Q_{m+|\alpha|} (\lambda - \mathscr{L}_2^t)^{-1}$ as a linear combination of operators of the type

$$[Q_1, [Q_2, \dots [Q_{m'}, (\lambda - \mathscr{L}_2^t)^{-1}] \dots]] Q_{m'+1} \cdots Q_{m+|\alpha|}, \quad m' \leqslant m + |\alpha|. \tag{2.130}$$

Let $\mathscr{R}_t$ be the family of operators

$$\mathscr{R}_t = \{[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \mathscr{L}_2^t] \dots]]\}.$$

Clearly, any commutator $[Q_1, [Q_2, \dots [Q_{m'}, (\lambda - \mathscr{L}_2^t)^{-1}] \dots]]$ is a linear combination of operators of the form

$$(\lambda - \mathscr{L}_2^t)^{-1} R_1 (\lambda - \mathscr{L}_2^t)^{-1} R_2 \cdots R_{m'} (\lambda - \mathscr{L}_2^t)^{-1} \tag{2.131}$$

with $R_1, \cdots, R_{m'} \in \mathscr{R}_t$.

By Proposition 2.15, the norm $\|\ \ \|_t^{1,-1}$ of the operators $R_j \in \mathscr{R}_t$ is uniformly bound by $C$.

By Theorem 2.14, we find that there exist $C > 0$, $N \in \mathbb{N}$ such that the norm $\|\ \ \|_t^{0,1}$ of operators (2.131) is dominated by $C(1 + |\lambda|^2)^N$. □

Let $\pi_B : TB \times_B TB \to B$ be the natural projection from the fiberwise product of $TB$ on $B$.

Let $e^{-u\mathscr{L}_2^t}(Z, Z')$, $(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})(Z, Z')$ be the smooth kernels of the operators $e^{-u\mathscr{L}_2^t}$, $\mathscr{L}_2^t e^{-u\mathscr{L}_2^t}$ with respect to $dv_{T_{x_0}B}(Z')$.

Note that $\mathscr{L}_2^t$ are families of differential operators with coefficients in $\mathrm{End}(\mathbf{E}_{B,x_0}) = \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}$. Thus we can view $e^{-u\mathscr{L}_2^t}(Z, Z')$, $(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})(Z, Z')$ as smooth sections of $\pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$ on $TB \times_B TB$.

Let $\nabla^{\mathrm{End}(\mathbf{E}_B)}$ be the connection on $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B$ induced by $\nabla^{\mathrm{Cliff}_B}$ and $\nabla^{E_B}$. And $\nabla^{\mathrm{End}(\mathbf{E}_B)}$, $h^E$ and $g^{TX}$ induce naturally a $\mathscr{C}^m$-norm for the parameter $x_0 \in X_G$.

As in Introduction, for $Z \in T_{x_0}B$, we will write $Z = Z^0 + Z^\perp$, with $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$.

In the following result, we adapt [**17**, Theorem 4.11] to the present situation. The new point is that the kernels here have the fast decay estimate along the normal direction $N_{G,x_0}$.

***Theorem 2.17***. — *There exists $C''>0$ such that for any $m,m',m'',r\in\mathbb{N}$, $u_0>0$, there exists $C>0$ such that for $t\in]0,t_0]$, $u\geqslant u_0$, $Z,Z'\in T_{x_0}B$,*

$$
\begin{aligned}
&\sup_{|\alpha|+|\alpha'|\leqslant m}(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}(Z,Z')\Big|_{\mathscr{C}^{m'}(X_G)}\\
&\qquad\leqslant C(1+|Z^0|+|Z'^0|)^{2(n+r+m'+1)+m}\exp\Big(\frac{1}{2}\nu u-\frac{2C''}{u}|Z-Z'|^2\Big),\\
&\sup_{|\alpha|+|\alpha'|\leqslant m}(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}(\mathscr{L}_2^te^{-u\mathscr{L}_2^t})(Z,Z')\Big|_{\mathscr{C}^{m'}(X_G)}\\
&\qquad\leqslant C(1+|Z^0|+|Z'^0|)^{2(n+r+m'+1)+m}\exp\Big(-\frac{1}{4}\nu u-\frac{2C''}{u}|Z-Z'|^2\Big),
\end{aligned}
\tag{2.132}
$$

*where $\mathscr{C}^{m'}(X_G)$ is the $\mathscr{C}^{m'}$ norm for the parameter $x_0\in X_G$.*

*Proof.* — By (2.120), for any $k\in\mathbb{N}^*$,

$$
\begin{aligned}
e^{-u\mathscr{L}_2^t}&=\frac{(-1)^{k-1}(k-1)!}{2\pi iu^{k-1}}\int_{\delta\cup\Delta}e^{-u\lambda}(\lambda-\mathscr{L}_2^t)^{-k}d\lambda,\\
\mathscr{L}_2^te^{-u\mathscr{L}_2^t}&=\frac{(-1)^{k-1}(k-1)!}{2\pi iu^{k-1}}\int_{\Delta}e^{-u\lambda}\Big[\lambda(\lambda-\mathscr{L}_2^t)^{-k}-(\lambda-\mathscr{L}_2^t)^{-k+1}\Big]d\lambda.
\end{aligned}
\tag{2.133}
$$

From Theorem 2.16, we deduce that if $Q\in\cup_{l=1}^m\mathcal{D}_t^l$, there are $N\in\mathbb{N}$, $C_m>0$ such that for any $\lambda\in\delta\cup\Delta$,

$$
\|Q(\lambda-\mathscr{L}_2^t)^{-m}\|_t^{0,0}\leqslant C_m(1+|\lambda|^2)^N.
\tag{2.134}
$$

Recall that $\mathscr{L}_t^2$ is self-adjoint with respect to $\|\ \|_{t,0}$. After taking the adjoint of (2.134), we get

$$
\|(\lambda-\mathscr{L}_2^t)^{-m}Q\|_t^{0,0}\leqslant C_m(1+|\lambda|^2)^N.
\tag{2.135}
$$

From (2.133), (2.134) and (2.135), we get if $Q,Q'\in\cup_{l=1}^m\mathcal{D}_t^l$,

$$
\begin{aligned}
&\|Qe^{-u\mathscr{L}_2^t}Q'\|_t^{0,0}\leqslant C_me^{\frac{1}{4}\nu u},\\
&\|Q(\mathscr{L}_2^te^{-u\mathscr{L}_2^t})Q'\|_t^{0,0}\leqslant C_me^{-\frac{1}{2}\nu u}.
\end{aligned}
\tag{2.136}
$$

Let $|\ \ |_m$ be the usual Sobolev norm on $\mathscr{C}^\infty(\mathbb{R}^{2n-n_0},\mathbf{E}_{B,x_0})$ induced by $h^{\mathbf{E}_{B,x_0}}=h^{(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}}$ and the volume form $dv_{T_{x_0}B}(Z)$ as in (2.114).

Observe that by (2.105), (2.114), there exists $C>0$ such that for $s\in\mathscr{C}^\infty(T_{x_0}B,\mathbf{E}_{B,x_0})$, $\operatorname{supp}(s)\subset B^{T_{x_0}B}(0,q)$, $m\geqslant 0$,

$$
\frac{1}{C}(1+q)^{-m}\|s\|_{t,m}\leqslant|s|_m\leqslant C(1+q)^m\|s\|_{t,m}.
\tag{2.137}
$$

Now (2.136), (2.137) together with Sobolev's inequalities imply that if $Q, Q' \in \cup_{l=1}^m \mathcal{D}_t^l$, for $\mathcal{K}_u(\mathscr{L}_2^t) = e^{-\frac{1}{4}\nu u} e^{-u\mathscr{L}_2^t}$ or $e^{\frac{1}{2}\nu u}\mathscr{L}_2^t e^{-u\mathscr{L}_2^t}$, we have

$$\sup_{|Z|,|Z'|\leqslant q} |Q_Z Q'_{Z'} \mathcal{K}_u(\mathscr{L}_2^t)(Z,Z')| \leqslant C(1+q)^{2n+2}. \tag{2.138}$$

By (2.77), (2.78) and (2.80),

$$\sum_{l=1}^{n_0} \Big|\frac{1}{t}\langle \widetilde{\mu}_{X_0}, f_{0,l}\rangle(tZ)\Big|^2 = |\frac{1}{t}\widetilde{\mu}_{X_0}|^2_{g^{TY}}(tZ) \geqslant C|Z^{\perp}|^2. \tag{2.139}$$

Thus by (2.105), (2.138), (2.139), we derive (2.132) with the exponentials $e^{\frac{1}{4}\nu u}$, $e^{-\frac{1}{2}\nu u}$ for the case when $r = m' = 0$ and $C'' = 0$, i.e.

$$\begin{aligned}\sup_{|\alpha|+|\alpha'|\leqslant m} (1+|Z^{\perp}|+|Z'^{\perp}|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\mathcal{K}_u(\mathscr{L}_2^t)(Z,Z')\Big| \\ \leqslant C(1+|Z^0|+|Z'^0|)^{2n+m+2}.\end{aligned} \tag{2.140}$$

To obtain (2.132) in general, we proceed as in the proof of [**4**, Theorem 11.14].

Note that the function $f$ is defined in (2.30). For $\varrho > 1$, put

$$K_{u,\varrho}(a) = \int_{-\infty}^{+\infty} \exp(iv\sqrt{2u}a)\exp(-\frac{v^2}{2})\Big(1 - f(\frac{1}{\varrho}\sqrt{2u}v)\Big)\frac{dv}{\sqrt{2\pi}}. \tag{2.141}$$

Then there exist $C', C_1 > 0$ such that for any $c > 0$, $m, m' \in \mathbb{N}$, there is $C > 0$ such that for $u \geqslant u_0$, $a \in \mathbb{C}, |\mathrm{Im}(a)| \leqslant c$, $\varrho > 1$, we have

$$|a|^m |K_{u,\varrho}^{(m')}(a)| \leqslant C\exp\Big(C'c^2u - \frac{C_1}{u}\varrho^2\Big). \tag{2.142}$$

For any $c > 0$, let $V_c$ be the image of $\{\lambda \in \mathbb{C}, |\mathrm{Im}(\lambda)| \leqslant c\}$ by the map $\lambda \to \lambda^2$. Then

$$V_c = \{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \geqslant \frac{1}{4c^2}\mathrm{Im}(\lambda)^2 - c^2\},$$

and $\delta \cup \Delta \subset V_c$ for $c$ large enough.

Let $\widetilde{K}_{u,\varrho}$ be the holomorphic function such that $\widetilde{K}_{u,\varrho}(a^2) = K_{u,\varrho}(a)$. By (2.142), for $\lambda \in V_c$,

$$|\lambda|^m |\widetilde{K}_{u,\varrho}^{(m')}(\lambda)| \leqslant C\exp\Big(C'c^2u - \frac{C_1}{u}\varrho^2\Big). \tag{2.143}$$

Using finite propagation speed of solutions of hyperbolic equations (cf. [**41**, §4.4], [**31**, Append. D]) and (2.141), we find that there exists a fixed constant (which depends on $\varepsilon$) $c' > 0$ such that

$$\widetilde{K}_{u,\varrho}(\mathscr{L}_2^t)(Z,Z') = e^{-u\mathscr{L}_2^t}(Z,Z') \quad \text{if } |Z - Z'| \geqslant c'\varrho. \tag{2.144}$$

By (2.143), we see that given $k \in \mathbb{N}$, there is a unique holomorphic function $\widetilde{K}_{u,\varrho,k}(\lambda)$ defined on a neighborhood of $V_c$ such that it verifies the same estimates as $\widetilde{K}_{u,\varrho}$ in (2.143) and $\widetilde{K}_{u,\varrho,k}(\lambda) \to 0$ as $\lambda \to +\infty$; moreover

$$\widetilde{K}_{u,\varrho,k}^{(k-1)}(\lambda)/(k-1)! = \widetilde{K}_{u,\varrho}(\lambda). \tag{2.145}$$

Thus as in (2.133),

$$
\begin{aligned}
\widetilde{K}_{u,\varrho}(\mathscr{L}_2^t) &= \frac{1}{2\pi i}\int_{\delta\cup\Delta} \widetilde{K}_{u,\varrho,k}(\lambda)(\lambda-\mathscr{L}_2^t)^{-k}d\lambda,\\
\mathscr{L}_2^t\widetilde{K}_{u,\varrho}(\mathscr{L}_2^t) &= \frac{1}{2\pi i}\int_{\Delta} \widetilde{K}_{u,\varrho,k}(\lambda)\Big[\lambda(\lambda-\mathscr{L}_2^t)^{-k}-(\lambda-\mathscr{L}_2^t)^{-k+1}\Big]d\lambda.
\end{aligned}
\tag{2.146}
$$

By (2.134), (2.135) and by proceeding as in (2.136)-(2.138), we find that for $\mathbf{K}_u(a)=\widetilde{K}_{u,\varrho}(a)$ or $a\widetilde{K}_{u,\varrho}(a)$, for $|Z|,|Z'|\leqslant q$,

$$
\begin{aligned}
\sup_{|\alpha|+|\alpha'|\leqslant m}(1+|Z^{\perp}|+|Z'^{\perp}|)^{2n+m+m''+2}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\mathbf{K}_u(\mathscr{L}_2^t)(Z,Z')\Big|&\\
\leqslant C(1+q)^{2n+2+m}\exp(C'c^2u-\frac{C_1}{u}\varrho^2).&
\end{aligned}
\tag{2.147}
$$

Setting $\varrho\in\mathbb{N}^*$, $|\varrho-\frac{1}{c'}|Z-Z'||<1$ in (2.147), we get for $\alpha,\alpha'$ verifying $|\alpha|+|\alpha'|\leqslant m$,

$$
\begin{aligned}
(1+|Z^{\perp}|+|Z'^{\perp}|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\mathbf{K}_u(\mathscr{L}_2^t)(Z,Z')\Big|&\\
\leqslant C(1+|Z^0|+|Z'^0|)^{2n+m+2}\exp(C'c^2u-\frac{C_1}{2c'^2u}|Z-Z'|^2).&
\end{aligned}
\tag{2.148}
$$

Take $\delta_1=\frac{C'c^2+\frac{1}{4}\nu}{C'c^2+\frac{1}{2}\nu}$, from $(2.140)^{\delta_1}\times$ $(2.148)^{1-\delta_1}$ and (2.144), we get (2.132) for $r=m'=0$.

To get (2.132) for $r\geqslant 1$, note that from (2.133), for $k\geqslant 1$

$$
\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}=\frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\delta\cup\Delta}e^{-u\lambda}\frac{\partial^r}{\partial t^r}(\lambda-\mathscr{L}_2^t)^{-k}d\lambda.
\tag{2.149}
$$

We have the similar equation for $\frac{\partial^r}{\partial t^r}(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})$.

Set

$$
I_{k,r}=\Big\{(\mathbf{k},\mathbf{r})=(k_i,r_i)\Big|\sum_{i=0}^{j}k_i=k+j,\sum_{i=1}^{j}r_i=r,\ k_i,r_i\in\mathbb{N}^*\Big\}.
\tag{2.150}
$$

Then there exist $a^{\mathbf{k}}_{\mathbf{r}}\in\mathbb{R}$ such that

$$
\begin{aligned}
A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)&=(\lambda-\mathscr{L}_2^t)^{-k_0}\frac{\partial^{r_1}\mathscr{L}_2^t}{\partial t^{r_1}}(\lambda-\mathscr{L}_2^t)^{-k_1}\cdots\frac{\partial^{r_j}\mathscr{L}_2^t}{\partial t^{r_j}}(\lambda-\mathscr{L}_2^t)^{-k_j},\\
\frac{\partial^r}{\partial t^r}(\lambda-\mathscr{L}_2^t)^{-k}&=\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}}a^{\mathbf{k}}_{\mathbf{r}}A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t).
\end{aligned}
\tag{2.151}
$$

We claim that $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)$ is well defined and for any $m\in\mathbb{N}$, $k>2(m+r+1)$, $Q,Q'\in\cup_{l=1}^m\mathcal{D}_t^l$, there exist $C>0$, $N\in\mathbb{N}$ such that for $\lambda\in\delta\cup\Delta$,

$$
\|QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q's\|_{t,0}\leqslant C(1+|\lambda|)^N\sum_{|\beta|\leqslant 2r}\|Z^{\beta}s\|_{t,0}.
\tag{2.152}
$$

In fact, by (2.109), $\frac{\partial^r}{\partial t^r}\mathscr{L}_2^t$ is a combination of

$$\frac{\partial^{r_1}}{\partial t^{r_1}}(g^{ij}(tZ)),\quad (\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i}),\quad \frac{\partial^{r_3}}{\partial t^{r_3}}(q(tZ)),\quad \frac{\partial^{r_4}}{\partial t^{r_4}}(t\langle\widetilde{\mu}^{E_{0,p}}, f_{0,l}(tZ)\rangle),$$

where $q$ runs over the functions $r^X$, etc., appearing in (2.109). Now $\frac{\partial^{r_1}}{\partial t^{r_1}}(q(tZ))$ (resp. $\frac{\partial^{r_1}}{\partial t^{r_1}}(t\langle\widetilde{\mu}^{E_{0,p}}, f_{0,l}\rangle(tZ))$, $\frac{\partial^{r_1}}{\partial t^{r_1}}\nabla_{t,e_i}$) ($r_1 \geqslant 1$) are functions of the type as $q'(tZ)Z^\beta$, $|\beta| \leqslant r_1$ (resp. $r_1+1$) (where $q'$, as $q$, runs over the functions $r^X$, etc., appearing in (2.109)), with $q'(Z)$ and its derivatives on $Z$ being bounded smooth functions on $Z$.

Let $\mathscr{R}'_t$ be the family of operators of the type

$$\mathscr{R}'_t = \{[f_{j_1}Q_{j_1}, [f_{j_2}Q_{j_2}, \dots [f_{j_l}Q_{j_l}, \mathscr{L}_2^t]\dots]]\}$$

with $f_{j_i}$ smooth bounded (with its derivatives) functions and $Q_{j_i} \in \mathcal{D}_t \cup \{Z_j\}_{j=1}^{2n-n_0}$.

Now for the operator $A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)Q'$, we will move first all the term $Z^\beta$ in $q'(tZ)Z^\beta$ as above to the right hand side of this operator, to do so, we always use the commutator trick, i.e., each time, we consider only the commutation for $Z_i$, not for $Z^\beta$ with $|\beta|>1$.

Then $A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)Q'$ is as the form $\sum_{|\beta|\leqslant 2r} L_\beta^t Q''_\beta Z^\beta$, and $Q''_\beta$ is obtained from $Q'$ and its commutation with $Z^\beta$.

Now we move all the terms $\nabla_{t,e_i}$, $\langle\frac{1}{t}\widetilde{\mu}, f_{0,l}\rangle(tZ)$ in $\frac{\partial^{r_j}\mathscr{L}_2^t}{\partial t^{r_j}}$ to the right hand side of the operator $L_\beta^t$.

Then as in the proof of Theorem 2.16, we get finally that $QA_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)Q'$ is as the form $\sum_\beta \mathscr{L}_\beta^t Z^\beta$ where $\mathscr{L}_\beta^t$ is a linear combination of operators of the form

$$Q(\lambda-\mathscr{L}_2^t)^{-k'_0}R_1(\lambda-\mathscr{L}_2^t)^{-k'_1}R_2\cdots R_{l'}(\lambda-\mathscr{L}_2^t)^{-k'_{l'}}Q'''Q'', \tag{2.153}$$

with $R_1,\cdots,R_{l'} \in \mathscr{R}'_t$, $Q''' \in \cup_{l=1}^{2r}\mathcal{D}_t^l$, $Q'' \in \cup_{l=1}^{m}\mathcal{D}_t^l$, $|\beta| \leqslant 2r$, and $Q''$ is obtained from $Q'$ and its commutation with $Z^\beta$.

By the argument as in (2.134) and (2.135), as $k>2(m+r+1)$, we can split the above operator to two parts

$$\begin{aligned}&Q(\lambda-\mathscr{L}_2^t)^{-k'_0}R_1(\lambda-\mathscr{L}_2^t)^{-k'_1}R_2\cdots R_i(\lambda-\mathscr{L}_2^t)^{-k''_i};\\ &(\lambda-\mathscr{L}_2^t)^{-(k'_i-k''_i)}\cdots R_{l'}(\lambda-\mathscr{L}_2^t)^{-k'_{l'}}Q'''Q'',\end{aligned}$$

and the $\|\ \ \|_t^{0,0}$-norm of each part is bounded by $C(1+|\lambda|^2)^N$.

Thus the proof of (2.152) is complete.

By (2.149), (2.151) and (2.152), we get the similar estimates (2.140), (2.148) for $\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}$, $\frac{\partial^r}{\partial t^r}(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})$ with the exponential $2n+m+2r+2$ instead of $2n+m+2$ therein.

Thus we get (2.132) for $m'=0$.

Finally, for $U \in TX_G$ a vector on $X_G$,

$$\nabla_U^{\pi^*\operatorname{End}(\mathbf{E}_B)}e^{-u\mathscr{L}_2^t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\delta\cup\Delta} e^{-u\lambda}\nabla_U^{\pi^*\operatorname{End}(\mathbf{E}_B)}(\lambda-\mathscr{L}_2^t)^{-k}d\lambda. \tag{2.154}$$

Now, by using the similar formula (2.151) for $\nabla_U^{\pi^* \operatorname{End}(\mathbf{E}_B)}(\lambda - \mathscr{L}_2^t)^{-k}$ by replacing $\frac{\partial^{r_1}\mathscr{L}_2^t}{\partial t^{r_1}}$ by $\nabla_U^{\pi^* \operatorname{End}(\mathbf{E}_B)}\mathscr{L}_2^t$, and remark that $\nabla_U^{\pi^* \operatorname{End}(\mathbf{E}_B)}\mathscr{L}_2^t$ is a differential operator on $T_{x_0}B$ with the same structure as $\mathscr{L}_2^t$.

Then by the above argument, we get (2.132) for $m' \geqslant 1$. □

Let $P_{0,t}$ be the orthogonal projection from $\mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ to the kernel of $\mathscr{L}_2^t$ with respect to $\langle\, ,\,\rangle_{t,0}$. Set

$$F_u(\mathscr{L}_2^t) = \frac{1}{2\pi i}\int_\Delta e^{-u\lambda}(\lambda - \mathscr{L}_2^t)^{-1}d\lambda. \tag{2.155}$$

By (2.120),

$$F_u(\mathscr{L}_2^t) = e^{-u\mathscr{L}_2^t} - P_{0,t} = \int_u^{+\infty} \mathscr{L}_2^t e^{-u_1\mathscr{L}_2^t}du_1. \tag{2.156}$$

Let $P_{0,t}(Z,Z')$, $F_u(\mathscr{L}_2^t)(Z,Z')$ be the smooth kernels of $P_{0,t}, F_u(\mathscr{L}_2^t)$ with respect to $dv_{T_{x_0}B}(Z')$.

***Corollary 2.18***. — *With the notation in Theorem 2.17,*

$$\begin{aligned}\sup_{|\alpha|+|\alpha'|\leqslant m}(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}F_u(\mathscr{L}_2^t)\,(Z,Z')\Big|_{\mathscr{C}^{m'}(P)} \\ \leqslant C(1+|Z^0|+|Z'^0|)^{2n+m+2m'+2r+2}\exp(-\frac{1}{8}\nu u - \sqrt{C''\nu}|Z-Z'|).\end{aligned} \tag{2.157}$$

*Proof.* — Note that $\frac{1}{8}\nu u + \frac{2C''}{u}|Z-Z'|^2 \geqslant \sqrt{C''\nu}|Z-Z'|$, thus

$$\begin{aligned}\int_u^{+\infty} e^{-\frac{1}{4}\nu u_1 - \frac{2C''}{u_1}|Z-Z'|^2}du_1 \leqslant e^{-\sqrt{C''\nu}|Z-Z'|}\int_u^{+\infty} e^{-\frac{1}{8}\nu u_1}du_1 \\ = \frac{8}{\nu}e^{-\frac{1}{8}\nu u - \sqrt{C''\nu}|Z-Z'|}.\end{aligned} \tag{2.158}$$

By (2.132), (2.156) and (2.158), we get (2.157). □

For $k$ large enough, set

$$\begin{aligned}F_{r,u} &= \frac{(-1)^{k-1}(k-1)!}{2\pi i\, r!u^{k-1}}\int_\Delta e^{-u\lambda}\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_\mathbf{r}^\mathbf{k} A_\mathbf{r}^\mathbf{k}(\lambda,0)d\lambda, \\ J_{r,u} &= \frac{(-1)^{k-1}(k-1)!}{2\pi i\, r!u^{k-1}}\int_{\delta\cup\Delta} e^{-u\lambda}\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_\mathbf{r}^\mathbf{k} A_\mathbf{r}^\mathbf{k}(\lambda,0)d\lambda, \\ F_{r,u,t} &= \frac{1}{r!}\frac{\partial^r}{\partial t^r}F_u(\mathscr{L}_2^t) - F_{r,u}, \quad J_{r,u,t} = \frac{1}{r!}\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t} - J_{r,u}.\end{aligned} \tag{2.159}$$

Certainly, as $t \to 0$, the limit of $\|\quad\|_{t,m}$ exists, and we denote it by $\|\quad\|_{0,m}$.

Theorems 2.19, 2.20 are the analogues of [**17**, Theorems 4.14, 4.15], we include the proofs for the sake of completeness.

***Theorem 2.19***. — *For any $r \geqslant 0$, $k > 0$, there exist $C > 0$, $N \in \mathbb{N}$ such that for* $t \in [0, t_0], \lambda \in \delta \cup \Delta$,

(2.160)
$$\Big\| \Big( \frac{\partial^r \mathscr{L}_2^t}{\partial t^r} - \frac{\partial^r \mathscr{L}_2^t}{\partial t^r}\Big|_{t=0} \Big) s \Big\|_{t,-1} \leqslant Ct \sum_{|\alpha| \leqslant r+3} \|Z^\alpha s\|_{0,1},$$
$$\Big\| \Big( \frac{\partial^r}{\partial t^r} (\lambda - \mathscr{L}_2^t)^{-k} - \sum_{(\mathbf{k},\mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) \Big) s \Big\|_{0,0} \leqslant Ct(1 + |\lambda|^2)^N \sum_{|\alpha| \leqslant 4r+3} \|Z^\alpha s\|_{0,0}.$$

*Proof*. — Note that by (2.105), (2.114), for $t \in [0, 1]$, $k \geqslant 1$

$$\|s\|_{t,0} \leqslant C \|s\|_{0,0}, \quad \|s\|_{t,k} \leqslant C \sum_{|\alpha| \leqslant k} \|Z^\alpha s\|_{0,k}. \tag{2.161}$$

An application of Taylor expansion for (2.109) leads to the following inequality, if $s, s'$ have compact support,

$$\Big| \Big\langle \Big( \frac{\partial^r \mathscr{L}_2^t}{\partial t^r} - \frac{\partial^r \mathscr{L}_2^t}{\partial t^r}\Big|_{t=0} \Big) s, s' \Big\rangle_{0,0} \Big| \leqslant Ct \|s'\|_{t,1} \sum_{|\alpha| \leqslant r+3} \|Z^\alpha s\|_{0,1}. \tag{2.162}$$

Thus we get the first inequality of (2.160).

Note that

$$(\lambda - \mathscr{L}_2^t)^{-1} - (\lambda - \mathscr{L}_2^0)^{-1} = (\lambda - \mathscr{L}_2^t)^{-1} (\mathscr{L}_2^t - \mathscr{L}_2^0)(\lambda - \mathscr{L}_2^0)^{-1}. \tag{2.163}$$

Now from (2.119), (2.162) and (2.163),

$$\big\| \big( (\lambda - \mathscr{L}_2^t)^{-1} - (\lambda - \mathscr{L}_2^0)^{-1} \big) s \big\|_{0,0} \leqslant Ct(1 + |\lambda|)^N \sum_{|\alpha| \leqslant 3} \|Z^\alpha s\|_{0,0}. \tag{2.164}$$

After taking the limit, we know that Theorems 2.14-2.16 still hold for $t = 0$.

Note that $\nabla_{0,e_j} = \nabla_{e_j} + \frac{1}{2} R_{x_0}^{L_B}(\mathcal{R}, e_j)$ by (2.105).

If we denote by $\mathscr{L}_{\lambda,t} = \lambda - \mathscr{L}_2^t$, then

$$\begin{aligned} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) - A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) = \sum_{i=1}^{j} \mathscr{L}_{\lambda,t}^{-k_0} \cdots \left( \frac{\partial^{r_i} \mathscr{L}_2^t}{\partial t^{r_i}} - \frac{\partial^{r_i} \mathscr{L}_2^t}{\partial t^{r_i}}\Big|_{t=0} \right) \mathscr{L}_{\lambda,0}^{-k_i} \cdots \mathscr{L}_{\lambda,0}^{-k_j} \\ + \sum_{i=0}^{j} \mathscr{L}_{\lambda,t}^{-k_0} \cdots \left( \mathscr{L}_{\lambda,t}^{-k_i} - \mathscr{L}_{\lambda,0}^{-k_i} \right) \left( \frac{\partial^{r_{i+1}} \mathscr{L}_2^t}{\partial t^{r_{i+1}}}\Big|_{t=0} \right) \cdots \mathscr{L}_{\lambda,0}^{-k_j}. \end{aligned} \tag{2.165}$$

Now from the first inequality of (2.160), (2.119), (2.151), (2.164) and (2.165), we get (2.160). □

***Theorem 2.20***. — *There exist $C > 0$, $N \in \mathbb{N}$ such that for $t \in ]0, t_0]$, $u \geqslant u_0$, $q \in \mathbb{N}$, $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| \leqslant q$,*

$$\begin{aligned} \Big| F_{r,u,t}(Z, Z') \Big| &\leqslant C t^{\frac{1}{2n - n_0 + 1}} (1 + q)^N e^{-\frac{1}{8}\nu u}, \\ \Big| J_{r,u,t}(Z, Z') \Big| &\leqslant C t^{\frac{1}{2n - n_0 + 1}} (1 + q)^N e^{\frac{1}{2}\nu u}. \end{aligned} \tag{2.166}$$

*Proof.* — Let $J^0_{x_0,q}$ be the vector space of square integrable sections of $\mathbf{E}_{B,x_0}$ over $\{Z\in T_{x_0}B, |Z|\leqslant q+1\}$.

If $s\in J^0_{x_0,q}$, put $\|s\|^2_{(q)}=\int_{|Z|\leqslant q+1}|s|^2_{h^{\mathbf{E}_{B,x_0}}}dv_{TB}(Z)$. Let $\|A\|_{(q)}$ be the operator norm of $A\in \mathscr{L}(J^0_{x_0,q})$ with respect to $\|\ \ \|_{(q)}$.

By (2.149), (2.159) and (2.160), we get: there exist $C>0, N\in\mathbb{N}$ such that for $t\in ]0,t_0], u\geqslant u_0$,

$$\begin{aligned}\|F_{r,u,t}\|_{(q)}&\leqslant Ct(1+q)^N e^{-\frac{1}{2}\nu u},\\ \|J_{r,u,t}\|_{(q)}&\leqslant Ct(1+q)^N e^{\frac{1}{4}\nu u}.\end{aligned}\tag{2.167}$$

Let $\phi:\mathbb{R}^{2n-n_0}\to[0,1]$ be a smooth function with compact support, equal 1 near 0, such that $\int_{T_{x_0}B}\phi(Z)dv_{T_{x_0}B}(Z)=1$.

Take $\varsigma\in]0,1]$.

By the proof of Theorem 2.17, $F_{r,u}$ verifies the similar inequality as in (2.157). Thus by (2.157), there exists $C>0$ such that if $|Z|,|Z'|\leqslant q$, $U,U'\in\mathbf{E}_{B,x_0}$,

$$\begin{aligned}\Big|\langle F_{r,u,t}(Z,Z')U,U'\rangle-\int_{T_{x_0}B\times T_{x_0}B}\langle F_{r,u,t}(Z-W,Z'-W')U,U'\rangle\\ \times\frac{1}{\varsigma^{4n-2n_0}}\phi(W/\varsigma)\phi(W'/\varsigma)dv_{T_{x_0}B}(W)dv_{T_{x_0}B}(W')\Big|\leqslant C\varsigma(1+q)^N e^{-\frac{1}{8}\nu u}|U||U'|.\end{aligned}\tag{2.168}$$

On the other hand, by (2.167),

$$\begin{aligned}\Big|\int_{T_{x_0}B\times T_{x_0}B}\langle F_{r,u,t}(Z-W,Z'-W')U,U'\rangle\frac{1}{\varsigma^{4n-2n_0}}\phi(W/\varsigma)\phi(W'/\varsigma)\\ dv_{T_{x_0}B}(W)dv_{T_{x_0}B}(W')\Big|\leqslant Ct\frac{1}{\varsigma^{2n-n_0}}(1+q)^N e^{-\frac{1}{2}\nu u}|U||U'|.\end{aligned}\tag{2.169}$$

By taking $\varsigma=t^{1/(2n-n_0+1)}$, we get (2.166).

In the same way, we get (2.166) for $J_{r,u,t}$. □

***Theorem 2.21***. — *There exists $C''>0$ such that for any $k,m,m',m''\in\mathbb{N}$, there exist $N\in\mathbb{N}$, $C>0$ such that if $t\in]0,t_0], u\geqslant u_0$, $Z,Z'\in T^H_{x_0}U$, $\alpha,\alpha\in\mathbb{Z}^{2n-n_0}$, $|\alpha|+|\alpha'|\leqslant m$,*

$$\begin{aligned}&(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\Big(F_u(\mathscr{L}^t_2)-\sum_{r=0}^k F_{r,u}t^r\Big)(Z,Z')\Big|_{\mathscr{C}^{m'}(X_G)}\\ &\qquad\leqslant Ct^{k+1}(1+|Z^0|+|Z'^0|)^{2(n+k+m'+2)+m}\exp\Big(-\frac{1}{8}\nu u-\sqrt{C''\nu}|Z-Z'|\Big),\\ &(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\Big(e^{-u\mathscr{L}^t_2}-\sum_{r=0}^k J_{r,u}t^r\Big)(Z,Z')\Big|_{\mathscr{C}^{m'}(X_G)}\\ &\qquad\leqslant Ct^{k+1}(1+|Z^0|+|Z'^0|)^{2(n+k+m'+2)+m}\exp\Big(\frac{1}{2}\nu u-\frac{2C''}{u}|Z-Z'|^2\Big).\end{aligned}\tag{2.170}$$

*Proof*. — By (2.159) and (2.166),

$$\frac{1}{r!}\frac{\partial^r}{\partial t^r}F_u(\mathscr{L}_2^t)|_{t=0} = F_{r,u}, \quad \frac{1}{r!}\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}|_{t=0} = J_{r,u}. \tag{2.171}$$

Now by Theorem 2.17 and (2.159), $J_{r,u}, F_{r,u}$ have the same estimates as $\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}$, $\frac{\partial^r}{\partial t^r}F_u(\mathscr{L}_2^t)$ in (2.132), (2.157).

Again from (2.132), (2.157), (2.159), (2.166), and the Taylor expansion

$$G(t) - \sum_{r=0}^{k}\frac{1}{r!}\frac{\partial^r G}{\partial t^r}(0)t^r = \frac{1}{k!}\int_0^t (t-t_0)^k\frac{\partial^{k+1}G}{\partial t^{k+1}}(t_0)dt_0, \tag{2.172}$$

we get (2.170). □

## 2.8. Evaluation of $J_{r,u}$

For $u>0$, we will write $u\Delta_j$ for the rescaled simplex $\{(u_1,\cdots,u_j)|\ 0\leqslant u_1\leqslant u_2\leqslant \cdots\leqslant u_j\leqslant u\}$.

Let $e^{-u\mathscr{L}_2^0}(Z,Z')$ be the smooth kernel of $e^{-u\mathscr{L}_2^0}$ with respect to $dv_{T_{x_0}B}(Z')$.

Recall that the $\mathcal{O}_r$'s have been defined in (2.101).

***Theorem 2.22***. — *For $r\geqslant 0$, we have*

$$J_{r,u} = \sum_{\sum_{i=1}^j r_i = r,\, r_i\geqslant 1}(-1)^j\int_{u\Delta_j} e^{-(u-u_j)\mathscr{L}_2^0}\mathcal{O}_{r_j}e^{-(u_j-u_{j-1})\mathscr{L}_2^0} \cdots\mathcal{O}_{r_1}e^{-u_1\mathscr{L}_2^0}du_1\cdots du_j, \tag{2.173}$$

*where the product in the integrand is the convolution product. Moreover,*

$$J_{r,u}(Z,Z') = (-1)^r J_{r,u}(-Z,-Z'). \tag{2.174}$$

*Proof*. — We introduce an even extra-variable $\sigma$ such that $\sigma^{r+1}=0$.

Set $[\quad]^{[r]}$ the coefficient of $\sigma^r$, $\mathscr{L}_\sigma = \mathscr{L}_2^0 + \sum_{j=1}^r \mathcal{O}_j\sigma^j$.

From (2.159), (2.171), we know

$$J_{r,u}(Z,Z') = \frac{1}{r!}\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}(Z,Z')\Big|_{t=0} = [e^{-u\mathscr{L}_\sigma}]^{[r]}(Z,Z'). \tag{2.175}$$

Now from (2.175) and the Volterra expansion of $e^{-u\mathscr{L}_\sigma}$ (cf. [**1**, §2.4]), we get (2.173).

We prove (2.174) by iteration.

By (1.18), for $x_0 \in X_G$, $U_1, U_2 \in T_{x_0}B$, $R^{L_B}_{x_0}(U_1, U_2) = R^L(U_1^H, U_2^H)$. From (2.6), (2.101), we get

(2.176)
$$\mathscr{L}_2^0 = -\sum_{j=1}^{2n-n_0} (\nabla_{e_j})^2 - \pi^2 \langle ((P^{T^HU}\mathbf{J}P^{T^HU})^2 + 4P^{T^HU}\mathbf{J}P^{TY}\mathbf{J}P^{T^HU})_{x_0}\mathcal{R}, \mathcal{R}\rangle \\ + 2\pi\sqrt{-1}\nabla_{P^{T^HU}\mathbf{J}P^{T^HU}\mathcal{R}} - 2\omega_{d,x_0} - \tau_{x_0}.$$

Here the matrix $((P^{T^HU}\mathbf{J}P^{T^HU})^2 + 4P^{T^HU}\mathbf{J}P^{TY}\mathbf{J}P^{T^HU})_{x_0}$ need not commute with $P^{T^HU}\mathbf{J}P^{T^HU}$. Thus [**3**, (6.37), (6.38)] does not apply directly here, and we could not get a precise formula for $e^{-u\mathscr{L}_2^0}$ as in [**17**, (4.106)].

By the uniqueness of the solution of heat equations and (2.176), we know

$$e^{-u\mathscr{L}_2^0}(Z, Z') = e^{-u\mathscr{L}_2^0}(-Z, -Z'). \tag{2.177}$$

By (2.173),

$$J_{0,u}(Z, Z') = e^{-u\mathscr{L}_2^0}(Z, Z'). \tag{2.178}$$

Thus we get (2.174) for $r = 0$.

If (2.174) holds for $r \leqslant k$, then by (2.173), (2.178),

$$J_{k+1,u} = -\sum_{j=1}^{k+1} \int_0^u e^{-(u-u_1)\mathscr{L}_2^0} \mathcal{O}_j J_{k+1-j,u_1} du_1. \tag{2.179}$$

By the iteration, Theorem 2.11 and (2.178), and note that $\nabla_{e_i}$ in $\mathcal{O}_j$ will change the parity of the polynomials we obtained, we get (2.174) for $r = k+1$. ☐

## 2.9. Proof of Theorem 0.2

By (2.156) and (2.170), for any $u > 0$ fixed, there exists $C_u > 0$ such that for $t = \frac{1}{\sqrt{p}}$, $Z, Z' \in T_{x_0}B$, $x_0 \in X_G$, $\alpha, \alpha' \in \mathbb{Z}^{2n-n_0}$, $|\alpha| + |\alpha'| \leqslant m$, we have

(2.180)
$$(1 + |Z^{\perp}| + |Z'^{\perp}|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \Big( P_{0,t} - \sum_{r=0}^{k} t^r (J_{r,u} - F_{r,u}) \Big)(Z, Z') \right|_{\mathscr{C}^{m'}(X_G)} \\ \leqslant C_u t^{k+1} (1 + |Z^0| + |Z'^0|)^{2(n+k+m'+2)+m} \exp(-\sqrt{C''\nu}|Z - Z'|).$$

Set

$$P^{(r)} = J_{r,u} - F_{r,u}. \tag{2.181}$$

Then $P^{(r)}$ does not depend on $u > 0$ by (2.180), as $P_{0,t}$ does not depend on $u$.

Moreover, by taking the limit of (2.157) as $t \to 0$,

$$(2.182) \quad (1+|Z^{\perp}|+|Z'^{\perp}|)^{m''}\Big|F_{r,u}(Z,Z')\Big|_{\mathscr{C}^{m'}(X_G)} \leqslant C(1+|Z^0|+|Z'^0|)^{2n+2r+2m'+2}\exp\left(-\frac{1}{8}\nu u-\sqrt{C''\nu}|Z-Z'|\right).$$

Thus

$$(2.183) \qquad J_{r,u}(Z,Z') = P^{(r)}(Z,Z') + F_{r,u}(Z,Z') = P^{(r)}(Z,Z') + \mathcal{O}(e^{-\frac{1}{8}\nu u}),$$

uniformly on any compact set of $T_{x_0}B \times T_{x_0}B$.

Especially, from (2.174), (2.183), we get

$$(2.184) \qquad P^{(r)}(Z,Z') = (-1)^r P^{(r)}(-Z,-Z').$$

By (2.100), for $Z, Z' \in T_{x_0}B$,

$$(2.185) \qquad P_{x_0,p}(Z,Z') = p^{n-\frac{n_0}{2}}\kappa^{-\frac{1}{2}}(Z)P_{0,t}(Z/t,Z'/t)\kappa^{-\frac{1}{2}}(Z').$$

We note in passing that, as a consequence of (2.180) and (2.185), we obtain the following estimate.

***Theorem 2.23***. — *For any $k, m, m', m'' \in \mathbb{N}$, there exists $C > 0$ such that for $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| \leqslant \varepsilon$, $x_0 \in X_G$,*

$$(2.186) \quad \sup_{|\alpha|+|\alpha'|\leqslant m}(1+\sqrt{p}|Z^{\perp}|+\sqrt{p}|Z'^{\perp}|)^{m''}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\left(p^{-n+\frac{n_0}{2}}P_{x_0,p}(Z,Z') - \sum_{r=0}^{k}P^{(r)}(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-r/2}\right)\right|_{\mathscr{C}^{m'}(X_G)} \leqslant Cp^{-(k+1-m)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m}\exp(-\sqrt{C''\nu}\sqrt{p}|Z-Z'|).$$

From (2.83), (2.84), (2.108) and (2.186), we get Theorem 0.2 without knowing the properties (0.12), (0.13) for $P^{(r)}$.

To prove the uniformity part of Theorem 0.2, we notice that in the proof of Theorem 2.17, we only use the derivatives of the coefficients of $\mathscr{L}_2^t$ with order $\leqslant 2n+m+m'+r+2$. Thus the constants in Theorems 2.17 and 2.20, (resp. Theorem 2.21) are uniformly bounded, if with respect to a fixed metric $g_0^{TX}$, the $\mathscr{C}^{2n+m+m'+r+4}$ (resp. $\mathscr{C}^{2n+m+m'+k+5}$)$-$norms on $X$ of the data ($g^{TX}$, $h^L$, $\nabla^L$, $h^E$, $\nabla^E$, $J$) are bounded (as by (2.109), the coefficients of $\mathscr{L}_2^t$ are functions of $g^{TX}$ (resp. $\nabla^L, \nabla^E$) and their derivatives with order $\leqslant 2$ (resp. 1)), and $g^{TX}$ is bounded below.

Moreover, taking derivatives with respect to the parameters we obtain a similar equation as (2.154), where $x_0 \in X_G$ plays now a role of a parameter. Thus the $\mathscr{C}^{m'}$$-$norm in (2.186) can also include the parameters if the $\mathscr{C}^{m'}$$-$norms (with respect to the parameter $x_0 \in X_G$) of the derivatives of above data with order $\leqslant 2n+k+m+5$ are bounded.

Thus we can take $C_{k,\,l}$ in (0.10) independent of $g^{TX}$ under our condition.

This achieves the proof of Theorem 0.2 except (0.12) and (0.13) which will be proved in Theorem 3.2 under the condition in Theorem 0.2.

# CHAPTER 3

# EVALUATION OF $P^{(r)}$

In this Chapter, inspired by the method in [**28**, §1.4, 1.5], we develop a direct and effective method to compute $P^{(r)}$. In particular, we get (0.12) and (0.13) under the condition in Theorem 0.2.

This section is organized as follows. In Section 3.1, we study the spectrum of the limiting operator $\mathscr{L}_2^0$. In Section 3.2, we get a direct method to evaluate $P^{(r)}$ in (0.12), especially, we prove (0.12) and (0.13). In Section 3.3, we compute explicitly $\mathcal{O}_1$ in (2.102), and get a general formula for $P^{(2)}$ by using the operators $\mathcal{O}_1$, $\mathcal{O}_2$. In Section 3.4, we compute explicitly an interesting example: the line bundle $\mathcal{O}(2)$ on $(\mathbb{C}P^1, 2\,\omega_{FS})$. We verify that Theorem 0.2 coincides with our computation here if 0 is a regular value of the moment map $\mu$, but it does not hold if 0 is a singular value.

We use the notations in Section 2.6, and we suppose that (3.2) is verified.

## 3.1. Spectrum of $\mathscr{L}_2^0$

Recall that $T^HP$ is the orthogonal complement of $TY$ in $(TP, g^{TP})$. Note that by (2.6) and (2.17), we have the following orthogonal splitting of vector bundles on $P=\mu^{-1}(0)$,

$$TP = T^HP \oplus TY, \qquad\qquad TX = T^HP \oplus TY \oplus \mathbf{J}TY. \tag{3.1}$$

In the rest of this Chapter, we suppose that on $P$

$$\mathbf{J}^2 TY = TY. \tag{3.2}$$

(2.8) and (3.2) imply that $-J\mathbf{J}$ preserves $TY$ and $\mathbf{J}TY$. Especially if $\mathbf{J}=J$ on $P$, then (3.2) holds.

By (2.8) , (2.17) and (3.1) , the condition (3.2) implies

$$\mathbf{J}\,TY = J\,TY, \quad \mathbf{J}\,T^HP = T^HP = J\,T^HP. \tag{3.3}$$

Thus $(\mathbf{J}TY)_B|_{X_G}$ is the orthogonal complement of $TX_G$ in $TB$, and $\mathbf{J}$ induces naturally $\mathbf{J}_G \in \mathrm{End}(TX_G)$. We will identify $(\mathbf{J}TY)_B|_{X_G}$ to the normal bundle of $X_G$ in $B$.

For $U, V \in T_{x_0}B$, $x_0 \in X_G$, by (3.2), we have

$$\omega(U^H, V^H) = \omega_G(P^{TX_G}U, P^{TX_G}V). \tag{3.4}$$

From the above discussion, for $x_0 \in X_G$, we can choose $\{w_j^0\}_{j=1}^{n-n_0}$, $\{e_j^\perp\}_{j=1}^{n_0}$ orthonormal basis of $T_{x_0}^{(1,0)}X_G$, $(\mathbf{J}TY)_{B,x_0} \subset TB$ such that

$$\begin{aligned} \mathbf{J}|_{T_{x_0}^{(1,0)}X_G} &= \frac{\sqrt{-1}}{2\pi}\mathrm{diag}(a_1, \cdots, a_{n-n_0}) \in \mathrm{End}(T_{x_0}^{(1,0)}X_G), \\ \mathbf{J}^2|_{(\mathbf{J}TY)_B} &= \frac{-1}{4\pi^2}\mathrm{diag}(a_1^{\perp,2}, \cdots, a_{n_0}^{\perp,2}) \in \mathrm{End}((\mathbf{J}TY)_{B,x_0}), \end{aligned} \tag{3.5}$$

with $a_j, a_j^\perp > 0$, and let $\{w^{0,j}\}_{j=1}^{n-n_0}$, $\{e^{\perp j}\}_{j=1}^{n_0}$ be their dual basis, then

$$e_{2j-1}^0 = \frac{1}{\sqrt{2}}(w_j^0 + \overline{w}_j^0) \quad \text{and} \quad e_{2j}^0 = \frac{\sqrt{-1}}{\sqrt{2}}(w_j^0 - \overline{w}_j^0)\,,$$

$j = 1, \ldots, n - n_0$, form an orthonormal basis of $T_{x_0}X_G$.

From now on, we use the coordinate in Section 2.6 induced by the above basis.

Denote by $Z^0 = (Z_1^0, \cdots, Z_{2n-2n_0}^0)$, $Z^\perp = (Z_1^\perp, \cdots, Z_{n_0}^\perp)$, then $Z = (Z^0, Z^\perp)$.

In what follows we will use the complex coordinates $z^0 = (z_1^0, \cdots, z_{n-n_0}^0)$, thus $Z^0 = z^0 + \overline{z}^0$, and $w_i^0 = \sqrt{2}\frac{\partial}{\partial z_i^0}$, $\overline{w}_i^0 = \sqrt{2}\frac{\partial}{\partial \overline{z}_i^0}$, and

$$e_{2i-1}^0 = \frac{\partial}{\partial z_i^0} + \frac{\partial}{\partial \overline{z}_i^0}, \quad e_{2i}^0 = \sqrt{-1}\left(\frac{\partial}{\partial z_i^0} - \frac{\partial}{\partial \overline{z}_i^0}\right). \tag{3.6}$$

We will also identify $z^0$ to $\sum_i z_i^0 \frac{\partial}{\partial z_i^0}$ and $\overline{z}^0$ to $\sum_i \overline{z}_i^0 \frac{\partial}{\partial \overline{z}_i^0}$ when we consider $z^0$ and $\overline{z}^0$ as vector fields. Remark that

$$\left|\frac{\partial}{\partial z_i^0}\right|^2 = \left|\frac{\partial}{\partial \overline{z}_i^0}\right|^2 = \frac{1}{2}, \quad \text{so that } |z^0|^2 = |\overline{z}^0|^2 = \frac{1}{2}|Z^0|^2\,. \tag{3.7}$$

It is very useful to rewrite $\mathscr{L}_2^0$ by using the creation and annihilation operators. Set

$$\begin{aligned} &b_i = -2\frac{\partial}{\partial z_i^0} + \frac{1}{2}a_i\overline{z}_i^0\,, \quad b_i^+ = 2\frac{\partial}{\partial \overline{z}_i^0} + \frac{1}{2}a_i z_i^0\,, \quad b = (b_1, \cdots, b_{n-n_0})\,; \\ &b_j^\perp = -\frac{\partial}{\partial Z_j^\perp} + a_j^\perp Z_j^\perp, \quad b_j^{\perp+} = \frac{\partial}{\partial Z_j^\perp} + a_j^\perp Z_j^\perp, \quad b^\perp = (b_1^\perp, \cdots, b_{n_0}^\perp). \end{aligned} \tag{3.8}$$

Then for any polynomial $g(Z^0, Z^\perp)$ on $Z^0$ and $Z^\perp$,

$$\begin{aligned} &[b_i, b_j^+] = b_i b_j^+ - b_j^+ b_i = -2a_i\delta_{ij}, && [b_i, b_j] = [b_i^+, b_j^+] = 0\,, \\ &[g, b_j] = 2\frac{\partial}{\partial z_j^0}g, && [g, b_j^+] = -2\frac{\partial}{\partial \overline{z}_j^0}g\,, \\ &[b_i^\perp, b_j^{\perp+}] = -2a_i^\perp\delta_{ij}, && [b_j^\perp, b_k^\perp] = [b_j^{\perp+}, b_k^{\perp+}] = 0, \\ &[g, b_j^\perp] = -[g, b_j^{\perp+}] = \frac{\partial}{\partial Z_j^\perp}g\,. \end{aligned} \tag{3.9}$$

Set

$$\mathscr{L} = \sum_{j=1}^{n-n_0} b_j b_j^+, \quad \mathscr{L}^\perp = \sum_{j=1}^{n_0} b_j^\perp b_j^{\perp +}, \quad \nabla_{0,\cdot} = \nabla_\cdot + \frac{1}{2} R_{x_0}^{L_B}(\mathcal{R}, \cdot). \tag{3.10}$$

From (0.1), (1.18) and (3.4), for $U, V \in T_{x_0}B$, we get

$$R_{x_0}^{L_B}(U,V) = -2\pi\sqrt{-1}\left\langle \mathbf{J} P^{TX_G} U, P^{TX_G} V \right\rangle. \tag{3.11}$$

By (2.50), (3.5), (3.8), (3.10) and (3.11), we have

$$\begin{aligned} & b_i = -2\nabla_{0,\frac{\partial}{\partial z_i^0}}, \quad b_i^+ = 2\nabla_{0,\frac{\partial}{\partial \overline{z}_i^0}}, \quad \nabla_{0,e_j^\perp} = \nabla_{e_j^\perp}, \\ & \tau_{x_0} = \sum_j a_j + \sum_j a_j^\perp. \end{aligned} \tag{3.12}$$

From (2.101), (3.10) and (3.12), we get

$$\begin{aligned} \mathscr{L}_2^0 &= -\sum_{j=1}^{2n-2n_0} (\nabla_{0,e_j^0})^2 - \sum_{j=1}^{n_0} \Big( (\nabla_{e_j^\perp})^2 - |a_j^\perp Z_j^\perp|^2 \Big) - 2\omega_{d,x_0} - \tau_{x_0} \\ &= \mathscr{L} + \mathscr{L}^\perp - 2\omega_{d,x_0}. \end{aligned} \tag{3.13}$$

By [**42**, §8.6], [**28**, Theorem 1.15] (cf. [**31**, Theorems 4.1.20, E.1.1]), we know

***Theorem 3.1***. — *The spectrum of the restriction of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n-2n_0})$ is given by*

$$\operatorname{Spec}\left(\mathscr{L}|_{L^2(\mathbb{R}^{2n-2n_0})}\right) = \Big\{ 2\sum_{i=1}^{n-n_0} \alpha_i^0 a_i \,:\, \alpha^0 = (\alpha_1^0, \cdots, \alpha_{n-n_0}^0) \in \mathbb{N}^{n-n_0} \Big\}, \tag{3.14}$$

*and an orthogonal basis of the eigenspace of $2\sum_{i=1}^{n-n_0} \alpha_i^0 a_i$ is given by*

$$b^{\alpha^0}\Big( (z^0)^\beta \exp\Big( -\frac{1}{4}\sum_i a_i |z_i^0|^2 \Big) \Big), \quad \textit{with } \beta \in \mathbb{N}^{n-n_0}. \tag{3.15}$$

*The spectrum of the restriction of $\mathscr{L}^\perp$ on $L^2(\mathbb{R}^{n_0})$ is given by*

$$\operatorname{Spec}\left(\mathscr{L}^\perp|_{L^2(\mathbb{R}^{n_0})}\right) = \Big\{ 2\sum_{i=1}^{n_0} \alpha_i^\perp a_i^\perp \,:\, \alpha^\perp = (\alpha_1^\perp, \cdots, \alpha_{n_0}^\perp) \in \mathbb{N}^{n_0} \Big\}, \tag{3.16}$$

*and the eigenspace of $2\sum_{i=1}^{n_0} \alpha_i^\perp a_i^\perp$ is one dimensional and an orthonormal basis is given by*

$$\Big( \prod_{i=1}^{n_0} \sqrt{\frac{\pi}{a_i^\perp}} (2a_i^\perp)^{\alpha_i^\perp} (\alpha_i^\perp !) \Big)^{-1/2} (b^\perp)^{\alpha^\perp} \exp\Big( -\frac{1}{2}\sum_i a_i^\perp |Z_i^\perp|^2 \Big). \tag{3.17}$$

Especially, the orthonormal basis of $\mathrm{Ker}(\mathscr{L}|_{L^2(\mathbb{R}^{2n-2n_0})})$; $\mathrm{Ker}(\mathscr{L}^\perp|_{L^2(\mathbb{R}^{n_0})})$ are

$$\begin{aligned}&\Big(\frac{a^\beta}{2^{|\beta|}\beta!}\prod_{i=1}^{n-n_0}\frac{a_i}{2\pi}\Big)^{\frac{1}{2}}\Big((z^0)^\beta\exp\Big(-\frac{1}{4}\sum_{j=1}^{n-n_0}a_j|z_j^0|^2\Big)\Big),\ \beta\in\mathbb{N}^{n-n_0};\\ &G^\perp(Z^\perp)=\Big(\prod_{i=1}^{n_0}\frac{a_i^\perp}{\pi}\Big)^{\frac{1}{4}}\exp\Big(-\frac{1}{2}\sum_{i=1}^{n_0}a_i^\perp|Z_i^\perp|^2\Big).\end{aligned}\tag{3.18}$$

Let $P_{\mathscr{L}}(Z^0,Z'^0)$, $P_{\mathscr{L}^\perp}(Z^\perp,Z'^\perp)$, $P(Z,Z')$ be the kernels of the orthogonal projections $P_{\mathscr{L}}$, $P_{\mathscr{L}^\perp}$, $P$ from $L^2(\mathbb{R}^{2n-2n_0})$, $L^2(\mathbb{R}^{n_0})$, $L^2(\mathbb{R}^{2n-n_0})$ onto $\mathrm{Ker}(\mathscr{L})$, $\mathrm{Ker}(\mathscr{L}^\perp)$, $\mathrm{Ker}(\mathscr{L}+\mathscr{L}^\perp)$ respectively.

From (3.18), we get

$$\begin{aligned}P_{\mathscr{L}}(Z^0,Z'^0)=&\Big(\prod_{i=1}^{n-n_0}\frac{a_i}{2\pi}\Big)\ \exp\Big(-\frac{1}{4}\sum_{i=1}^{n-n_0}a_i\big(|z_i^0|^2+|z_i'^0|^2-2z_i^0\overline{z}_i'^0\big)\Big),\\ P_{\mathscr{L}^\perp}(Z^\perp,Z'^\perp)=&\Big(\prod_{i=1}^{n_0}\sqrt{\frac{a_i^\perp}{\pi}}\Big)\exp\Big(-\frac{1}{2}\sum_{i=1}^{n_0}a_i^\perp(|Z_i^\perp|^2+|Z_i'^\perp|^2)\Big),\\ P(Z,Z')=&P_{\mathscr{L}}(Z^0,Z'^0)P_{\mathscr{L}^\perp}(Z^\perp,Z'^\perp).\end{aligned}\tag{3.19}$$

Let $P^N$ be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0})$ onto $N=\mathrm{Ker}(\mathscr{L}_2^0)$. Let $P^N(Z,Z')$ be the associated kernel.

Recall that the projection $I_{\mathbb{C}\otimes E_B}$ from $(\Lambda(T^{*(0,1)}X)\otimes E)_B$ onto $\mathbb{C}\otimes E_B$ is defined in Introduction.

By (2.8), (2.10), (2.50) and (3.5),

$$-\omega_{d,x_0}\geqslant\nu_0\quad\text{on }\Lambda^{>0}(T^{*(0,1)}X),\tag{3.20}$$

thus

$$P^N(Z,Z')=P(Z,Z')I_{\mathbb{C}\otimes E_B}.\tag{3.21}$$

If $\mathbf{J}=J$ on $P$, then by (3.19) and (3.21),

$$\begin{aligned}P^N(Z,Z')=&\exp\Big(-\frac{\pi}{2}\sum_{i=1}^{n-n_0}\big(|z_i^0|^2+|z_i'^0|^2-2z_i^0\overline{z}_i'^0\big)\Big)\\ &\times 2^{\frac{n_0}{2}}\exp\Big(-\pi\big(|Z^\perp|^2+|Z'^\perp|^2\big)\Big)I_{\mathbb{C}\otimes E_B},\\ P^N((0,Z^\perp),(0,Z^\perp))=&2^{\frac{n_0}{2}}\exp\Big(-2\pi|Z^\perp|^2\Big)I_{\mathbb{C}\otimes E_B}.\end{aligned}\tag{3.22}$$

## 3.2. Evaluation of $P^{(r)}$: a proof of (0.12) and (0.13)

Recall that $\delta$ is the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\nu/4$.

By (2.120),

$$P_{0,t} = \frac{1}{2\pi i}\int_{\delta}(\lambda - \mathscr{L}_2^t)^{-1}d\lambda. \tag{3.23}$$

Let $f(\lambda,t)$ be a formal power series with values in $\mathrm{End}(L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}))$

$$f(\lambda,t) = \sum_{r=0}^{\infty} t^r f_r(\lambda), \quad f_r(\lambda)\in \mathrm{End}(L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})). \tag{3.24}$$

By (2.102), consider the equation of formal power series for $\lambda\in\delta$,

$$(\lambda - \mathscr{L}_2^0 - \sum_{r=1}^{\infty} t^r\mathscr{O}_r)f(\lambda,t) = \mathrm{Id}_{L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})}\,. \tag{3.25}$$

Let $N^{\perp}$ be the orthogonal space of $N$ in $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})$, and $P^{N^{\perp}}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})$ onto $N^{\perp}$.

We decompose $f(\lambda,t)$ according to the splitting $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}) = N\oplus N^{\perp}$,

$$g_r(\lambda) = P^N f_r(\lambda), \quad f_r^{\perp}(\lambda) = P^{N^{\perp}}f_r(\lambda). \tag{3.26}$$

Using Theorem 3.1, (3.13), (3.20), (3.26) and identifying the powers of $t$ in (3.25), we find that

$$\begin{aligned}
&g_0(\lambda) = \frac{1}{\lambda}P^N, \quad f_0^{\perp}(\lambda) = (\lambda-\mathscr{L}_2^0)^{-1}P^{N^{\perp}},\\
&f_r^{\perp}(\lambda) = (\lambda-\mathscr{L}_2^0)^{-1}\sum_{j=1}^{r}P^{N^{\perp}}\mathscr{O}_j f_{r-j}(\lambda),\\
&g_r(\lambda) = \frac{1}{\lambda}\sum_{j=1}^{r}P^N\mathscr{O}_j f_{r-j}(\lambda).
\end{aligned} \tag{3.27}$$

Recall that $P^{(r)}$ ($r\in\mathbb{N}$) is defined in (2.181) and (2.186).

***Theorem 3.2***. — *There exist $J_r(Z,Z')$ polynomials in $Z,Z'$ with the same parity as $r$, and $\deg J_r(Z,Z')\leqslant 3r$, whose coefficients are polynomials in $R^{TB}$, $R^{\mathrm{Cliff}_B}$, $R^{E_B}$, $r^X$, $\mathrm{Tr}[R^{T^{(1,0)}X}]$, $R^E$ (resp. $A$, $\mu^E$, $\mu^{\mathrm{Cliff}}$; resp. $h$, $R^L$, $R^{L_B}$; resp. $\mu$) and their derivatives at $x_0$ up to order $r-2$ (resp. $r-1$; resp. $r$; resp. $r+1$), and in the inverses of the linear combination of the eigenvalues of $\mathbf{J}$ at $x_0$, such that*

$$P^{(r)}(Z,Z') = J_r(Z,Z')P(Z,Z'). \tag{3.28}$$

*Moreover,*

$$P^{(0)}(Z,Z') = P^N(Z,Z') = P(Z,Z')I_{\mathbb{C}\otimes E_B}. \tag{3.29}$$

*Proof.* — By (3.23), for $\sigma > 0$, by combining Theorems 2.13-2.16 and the argument as in [**28**, §1.3], we get another proof of the existence of the asymptotic expansion of $P_{0,t}(Z, Z')$ for $|Z|, |Z'| \leqslant \sigma$ when $t \to 0$.

By (2.83), (2.84) and (2.185), this gives another proof of Theorems 0.2, 2.23 for $|Z|, |Z'| \leqslant \sigma/\sqrt{p}$. Moreover, by (2.149), (2.159) and (3.26),

$$P^{(r)} = \frac{1}{2\pi i}\int_\delta g_r(\lambda)d\lambda + \frac{1}{2\pi i}\int_\delta f_r^\perp(\lambda)d\lambda. \tag{3.30}$$

From (3.27), (3.30), we get (3.29).

Generally, from Theorems 2.11, 3.1, (3.9), (3.27), (3.30) and the residue formula, we conclude Theorem 3.2. □

*Proof of* (0.12) *and* (0.13). — As $\mathbf{J} = J$ on $\mu^{-1}(0)$, the condition (3.2) is verified.

From Theorem 3.2, (3.22), we get (0.12) and (0.13). □

From Theorem 3.1, (3.27), (3.30), and the residue formula, we can get $P^{(r)}$ by using the operators $(\mathscr{L}_2^0)^{-1}$, $P^N, P^{N^\perp}, \mathcal{O}_k$ $(k \leqslant r)$.

This gives us a direct method to compute $P^{(r)}$ in view of Theorem 3.1. In particular,

$$P^{(1)} = -\,P^N\mathcal{O}_1P^{N^\perp}(\mathscr{L}_2^0)^{-1}P^{N^\perp} - P^{N^\perp}(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N, \tag{3.31}$$

and

(3.32)

$$\begin{aligned}
P^{(2)} =& \frac{1}{2\pi i}\int_\delta \Big[(\lambda - \mathscr{L}_2^0)^{-1}P^{N^\perp}(\mathcal{O}_1f_1 + \mathcal{O}_2f_0)(\lambda) + \frac{1}{\lambda}P^N(\mathcal{O}_1f_1 + \mathcal{O}_2f_0)(\lambda)\Big]d\lambda\\
=& \frac{1}{2\pi i}\int_\delta \Big\{(\lambda - \mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\mathcal{O}_1\Big((\lambda - \mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1 + \frac{1}{\lambda}P^N\mathcal{O}_1\Big) + \mathcal{O}_2\Big]\\
&+ \frac{1}{\lambda}P^N\Big[\mathcal{O}_1\Big((\lambda - \mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1 + \frac{1}{\lambda}P^N\mathcal{O}_1\Big) + \mathcal{O}_2\Big]\Big\}(\lambda - \mathscr{L}_2^0)^{-1}d\lambda\\
=& (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N - P^{N^\perp}(\mathscr{L}_2^0)^{-2}\mathcal{O}_1P^N\mathcal{O}_1P^N\\
&+ (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp} - (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2P^N\\
&+ P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp} - P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-2}P^{N^\perp}\mathcal{O}_1P^N\\
&- P^N\mathcal{O}_1P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-2}P^{N^\perp} - P^N\mathcal{O}_2(\mathscr{L}_2^0)^{-1}P^{N^\perp}.
\end{aligned}$$

In the next Section we will prove $P^N\mathcal{O}_1P^N = 0$, thus the second and seventh terms in (3.32) are zero.

## 3.3. A formula for $\mathcal{O}_1$

We will use the notation in Chapter 1. All tensors in this Section will be evaluated at the base point $x_0 \in X_G$.

For $\psi$ a tensor on $X$, we denote by $\nabla^X\psi$ its covariant derivative induced by $\nabla^{TX}$.

If $\psi_1$ is a $G$-equivariant tensor, then we can consider it as a tensor on $B = U/G$ with the covariant derivative $\nabla^B \psi_1$, we will denote by

$$(\nabla^B \nabla^B \psi_1)_{(c_j e_j, c'_k e_k)} := c_j c'_k (\nabla^B_{e_j} \nabla^B_{e_k} \psi_1)_{x_0},$$

etc.

We denote by $\{e_a\}$ an orthonormal basis of $(TX, g^{TX})$.

To simplify the notation, we often denote by $U$ the lift $U^H \in T^H X$ of $U \in TB$.

Recall that $\widetilde{\mu} \in TY$ is defined by (1.14) and the moment map $\mu$ (2.16), and that $A$ is the second fundamental form of $X_G$ defined by (0.10).

***Lemma 3.3***. — *The following identities hold,*

$$\begin{aligned}
&(\nabla^{TY}_{\mathcal{R}} \widetilde{\mu})_{x_0} = -\mathbf{J}\mathcal{R}^{\perp}, \\
&(\nabla^{TY}_{\cdot} \nabla^{TY}_{\cdot} \widetilde{\mu})_{(\mathcal{R},\mathcal{R})} := (\nabla^{TY}_{e^H_j} \nabla^{TY}_{e^H_i} \widetilde{\mu})_{x_0} Z_j Z_i \\
&\qquad = -P^{TY}\big((\nabla^X_{\mathcal{R}^0}\mathbf{J})(\mathcal{R}^0 + 2\mathcal{R}^{\perp}) + (\nabla^X_{\mathcal{R}^{\perp}}\mathbf{J})\mathcal{R}^{\perp}\big) \\
&\qquad\quad - \mathbf{J}A(\mathcal{R}^0)\mathcal{R}^0 - \frac{1}{2} T(\mathcal{R}^0, \mathbf{J}\mathcal{R}^0) + T(\mathcal{R}^{\perp}, \mathbf{J}\mathcal{R}^{\perp}).
\end{aligned} \tag{3.33}$$

*Proof*. — Recall that $P^{TY}, P^{T^H X}$ are the orthogonal projections from $TX$ onto $TY, T^H X$ defined in Section 1.1. Note that on $P$, by (3.3),

$$\mathbf{J} e_i^{\perp,H} \in TY, \quad \mathbf{J} e_i^{0,H} = (\mathbf{J}_G e_i^0)^H \in T^H P. \tag{3.34}$$

By (1.14) and (2.17), for $K \in \mathfrak{g}$,

$$-\left\langle \mathbf{J} e_i^H, K^X \right\rangle = \nabla_{e_i^H} \mu(K) = \left\langle \nabla^{TY}_{e_i^H} \widetilde{\mu}, K^X \right\rangle + \left\langle \widetilde{\mu}, \nabla^{TY}_{e_i^H} K^X \right\rangle. \tag{3.35}$$

From (1.4), (1.5), (1.6) and (3.35),

$$\nabla^{TY}_{e_i^H} \widetilde{\mu} = -P^{TY} \mathbf{J} e_i^H - \frac{1}{2} \dot{g}^{TY}_{e_i^H} \widetilde{\mu} = -P^{TY} \mathbf{J} e_i^H - T(e_i^H, \widetilde{\mu}). \tag{3.36}$$

From (3.36) and the fact that $\widetilde{\mu} = 0$ on $P$, one gets the first equation in (3.33).

Now for $W$ (resp. $Y$) a smooth section of $TX$ (resp. $TY$), by (1.8),

$$\begin{aligned}
\left\langle \nabla^{TY}_{e_j^H} P^{TY} W, Y \right\rangle &= e_j^H \langle W, Y \rangle - \left\langle P^{TY} W, \nabla^{TY}_{e_j^H} Y \right\rangle \\
&= \left\langle \nabla^{TX}_{e_j^H} W, Y \right\rangle + \frac{1}{2} \left\langle T(e_j^H, P^{T^H X} W), Y \right\rangle.
\end{aligned} \tag{3.37}$$

By (3.37),

$$\nabla^{TY}_{e_j^H} P^{TY} W = P^{TY} \nabla^{TX}_{e_j^H} W + \frac{1}{2} T(e_j^H, P^{T^H X} W). \tag{3.38}$$

By (3.36) and (3.38),

$$\begin{aligned}
\nabla^{TY}_{e_j^H} \nabla^{TY}_{e_i^H} \widetilde{\mu} = &-P^{TY} (\nabla^X_{e_j^H} \mathbf{J}) e_i^H - P^{TY} \mathbf{J} \nabla^{TX}_{e_j^H} e_i^H \\
&- \frac{1}{2} T(e_j^H, P^{T^H X} \mathbf{J} e_i^H) - \frac{1}{2} (\nabla^{TY}_{e_j^H} \dot{g}^{TY}_{e_i^H}) \widetilde{\mu} - \frac{1}{2} \dot{g}^{TY}_{e_i^H} (\nabla^{TY}_{e_j^H} \widetilde{\mu}).
\end{aligned} \tag{3.39}$$

By (1.3) and (1.7), for $U_1, U_2$ sections of $TB$ on $B$,

$$\nabla^{TX}_{U_2^H} U_1^H = (\nabla^{TB}_{U_2} U_1)^H - \frac{1}{2} T(U_2^H, U_1^H). \tag{3.40}$$

By the definition of our basis $\{e_i^0, e_j^\perp\}$ in Section 2.6,

$$\begin{aligned}&(\nabla^{TB}_{e_i^0} e_j^0)_{x_0} = A(e_i^0) e_j^0,\\ &(\nabla^{TB}_{e_i^0} e_j^\perp)_{x_0} = (\nabla^{TB}_{e_j^\perp} e_i^0)_{x_0} = A(e_i^0) e_j^\perp, \quad (\nabla^{TB}_{e_j^\perp} e_i^\perp)_{x_0} = 0.\end{aligned} \tag{3.41}$$

Thus by (1.6), (3.2), (3.36), (3.39), (3.40), (3.41) and the facts that $A$ exchanges $N_G$ and $TX_G$ on $X_G$, and that $\widetilde{\mu} = 0$ on $P$, we get

$$(\nabla^{TY}_{\cdot} \nabla^{TY}_{\cdot} \widetilde{\mu})_{(\mathcal{R},\mathcal{R})} = -P^{TY} (\nabla^X_{\mathcal{R}} \mathbf{J}) \mathcal{R} - \mathbf{J} A(\mathcal{R}^0) \mathcal{R}^0 - \frac{1}{2} T(\mathcal{R}, \mathbf{J}\mathcal{R}^0) + T(\mathcal{R}, \mathbf{J}\mathcal{R}^\perp). \tag{3.42}$$

We use the closeness of $\omega$ to complete the proof of (3.33).

From (0.2), for $U, V, W \in TX$,

$$\langle (\nabla^X_U \mathbf{J}) V, W \rangle = (\nabla^X_U \omega)(V, W), \tag{3.43}$$

thus

$$\langle (\nabla^X_U \mathbf{J}) V, W \rangle + \langle (\nabla^X_V \mathbf{J}) W, U \rangle + \langle (\nabla^X_W \mathbf{J}) U, V \rangle = d\omega(U, V, W) = 0. \tag{3.44}$$

By (1.3), (1.7), (3.34) and (3.44) for $Y$ a smooth section of $TY$, at $x_0$,

$$\langle \mathbf{J} \nabla^{TX}_Y e_j^0, e_i^\perp \rangle = -\langle \nabla^{TX}_Y e_j^0, \mathbf{J} e_i^\perp \rangle = -\langle T(e_j^0, \mathbf{J} e_i^\perp), Y \rangle$$

and

$$\begin{aligned}\langle T(e_i^\perp, \mathbf{J} e_j^0), Y \rangle &= -2 \langle \nabla^{TX}_Y (\mathbf{J} e_j^0), e_i^\perp \rangle\\ &= -2 \langle (\nabla^X_Y \mathbf{J}) e_j^0, e_i^\perp \rangle + 2 \langle T(e_j^0, \mathbf{J} e_i^\perp), Y \rangle\\ &= 2 \langle (\nabla^X_{e_j^0} \mathbf{J}) e_i^\perp, Y \rangle - 2 \langle (\nabla^X_{e_i^\perp} \mathbf{J}) e_j^0, Y \rangle + 2 \langle T(e_j^0, \mathbf{J} e_i^\perp), Y \rangle.\end{aligned} \tag{3.45}$$

From (3.42), (3.45), we get the second equation of (3.33). ☐

The following formula extends [**29**, Theorem 2.2] to the group action case.

***Theorem 3.4***. — *The following identity holds,*

$$\begin{aligned}\mathcal{O}_1 = &- \frac{2}{3} (\partial_j R^{L_B})_{x_0} (\mathcal{R}, e_i) Z_j \nabla_{0,e_i} - \frac{1}{3} (\partial_i R^{L_B})_{x_0} (\mathcal{R}, e_i)\\ &- 2 \left\langle A(e_i^0) e_j^0, \mathcal{R}^\perp \right\rangle \nabla_{0,e_i^0} \nabla_{0,e_j^0} - \pi \sqrt{-1} \left\langle (\nabla^X_{\mathcal{R}} \mathbf{J}) e_a, e_b \right\rangle c(e_a)\, c(e_b)\\ &+ 4\pi^2 \left\langle (\nabla^X_{\mathcal{R}^0} \mathbf{J})(\mathcal{R}^0 + 2\mathcal{R}^\perp) + (\nabla^X_{\mathcal{R}^\perp} \mathbf{J}) \mathcal{R}^\perp - T(\mathcal{R}^\perp, \mathbf{J}\mathcal{R}^\perp), \mathbf{J}\mathcal{R}^\perp \right\rangle\\ &+ 4\pi^2 \left\langle \mathbf{J} A(\mathcal{R}^0) \mathcal{R}^0 + \frac{1}{2} T(\mathcal{R}^0, \mathbf{J}\mathcal{R}^0), \mathbf{J}\mathcal{R}^\perp \right\rangle\\ &+ 4\pi \sqrt{-1} \left\langle \widetilde{\mu}^{\mathrm{Cliff}} + \widetilde{\mu}^E, \mathbf{J}\mathcal{R}^\perp \right\rangle.\end{aligned} \tag{3.46}$$

*Proof.* — For $\psi \in (T^*X \otimes \mathrm{End}(\Lambda(T^{*(0,1)}X)))_B \simeq (T^*X \otimes (C(TX) \otimes_{\mathbb{R}} \mathbb{C}))_B$, where $C(TX)$ is the Clifford algebra bundle of $TX$, we denote by $\nabla^X \psi$ the covariant derivative of $\psi$ induced by $\nabla^{TX}$.

From $[\nabla^{\mathrm{Cliff}}_W, c(e_a)] = c(\nabla^{TX}_W e_a)$, (cf. also [**31**, Prop. 1.3.1]), we observe that for $W \in TB$,

$$\begin{aligned}\nabla^X_W(\psi(e_a)c(e_a)) &= (\nabla^X_W\psi)(e_a)c(e_a) + \psi(\nabla^{TX}_{W^H}e_a)c(e_a) + \psi(e_a)c(\nabla^{TX}_{W^H}e_a)\\ &= (\nabla^X_W\psi)(e_a)c(e_a).\end{aligned} \tag{3.47}$$

Thus by (2.50) and (3.47), for $k \geqslant 2$,

$$\begin{aligned}-(2\omega_d+\tau)(tZ) &= \frac{1}{2}\left(R^L(e_a,e_b)\,c(e_a)\,c(e_b)\right)(tZ)\\ &= \frac{1}{2}\sum_{r=0}^{k}\frac{\partial^r}{\partial t^r}\left[(R^L(e_a,e_b)\,c(e_a)\,c(e_b))(tZ)\right]\Big|_{t=0}\frac{t^r}{r!} + \mathscr{O}(t^{k+1})\\ &= \frac{1}{2}\Big(R^L_{x_0} + t(\nabla^X_{\mathcal{R}}R^L)_{x_0}\Big)(e_a,e_b)\,c(e_a)\,c(e_b) + \mathscr{O}(t^2).\end{aligned} \tag{3.48}$$

By Lemma 3.3 and (2.110), we have

$$\begin{aligned}-t^2\langle\widetilde{\mu}^{E_p},\widetilde{\mu}^{E_p}\rangle(tZ) =& 4\pi^2\sum_{k=2}^{3}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big(|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big)\Big|_{t=0}t^{k-2}\\ &+ 4\pi\sqrt{-1}t\left\langle\widetilde{\mu}^{\mathrm{Cliff}} + \widetilde{\mu}^E, \mathbf{J}\mathcal{R}^\perp\right\rangle_{x_0} + \mathscr{O}(t^2).\end{aligned} \tag{3.49}$$

The following two formulas are clear,

$$\begin{aligned}\frac{1}{2}\,\frac{\partial^2}{\partial t^2}|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big|_{t=0} &= \frac{1}{2}\,\Big(\nabla\nabla|\widetilde{\mu}|^2_{g^{TY}}(Z)\Big)_{(\mathcal{R},\mathcal{R})}\Big|_{Z=0} = |\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}|^2,\\ \frac{1}{3!}\,\frac{\partial^3}{\partial t^3}|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big|_{t=0} &= \frac{1}{6}\,\Big(\nabla\nabla\nabla|\widetilde{\mu}|^2_{g^{TY}}(Z)\Big)_{(\mathcal{R},\mathcal{R},\mathcal{R})}\Big|_{Z=0}\\ &= \langle(\nabla^{TY}\nabla^{TY}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})}, \nabla^{TY}_{\mathcal{R}}\widetilde{\mu}\rangle.\end{aligned} \tag{3.50}$$

From Lemma 3.3, (3.49) and (3.50), we see that the contribution from $-t^2\langle\widetilde{\mu}^{E_p},\widetilde{\mu}^{E_p}\rangle(tZ)$ forms the last three terms of (3.46).

By (2.103), (2.105) and (3.10), we have

$$\nabla_{t,e_i} = \nabla_{0,e_i} + \frac{t}{3}(\partial_j R^{L_B})_{x_0} Z_j(\mathcal{R},e_i) - \frac{t}{2}(\frac{1}{\kappa}\nabla_{e_i}\kappa)(tZ) + \mathscr{O}(t^2). \tag{3.51}$$

By $g_{ij}(Z) = \theta^k_i(Z)\theta^k_j(Z)$ and (2.94)-(2.96), we know

$$\begin{aligned}g_{ij}(Z) &= \begin{cases}\delta_{ij} - 2\left\langle A(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle + \mathscr{O}(|Z|^2) & \text{for } 1 \leqslant i,j \leqslant 2(n-n_0),\\ \delta_{ij} + \mathscr{O}(|Z|^2) & \text{otherwise;}\end{cases}\\ \kappa(Z) &= \det(g_{ij}(Z))^{1/2} = 1 - \langle A(e^0_i)e^0_i, \mathcal{R}^\perp\rangle + \mathscr{O}(|Z|^2).\end{aligned} \tag{3.52}$$

From (3.41), (3.51) and (3.52), the first three terms of the right hand side of (3.46) is the coefficient $t^1$ of the Taylor expansion of $-g^{ij}(tZ)(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TB}_{e_i}e_j(tZ)})$.

By (2.109), (3.43), (3.48) and the above argument, the proof of Theorem 3.4 is complete. □

***Theorem 3.5***. — *We have the relation*

$$P^N \mathscr{O}_1 P^N = 0. \tag{3.53}$$

*Proof*. — By (3.8) and (3.19),

$$\begin{aligned} & b_i^+ P^N = b_i^{\perp +} P^N = 0\,, \quad (b_i^\perp P^N)(Z,Z') = 2a_i^\perp Z_i^\perp P^N(Z,Z'),\\ & (b_i P^N)(Z,Z') = a_i(\overline{z}_i^0 - \overline{z}_i'^0)P^N(Z,Z'). \end{aligned} \tag{3.54}$$

We learn from (3.54) that for any polynomial $g(Z^\perp)$ in $Z^\perp$, we can write $g(Z^\perp)P^N(Z,Z')$ as sums of $g_{\beta^\perp}(b^\perp)^{\beta^\perp}P^N(Z,Z')$ with constants $g_{\beta^\perp}$. By Theorem 3.1,

$$P_{\mathscr{L}^\perp}(b^\perp)^{\alpha^\perp} g(Z^\perp)P^N = 0\,, \quad \text{for } |\alpha^\perp| > 0. \tag{3.55}$$

Let $\{w_a\}$ be an orthonormal basis of $(T^{(1,0)}X, g^{TX})$.

Note that if $f, g$ are two $\mathbb{C}$-linear forms, then

$$f(e_a)g(e_a) = f(w_a)g(\overline{w}_a) + f(\overline{w}_a)g(w_a).$$

Thus by Theorem 3.1, (2.9), (3.21) and (3.54),

$$\begin{aligned} P^N \left\langle (\nabla^X_{\mathcal{R}}\mathbf{J})e_a, e_b\right\rangle c(e_a)\, c(e_b)P^N &= -2P^N \left\langle (\nabla^X_{\mathcal{R}}\mathbf{J})w_a, \overline{w}_a\right\rangle P^N\\ &= -2P^N \left\langle (\nabla^X_{\mathcal{R}^0}\mathbf{J})w_a, \overline{w}_a\right\rangle P^N = \sqrt{-1}P^N \operatorname{Tr}|_{TX}[J(\nabla^X_{\mathcal{R}^0}\mathbf{J})]P^N. \end{aligned} \tag{3.56}$$

By (3.8), (3.12), (3.21), (3.46), (3.54)-(3.56), we get

$$\begin{aligned} P^N \mathscr{O}_1 P^N = P^N\Big\{ & \frac{2}{3}(\partial_{\mathcal{R}} R^{L_B})_{x_0}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i^0})b_i - \frac{1}{3}(\partial_{e_i^0} R^{L_B})_{x_0}(\mathcal{R}, e_i^0)\\ & + \frac{1}{3}(\partial_{\mathcal{R}} R^{L_B})_{x_0}(\mathcal{R}, e_j^\perp)b_j^\perp - \frac{1}{3}(\partial_{e_j^\perp} R^{L_B})_{x_0}(\mathcal{R}^0, e_j^\perp)\\ & + \pi \operatorname{Tr}|_{TX}[J(\nabla^X_{\mathcal{R}^0}\mathbf{J})] + 8\pi^2 \left\langle (\nabla^X_{\mathcal{R}^0}\mathbf{J})\mathcal{R}^\perp, \mathbf{J}\mathcal{R}^\perp\right\rangle \Big\} P^N. \end{aligned} \tag{3.57}$$

By (3.9), (3.54) and (3.55),

$$P^N Z_j^\perp Z_k^\perp P^N = \frac{1}{2a_k^\perp} P^N Z_j^\perp b_k^\perp P^N = \frac{1}{2a_k^\perp}\delta_{jk}P^N. \tag{3.58}$$

For $\psi$ a tensor on $X_G$, let $\nabla^{X_G}\psi$ be the covariant derivative of $\psi$ induced by the Levi-Civita connection $\nabla^{TX_G}$.

For $U, V, W \in T_{x_0}X_G$, by (3.2), (3.3) and (3.11), we have

$$(\partial_U R^{L_B})_{x_0}(V,W) = -2\pi\sqrt{-1}\left\langle (\nabla^{X_G}_U \mathbf{J}_G)V, W\right\rangle = -2\pi\sqrt{-1}\left\langle (\nabla^X_U \mathbf{J})V, W\right\rangle. \tag{3.59}$$

From (2.8), (3.2), (3.5), we know that

$$\mathbf{J}e_j^\perp = \frac{a_j}{2\pi}Je_j^\perp. \tag{3.60}$$

By Theorem 3.1, (1.18), (2.8), (3.9), (3.44) and (3.54)-(3.60), we get

$$
\begin{aligned}
(3.61)\quad P^N\mathcal{O}_1P^N = P^N\Big\{ &- \frac{4\pi\sqrt{-1}}{3}\Big[2\left\langle(\nabla^X_{\mathcal{R}^0}\mathbf{J})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0}\right\rangle + \left\langle(\nabla^X_{\frac{\partial}{\partial z_i^0}}\mathbf{J})\mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_i^0}\right\rangle \\
&- \left\langle(\nabla^X_{\frac{\partial}{\partial \overline{z}_i^0}}\mathbf{J})\mathcal{R}^0, \tfrac{\partial}{\partial z_i^0}\right\rangle\Big] + \pi\,\mathrm{Tr}\,|_{TX}[J(\nabla^X_{\mathcal{R}^0}\mathbf{J})] + 2\pi\left\langle(\nabla^X_{\mathcal{R}^0}\mathbf{J})e_j^\perp, Je_j^\perp\right\rangle\Big\}P^N \\
&= \pi P^N\Big[-4\sqrt{-1}\left\langle(\nabla^X_{\mathcal{R}^0}\mathbf{J})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0}\right\rangle \\
&\qquad + \mathrm{Tr}\,|_{TX}[J(\nabla^X_{\mathcal{R}^0}\mathbf{J})] - 2\left\langle J(\nabla^X_{\mathcal{R}^0}\mathbf{J})e_j^\perp, e_j^\perp\right\rangle\Big]P^N = 0.
\end{aligned}
$$

The proof of Theorem 3.5 is complete. □

From (3.32) and Theorem 3.5, we get the following general formula which will be used in Chapter 5,

$$
\begin{aligned}
P^{(2)} =&(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N - (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2P^N \\
&+ P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp} - P^N\mathcal{O}_2(\mathscr{L}_2^0)^{-1}P^{N^\perp} \qquad (3.62)\\
&+ (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp} - P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-2}P^{N^\perp}\mathcal{O}_1P^N.
\end{aligned}
$$

## 3.4. Example $(\mathbb{C}P^1, 2\,\omega_{FS})$

Let $\omega_{FS}$ be the Kähler form associated to the Fubini-Study metric $g^{T\mathbb{C}P^1}_{FS}$ on $\mathbb{C}P^1$. We will use the metric $g^{T\mathbb{C}P^1} = 2\,g^{T\mathbb{C}P^1}_{FS}$ on $\mathbb{C}P^1$ in this Section.

Let $L$ be the holomorphic line bundle $\mathcal{O}(2)$ on $\mathbb{C}P^1$. Recall that $\mathcal{O}(-1)$ is the tautological line bundle of $\mathbb{C}P^1$.

We will use the homogeneous coordinate $(z_0, z_1) \in \mathbb{C}^2$ for $\mathbb{C}P^1 \simeq (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$.

Denote by $U_i = \{[z_0, z_1] \in \mathbb{C}P^1; z_i \neq 0\}$, $(i = 0, 1)$, the open subsets of $\mathbb{C}P^1$, and the two coordinate charts are defined by $\phi_i : U_i \simeq \mathbb{C}$, $\phi_i([z_0, z_1]) = \frac{z_j}{z_i}$, $j \neq i$.

For any $i_0, i_1 \in \mathbb{N}$, $z_0^{i_0}z_1^{i_1}$ is naturally identified to a holomorphic section of $\mathcal{O}(-i_0 - i_1)^*$ on $\mathbb{C}P^1$. For any $k \in \mathbb{N}$, we have

$$
H^0(\mathbb{C}P^1, \mathcal{O}(k)) = \mathbb{C}\{s_{k,i_0} := z_0^{i_0}z_1^{i_1},\ i_0 + i_1 = k, \text{ and } i_0, i_1 \in \mathbb{N}\}. \tag{3.63}
$$

On $U_i$, the trivialization of the line bundle $L$ is defined by $L \ni s \to s/z_i^2$, here $z_i^2$ is considered as a holomorphic section of $\mathcal{O}(2)$.

In the following, we will work on $\mathbb{C}$ by using $\phi_0 : U_0 \to \mathbb{C}$. Then for $z \in \mathbb{C}$,

$$
\omega_{FS}(z) = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial\log((1 + |z|^2)^{-1}) = \frac{\sqrt{-1}}{2\pi}\frac{dz \wedge d\overline{z}}{(1 + |z|^2)^2}. \tag{3.64}
$$

Let $h^L$ be the smooth Hermitian metric on $L$ on $\mathbb{C}P^1$ defined by for $z \in \mathbb{C}$,

$$
|s_{2,0}|^2_{h^L}(z) = (1 + |z|^2)^{-2}. \tag{3.65}
$$

Let $\nabla^L$ be the holomorphic Hermitian connection of $(L, h^L)$ with its curvature $R^L$.

By (3.64) and (3.65), under our trivialization on $\mathbb{C}$

$$\begin{aligned}\nabla^L &= \overline{\partial} + \partial + \partial \log(|s_{2,0}|^2_{h^L}),\\ \frac{\sqrt{-1}}{2\pi} R^L &= \frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial \log |s_{2,0}|^2_{h^L} = 2\,\omega_{FS} =: \omega.\end{aligned} \tag{3.66}$$

Let $K$ be the canonical basis of $\operatorname{Lie} S^1 = \mathbb{R}$, i.e. for $t \in \mathbb{R}, \exp(tK) = e^{2\pi\sqrt{-1}t} \in S^1$.

We define an $S^1$-action on $\mathbb{C}P^1$ by $g \cdot [z_0, z_1] = [z_0, gz_1]$ for $g \in S^1$.

On our local coordinate $U_0$, $g \cdot z = gz$, and the vector field $K^{\mathbb{C}P^1}$ on $\mathbb{C}P^1$ induced by $K$ is

$$K^{\mathbb{C}P^1}(z) := \tfrac{\partial}{\partial t} \exp(-tK) \cdot z|_{t=0} = -2\pi\sqrt{-1}\Big(z\tfrac{\partial}{\partial z} - \overline{z}\tfrac{\partial}{\partial \overline{z}}\Big). \tag{3.67}$$

Set

$$\mu(K)([z_0, z_1]) = \frac{2|z_0|^2}{|z_0|^2 + |z_1|^2} - 1.$$

Then, on $\mathbb{C}$,

$$\mu(K) = 2|z|^2\,(1 + |z|^2)^{-1} - 1. \tag{3.68}$$

By (3.64), (3.67) and (3.68), we verify easily that $\mu$ is a moment map associated to the $S^1$-action on $(\mathbb{C}P^1, \omega)$ in the sense of (2.17).

The $\operatorname{Lie} S^1$-action on the sections of $L$ defined by (2.16) induces a holomorphical $S^1$-action on $L$. In particular, from (3.66)-(3.68),

$$\tfrac{\partial}{\partial t} \exp(tK) \cdot s_{2,j}|_{t=0} =: L_K s_{2,j} = 2\pi\sqrt{-1}(1 - j)\, s_{2,j}. \tag{3.69}$$

By (3.69), the $S^1$-invariant sub-space of $H^0(\mathbb{C}P^1, L^p)$ and $\mu^{-1}(0)$ are

$$H^0(\mathbb{C}P^1, L^p)^{S^1} = \mathbb{C}\, s_{2p,p}, \quad \mu^{-1}(0) = \{z \in \mathbb{C}, |z| = 1\}, \tag{3.70}$$

and $S^1$ acts freely on $\mu^{-1}(0)$, thus $(\mathbb{C}P^1)_{S^1} = \{\mathrm{pt}\}$.

Under our trivialization of $L$, $s_{2p,j} \in H^0(\mathbb{C}P^1, L^p)$ is the function $z^j$, and from (3.65),

$$\|s_{2p,j}\|^2_{L^2} = \int_{\mathbb{C}} \frac{|z|^{2j}}{(1 + |z|^2)^{2p}} 2\,\omega_{FS} = \int_0^\infty \frac{2t^j\,dt}{(1 + t)^{2p+2}} = \frac{2\,j!\,(2p - j)!}{(2p + 1)!}. \tag{3.71}$$

Thus $(\frac{(2p+1)!}{2\,(p!)^2})^{1/2} s_{2p,p}$ is an orthonormal basis of $H^0(\mathbb{C}P^1, L^p)^{S^1}$.

Let $\overline{\partial}^{L^p*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p}$. For $p \geqslant 1$, the spin$^c$ Dirac operator $D_p$ in (2.14) and its kernel are given by

$$D_p = \sqrt{2}\left(\overline{\partial}^{L^p} + \overline{\partial}^{L^p*}\right), \quad \operatorname{Ker} D_p = H^0(\mathbb{C}P^1, L^p). \tag{3.72}$$

Finally, by Def. 2.3, for $p \geqslant 1$, we get

$$\begin{aligned} P^G_p(z, z') &= \frac{(2p + 1)!}{2\,(p!)^2} s_{2p,p}(z) \otimes s_{2p,p}(z')^*,\\ P^G_p(z, z) &= \frac{(2p + 1)!}{2\,(p!)^2} |s_{2p,p}|^2_{h^{L^p}}(z) = \frac{(2p + 1)!}{2\,(p!)^2} \frac{|z|^{2p}}{(1 + |z|^2)^{2p}}.\end{aligned} \tag{3.73}$$

Note that our trivialization by $s_{2,0}$ is not unitary, thus we do not see directly the off-diagonal decay (0.14) from (3.73).

Here we will only verify that (3.73) is compatible with (0.13), (0.15) and (0.16).

Recall that Stirling's formula [**42**, Chap. 3, (A.40)] tells us that as $p \to +\infty$,

$$p! = (2\pi p)^{1/2} p^p\, e^{-p} \left(1 + \mathscr{O}\left(\frac{1}{p}\right)\right). \tag{3.74}$$

By (3.74),

$$\frac{(2p+1)!}{2\,(p!)^2} = \frac{\sqrt{p}}{\sqrt{\pi}\,e}\, 2^{2p}\Big(1+\frac{1}{2p}\Big)^{2p}\Big(1+\mathscr{O}(\frac{1}{p})\Big) = \sqrt{\frac{p}{\pi}}\, 2^{2p}\Big(1+\mathscr{O}(\frac{1}{p})\Big). \tag{3.75}$$

Now, $\mathbb{C}^*$ is an open neighborhood of $\mu^{-1}(0)$ and $B = \mathbb{C}^*/S^1 \simeq \mathbb{R}^+$ by mapping $z \in \mathbb{C}^*$ to $r = |z| \in \mathbb{R}^+$.

By (3.64), the metrics on $\{|z| = r\} = \{re^{2\pi\sqrt{-1}\theta}; \theta \in \mathbb{R}/\mathbb{Z}\}$, $B \simeq \mathbb{R}^+$ induced by $\omega = 2\,\omega_{FS}$ is

$$8\pi\, r^2\,(1+r^2)^{-2}\, d\theta \otimes d\theta, \qquad g^{TB} = \frac{2}{\pi}(1+r^2)^{-2}\, dr \otimes dr. \tag{3.76}$$

From (3.76), the fiberwise volume function $h^2(r)$ in (0.10) on $\mathbb{R}^+$ is

$$h^2(r) = \sqrt{8\pi}\, r\,(1+r^2)^{-1}. \tag{3.77}$$

From (3.73), (3.75) and (3.77), we get for $|z| = r$,

$$h^2(r) P^G_p(z,z) = \sqrt{8\pi}\frac{(2p+1)!}{2\,(p!)^2}\Big(\frac{r}{1+r^2}\Big)^{2p+1} = \sqrt{2\,p}\Big(\frac{2r}{1+r^2}\Big)^{2p+1}\Big(1+\mathscr{O}(\frac{1}{p})\Big). \tag{3.78}$$

When $|z| = 1$, from (3.78), we re-find (0.15) and (0.16).

From (3.76), $\sqrt{2\pi}\frac{\partial}{\partial r}$ is an orthonormal basis of $(B, g^{TB})$ at $r = 1$, thus the normal coordinate $Z^\perp$ has the form $r - 1 = \sqrt{2\pi}(Z^\perp + \mathscr{O}(|Z^\perp|^2)$. Thus

$$(2r\,(1+r^2)^{-1})^{2p+1} = e^{(2p+1)\log(1-\pi(Z^\perp)^2 + \mathscr{O}(|Z^\perp|^3))} = e^{-2\pi p (Z^\perp)^2} + \cdots. \tag{3.79}$$

This means that (3.78), (3.79) are compatible with (0.13) and (3.22).

If we consider the sub-space $H^0(\mathbb{C}P^1, L^p)_p$ of $H^0(\mathbb{C}P^1, L^p)$ with the weight $p$ of $S^1$-action, then by (2.16) as in (3.69), and (3.71), $\sqrt{p+\frac{1}{2}}\, s_{2p,0}$ is an orthonormal basis of $H^0(\mathbb{C}P^1, L^p)_p$.

Thus the smooth kernel $P^p_p(z,z')$ of the orthogonal projection from $\mathscr{C}^\infty(\mathbb{C}P^1, L^p)$ onto $H^0(\mathbb{C}P^1, L^p)_p$ is

$$\begin{aligned} P^p_p(z,z') &= (p+\frac{1}{2})\, s_{2p,0}(z) \otimes s_{2p,0}(z')^*, \\ P^p_p(z,z) &= (p+\frac{1}{2})(1+|z|^2)^{-2p}. \end{aligned} \tag{3.80}$$

Note that $\mu^{-1}(-1) = \{0\}$, i.e. $-1$ is a singular value of $\mu$.

Let $\mu_1$ be the moment map defined by $\mu_1(K) = \mu(K) + 1$, then $H^0(\mathbb{C}P^1, L^p)_p$ is the corresponding $S^1$-invariant holomorphic sections of $L^p$ with respect to the corresponding $S^1$-action.

Thus 0 is a singular value of $\mu_1$ and this explains why we have a factor $p$ in (3.80) instead of $p^{1/2}$ in (3.78).

# CHAPTER 4

# APPLICATIONS

This Chapter is organized as follows. In Section 4.1, we explain Theorem 4.1, the version of Theorem 0.2 when we only assume that $\mu$ is regular at 0. In Section 4.2, we explain how to handle the $\vartheta$-weight Bergman kernel. In Section 4.3, we deduce (0.15), and (0.16) from [**17**, Theorem 4.18′]. In Section 4.4, we review the characterization of the Toeplitz operators established in [**30**], and only Lemma 4.6 is new. In Section 4.5, we explain Theorem 0.2 implies Toeplitz operator type properties on $X_G$. In Section 4.6, we extend our results for non-compact manifolds and for covering spaces. In Section 4.7, we explain that the relation on the $G$-invariant Bergman kernel on $X$ and the Bergman kernel on $X_G$.

We use the notation in Introduction.

## 4.1. Orbifold case

We will use the notation for the orbifold as in [**26**, §1], [**17**, §4.2], [**31**, §5.4] and we recall briefly here.

Let $M$ be an orbifold, by definition, there exist a connected open covering $\{U\}$ of $M$ and a ramified covering $\tau_U : \widetilde{U} \to U$ which is $H_U$-equivariant and induces a homeomorphism $U \sim \widetilde{U}/H_U$, here $H_U$ is a finite group acting effectively on the connected smooth manifold $\widetilde{U}$, moreover, these ramified coverings are compatible. Especially, for any $x \in M$, there exist a small neighborhood $U_x \subset M$, a finite group $H_x$ acting linearly and effectively on $\mathbb{R}^m$ and $\widetilde{U}_x \subset \mathbb{R}^m$ an $H_x$-open set such that $\widetilde{U}_x \overset{\tau_x}{\to} \widetilde{U}_x/H_x = U_x$ and $\{0\} = \tau_x^{-1}(x) \in \widetilde{U}_x$.

Any additional structure on $M$ is induced by a corresponding $H_x$-invariant structure on $\widetilde{U}_x$. In this way, we can define an oriented, Riemannian, almost-complex or complex structure on $M$.

An orbifold vector bundle $\mathcal{E}$ over $M$ is an orbifold defined by an $H_x^{\mathcal{E}}$-equivariant (Here $H_x^{\mathcal{E}}$ is a finite group) vector bundle $\widetilde{\mathcal{E}}_{U_x}$ on $\widetilde{U}_x$ such that $H_x = H_x^{\mathcal{E}}/K_x^{\mathcal{E}}$, here

$K_x^{\mathcal{E}} = \{g \in H_x^{\mathcal{E}}, g$ acts on $\widetilde{U}_x$ as Id$\}$, and $(H_x^{\mathcal{E}}, \widetilde{\mathcal{E}}_{U_x}) \to \widetilde{\mathcal{E}}_{U_x}/H_x^{\mathcal{E}}$ defines the orbifold structure on $\mathcal{E}$. If $K_x^{\mathcal{E}} = \{e\}$ for any $x \in X$, then we call $\mathcal{E}$ a proper orbifold vector bundle. Let $\widetilde{\mathcal{E}_{U_x}^{\mathrm{pr}}}$ be the maximal $K_x^{\mathcal{E}}$-invariant sub-bundle of $\widetilde{\mathcal{E}}_{U_x}$ on $\widetilde{U}_x$, then $(G_{U_x}, \widetilde{\mathcal{E}_{U_x}^{\mathrm{pr}}})$ defines a proper orbifold vector bundle on $X$, denote it by $\mathcal{E}^{\mathrm{pr}}$.'

Now we go back to the hypotheses in the Introduction. In this Section, we only suppose that $0 \in \mathfrak{g}^*$ is a regular value of $\mu$, then $G$ acts only infinitesimal freely on $P = \mu^{-1}(0)$, thus $X_G = P/G$ is a compact symplectic orbifold.

Let $G^0 = \{g \in G, g \cdot x = x \text{ for any } x \in P\}$, then $G^0$ is a finite normal sub-group of $G$ and the group $G/G_0$ acts effectively on $P$.

Let $U$ be a $G$-neighborhood of $P = \mu^{-1}(0)$ in $X$ such that $G$ acts infinitesimal freely on $\overline{U}$, the closure of $U$. From the construction in Section 1.2, any $G$-equivariant vector bundle $F$ on $U$ induces an orbifold vector bundle $F_B$ on the orbifold $B = U/G$.

The function $h$ in (0.10) is only $\mathscr{C}^\infty$ on the regular part of the orbifold $B$, and we extend continuously $h$ to $U/G$ from its regular part, which is $\mathscr{C}^\infty$ and we denote it by $\widehat{h}$, then $\widehat{h}$ is also $\mathscr{C}^\infty$ on $U$.

As we work on $P$ in Section 2.4, 2.5, we need not to modify this part. Especially, Theorem 0.1 still holds.

We need to modify Section 2.6 as follows.

Observe first that the construction in Section 1.1 works well if we only assume that $G$ acts locally freely on $X$ therein.

We identify the normal bundle $N$ of $P$ in $U$, to the orthogonal complement of $TP$. Denote by $\nabla^{T^HU}$ the connection on $T^HU$ as in Section 1.1, and on $P$, let $\nabla^N$, $\nabla^{T^HP}$ be the connections on $N, T^HP$ as in (0.9), and let ${}^0\nabla^{T^HU} = \nabla^N \oplus \nabla^{T^HP}$ be the connection on $T^HU = N \oplus T^HP$.

For $y_0 \in P$, $W \in T^HU$ (resp. $T^HP$), we define $\mathbb{R} \ni t \to x_t = \exp_{y_0}^{T^HU}(tW) \in U$ (resp. $\exp_{y_0}^{T^HP}(tW) \in P$) the curve such that $x_t|_{t=0} = y_0$, $\frac{dx}{dt}|_{t=0} = W$, $\frac{dx}{dt} \in T^HU$, $\nabla^{T^HU}_{\frac{dx}{dt}} \frac{dx}{dt} = 0$ (resp. $\frac{dx}{dt} \in T^HP$, $\nabla^{T^HP}_{\frac{dx}{dt}} \frac{dx}{dt} = 0$).

By proceeding as in Section 2.6, we identify $B^{T^HU}(y_0, \varepsilon)$ to a subset of $U$ as following, for $Z \in B^{T^HU}(y_0, \varepsilon)$, $Z = Z^0 + Z^\perp$, $Z^0 \in T^H_{y_0}P$, $Z^\perp \in N_{y_0}$, we identify $Z$ with $\exp^{T^HU}_{\exp^{T^HP}_{y_0}(Z^0)}(\tau_{Z^0} Z^\perp)$.

Set $G_{y_0} = \{g \in G, gy_0 = y_0\}$, then $G \cdot B^{T^HU}(y_0, \varepsilon) = G \times_{G_{y_0}} B^{T^HU}(y_0, \varepsilon)$ is a $G$-neighborhood of $Gy_0$, and $(G_{y_0}, B^{T^HU}(y_0, \varepsilon))$ is a local coordinate of $B$.

As the construction in Section 2.6 is $G_{y_0}$-equivariant, we extend the geometric objects on $G \times_{G_{y_0}} B^{T^HU}(y_0, \varepsilon)$ to $G \times_{G_{y_0}} \mathbb{R}^{2n-n_0} = X_0$.

Thus we get the corresponding geometric objects on $G \times \mathbb{R}^{2n-n_0}$ by using the covering $G \times \mathbb{R}^{2n-n_0} \to G \times_{G_{y_0}} \mathbb{R}^{2n-n_0}$, especially, $\widehat{\mathcal{L}}_p^{X_0}$ (where we use the $\widehat{\cdot}$ notation to indicate the modification) is defined similarly on $G \times \mathbb{R}^{2n-n_0}$, and Theorem 2.5 holds for $\widehat{\mathcal{L}}_p^{X_0}$.

Let $\widehat{\pi}_G : G\times\mathbb{R}^{2n-n_0}\to\mathbb{R}^{2n-n_0}$ be the natural projection and as in (1.20), (2.82), we define

$$\widehat{\Phi}=\widehat{h}\widehat{\pi}_G : \mathscr{C}^\infty(G\times\mathbb{R}^{2n-n_0},E_{0,p})^G\longrightarrow\mathscr{C}^\infty(\mathbb{R}^{2n-n_0},(E_{0,p})_{\mathbb{R}^{2n-n_0}}),$$

then the operator $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$ is well-defined on $T^H_{y_0}U\simeq\mathbb{R}^{2n-n_0}$.

Let $g^{T^HX_0}$ be the metric on $\mathbb{R}^{2n-n_0}$ induced by $g^{TX_0}$, and let $dv_{T^HX_0}$ be the Riemannian volume form on $(\mathbb{R}^{2n-n_0},g^{T^HX_0})$.

Let $P_{y_0,p}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes L^p\otimes E)_{y_0})$ onto $\operatorname{Ker}(\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1})$ on $\mathbb{R}^{2n-n_0}$. Let $P_{y_0,p}(Z,Z')$ $(Z,Z'\in\mathbb{R}^{2n-n_0})$ be the smooth kernel of $P_{y_0,p}$ with respect to $dv_{T^HX_0}(Z')$.

Let $P^G_{0,p}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_0,L_0^p\otimes E_0)$ on $(\operatorname{Ker}D_p^{X_0})^G$, and let $P^G_{0,p}(x,x')$ be the smooth kernel of $P^G_{0,p}$ with respect to the Riemannian volume form $dv_{X_0}(x')$.

Let $P_p^{X_0/G}(y,y')$ $(y,y'\in X_0/G)$ be the smooth kernel associated to the operator on $X_0/G$ induced by $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$ as $P_{x_0,p}$ in (2.83).

Note that our trivialization of the restriction of $L$ on $B^{T^HU}(y_0,\varepsilon)$ as in Section 2.6 is not $G_{y_0}$-invariant, except that $G_{y_0}$ acts trivially on $L_{y_0}$.

For $x,x'\in X_0$, with their representatives $\widetilde{x},\widetilde{x}'\in\mathbb{R}^{2n-n_0}$, we have

$$\widehat{h}(x)\widehat{h}(x')P^G_{0,p}(x,x')=P_p^{X_0/G}(\pi(x),\pi(x'))=\frac{1}{|G^0|}\sum_{g\in G_{y_0}}(g,1)\cdot P_{y_0,p}(g^{-1}\widetilde{x},\widetilde{x}'). \tag{4.1}$$

Here $|G^0|$ is the cardinal of $G^0$. The second equation of (4.1) is from direct computation (cf. [**17**, (5.19)], [**31**, (5.4.17)]).

As we work on $G\times\mathbb{R}^{2n-n_0}$, for the operator $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$, Prop. 2.9 and Sections 2.7-2.9 still hold.

From Theorem 2.23 for $P_{y_0,p}$ and (4.1), we get

***Theorem 4.1***. — *Theorem 0.1 still holds.*

*Under the same notation in Theorems 0.2, 2.23, for $\alpha,\alpha'\in\mathbb{N}^{2n-n_0}$, $|\alpha|+|\alpha'|\leqslant m$, we have*

(4.2)
$$\begin{aligned}(1+\sqrt{p}|Z^\perp|+\sqrt{p}|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\Big(p^{-n+\frac{n_0}{2}}(\widehat{h}\kappa^{\frac12})(Z)(\widehat{h}\kappa^{\frac12})(Z')P^G_p\circ\Psi(Z,Z')\\ -\frac{1}{|G^0|}\sum_{r=0}^k\sum_{g\in G_{y_0}}(g,1)\cdot P^{(r)}_{y_0}(g^{-1}\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}\Big)\Big|_{\mathscr{C}^{m'}(P)}\\ \leqslant Cp^{-(k+1-m)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m}\\ \times\exp(-\sqrt{C''\nu p}\inf_{g\in G_{y_0}}|g^{-1}Z-Z'|)+\mathscr{O}(p^{-\infty}).\end{aligned}$$

If $Z = Z' = Z^0$, then for $g \in G_{y_0}$, such that $gZ^0 = Z^0$, we use Theorem 2.23 for $Z = Z' = 0$ with the base point $Z^0$, and for the rest element in $G_{y_0}$, we use Theorem 2.23 for $Z = Z' = Z^0$ with the base point $y_0$, then we get

$$\begin{aligned}(4.3)\quad \Big| p^{-n+\frac{n_0}{2}} (\widehat{h}^2\kappa)(Z^0) P_p^G \circ \Psi(Z^0, Z^0) \\ - \frac{1}{|G^0|} \sum_{r=0}^{k} \sum_{g\in G_{y_0}, gZ^0=Z^0} (g,1)\cdot P_{Z^0}^{(2r)}(0,0) p^{-r} \\ - \frac{1}{|G^0|} \sum_{r=0}^{2k} \sum_{g\in G_{y_0}, gZ^0\neq Z^0} (g,1)\cdot P_{y_0}^{(r)}(g^{-1}\sqrt{p}Z^0, \sqrt{p}Z^0) p^{-\frac{r}{2}} \Big| \\ \leqslant C p^{-(2k+1)/2} \Big(1 + (1+\sqrt{p}|Z^0|)^{2(n+2k+2)} \exp(-\sqrt{C'''\nu p}|Z^0|)\Big).\end{aligned}$$

Note that if $g \in G_{y_0}$ acts as the multiplication by $e^{i\theta}$ on $L_{y_0}$, then $(g,1)\cdot P_{y_0}^{(r)}$, $(g,1)\cdot P_{Z^0}^{(r)}$ in (4.3) have a factor $e^{i\theta p}$ which depends on $p$.

Of course, after replacing $L$ by some power of $L$, we can assume that $G_{y_0}$ acts as identity on $L$ for any $y_0 \in P$, in this case, $(g,1)\cdot P_{y_0}^{(r)}(g^{-1}Z^0, Z^0)$, $(g,1)\cdot P_{Z^0}^{(r)}(0,0)$ do not depend on $p$.

From Theorem 3.2 and (4.3), if the singular set of $X_G$ is not empty, analogous to the usual orbifold case [**17**, (5.27)], $p^{-n+\frac{n_0}{2}}\ P_p^G(y_0,y_0)$, $(y_0 \in P)$ does not have a uniform asymptotic expansion in the form $\sum_{r=0}^{\infty} c_r(y_0)p^{-r}$.

## 4.2. $\vartheta$-weight Bergman kernel on $X$

In this section, we assume that $G$ acts on $P = \mu^{-1}(0)$ freely.

Let $\mathcal{V}$ be a finite dimensional irreducible representation of $G$, we denote it by $\rho^{\mathcal{V}} : G \to \mathrm{End}(\mathcal{V})$. Let $\vartheta$ be the highest weight of the representation $\mathcal{V}$. Let $\mathcal{V}^*$ be the trivial vector bundle on $X$ with $G$-action $\rho^{\mathcal{V}^*}$ induced by $\rho^{\mathcal{V}}$.

Let $P_p^{\mathcal{V}}$ be the orthogonal projection from $\Omega^{0,\bullet}(X, L^p\otimes E)$ on $\mathrm{Hom}_G(\mathcal{V}, \mathrm{Ker}\, D_p)\otimes \mathcal{V} \subset \mathrm{Ker}\, D_p$. Let $P_p^{\mathcal{V}}(x,x')$, $(x,x' \in X)$, be the smooth kernel of $P_p^{\mathcal{V}}$ with respect to $dv_X(x')$.

We call $P_p^{\mathcal{V}}(x,x')$ the $\vartheta$-weight Bergman kernel of $D_p$.

We explain now the asymptotic expansion of $P_p^{\mathcal{V}}(x,x')$ as $p\to\infty$.

We will consider the corresponding objects in Chapters 1-3 by replacing $E$ by $E\otimes \mathcal{V}^*$. Especially, we denote by $D_p^{\mathcal{V}^*}$ the corresponding spin$^c$ Dirac operator associated to the bundle $L^p\otimes E\otimes \mathcal{V}^*$.

Certainly, all results in Chapters 1-3 still hold for the bundle $E\otimes\mathcal{V}^*$.

Let $P_p^{\vartheta}$ be the orthogonal projection from $\mathscr{C}^\infty(X, E_p\otimes\mathcal{V}^*)$ onto $(\mathrm{Ker}\, D_p^{\mathcal{V}^*})^G$, and $P_p^{\vartheta}(x,x')$, $(x,x'\in X)$ the smooth kernel of $P_p^{\vartheta}$ with respect to $dv_X(x')$.

As $\mathcal{V}$ is an irreducible representation of $G$, we get

$$\operatorname{Ker} D_p^{\mathcal{V}^*} = (\operatorname{Ker} D_p) \otimes \mathcal{V}^*, \quad (\operatorname{Ker} D_p^{\mathcal{V}^*})^G = \operatorname{Hom}_G(\mathcal{V}, \operatorname{Ker} D_p). \tag{4.4}$$

Let $\{v_i\}$ be an orthonormal basis of $\mathcal{V}$ with respect to a $G$-invariant metric on $\mathcal{V}$ and $\{v_i^*\}$ the corresponding dual basis.

Let $dg$ be a Haar measure on $G$. By Schur Lemma,

$$\int_G g \cdot (v_j \otimes v_i^*) dg = \frac{1}{\dim_{\mathbb{C}} \mathcal{V}} \delta_{ij} \operatorname{Id}_{\mathcal{V}}. \tag{4.5}$$

Thus if $W$ is a finite dimensional representation of $G$ with the highest weight $\vartheta$, then for any $s \in W$, we have

$$s = \sum_i (\dim_{\mathbb{C}} \mathcal{V}) \Big( \int_G g \cdot (s \otimes v_i^*) dg \Big) \otimes v_i \in \operatorname{Hom}_G(\mathcal{V}, W) \otimes \mathcal{V} = W. \tag{4.6}$$

From (4.6) and the $G \times G$-invariance of the kernel $P_p^\vartheta(x, x')$, we get

$$\begin{aligned} &P_p^{\mathcal{V}}(x, x') = (\dim_{\mathbb{C}} \mathcal{V}) \sum_i (P_p^\vartheta(x, x') v_i^*, v_i), \\ &P_p^{\mathcal{V}}(x, x) = (\dim_{\mathbb{C}} \mathcal{V}) \operatorname{Tr}_{\mathcal{V}^*} P_p^\vartheta(x, x) \in \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x. \end{aligned} \tag{4.7}$$

In fact, let $\{\psi_j\}$ be an orthonormal basis of $(\operatorname{Ker} D_p^{\mathcal{V}^*})^G$, then $P_p^\vartheta(x, x') = \sum_j \psi_j(x) \otimes \psi_j(x')^*$, and for any $j$ fixed, in view of the second equality in (4.4), one sees that

$$\psi_j^* \psi_j \in \operatorname{End}_G(\mathcal{V}) \quad \text{and} \quad \operatorname{Tr}_{\mathcal{V}}[\psi_j^* \psi_j] = \|\psi_j\|_{L^2}^2 = 1. \tag{4.8}$$

Thus by Schur Lemma,

$$\psi_j^* \psi_j = \frac{1}{\dim_{\mathbb{C}} \mathcal{V}} \operatorname{Id}_{\mathcal{V}} \tag{4.9}$$

and $\{(\dim_{\mathbb{C}} \mathcal{V})^{\frac{1}{2}} \psi_j v_i\}$ is an orthonormal basis of $\operatorname{Hom}_G(\mathcal{V}, \operatorname{Ker} D_p) \otimes \mathcal{V} \subset \operatorname{Ker} D_p$.

Let $U$ be a $G$-neighborhood of $P = \mu^{-1}(0)$ as in Theorem 0.2, $P_p^\vartheta$ is viewed as a smooth section of $\mathrm{pr}_1^*(E_p \otimes \mathcal{V}^*)_B \otimes \mathrm{pr}_2^*(E_p \otimes \mathcal{V}^*)_B^*$ on $B \times B$, or as a $G \times G$-invariant smooth section of $\mathrm{pr}_1^*(E_p \otimes \mathcal{V}^*) \otimes \mathrm{pr}_2^*(E_p \otimes \mathcal{V}^*)^*$ on $U \times U$.

Moreover, $v_i$, $v_i^*$ are smooth (not $G$-invariant) sections of $U \times \mathcal{V}$, $U \times \mathcal{V}^*$ on $U$. Thus from (4.7), $P_p^{\mathcal{V}}$ is not a $G \times G$-invariant section of $\mathrm{pr}_1^*(E_p) \otimes \mathrm{pr}_2^*(E_p^*)$ on $U \times U$.

Now (2.83), (2.84), (2.108) and (2.186) (cf. also Theorem 0.2) apply well to the bundle $E \otimes \mathcal{V}^*$, thus we get the asymptotic expansion of $P_p^\vartheta(x, x')$ as $p \to +\infty$, and the leading term in the expansion of

$$p^{-n+\frac{n_0}{2}} (h\kappa^{\frac{1}{2}})(x)(h\kappa^{\frac{1}{2}})(x') P_p^\vartheta(x, x') \text{ is } P(\sqrt{p}Z, \sqrt{p}Z') I_{\mathbb{C} \otimes (E \otimes \mathcal{V}^*)_B}.$$

By (4.7), the leading term of the asymptotic expansion of

$$p^{-n+\frac{n_0}{2}} (h\kappa^{\frac{1}{2}})(x)(h\kappa^{\frac{1}{2}})(x') P_p^{\mathcal{V}}(x, x') \text{ is } (\dim_{\mathbb{C}} \mathcal{V})^2 P(\sqrt{p}Z, \sqrt{p}Z') I_{\mathbb{C} \otimes E_B}. \tag{4.10}$$

Let $\Theta$ be the curvature of $P \to X_G$ as in Section 1.1. Let $\rho_*^{\mathcal{V}^*}$ denote the differential of $\rho^{\mathcal{V}^*}$. By (1.18),

$$R^{(E\otimes\mathcal{V}^*)_G} = R^{E_G} + \rho_*^{\mathcal{V}^*}(\Theta). \tag{4.11}$$

In the same way, we can define $\mathscr{I}_p^{\mathcal{V}}$ a section of $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B$ on $X_G$ by (0.17) for $P_p^{\mathcal{V}}$. From (0.25) (which will be proved in Chapter 5), (4.7), (4.10) and (4.11), we get

***Theorem 4.2.*** — *Under the condition of Theorem 0.6, the first coefficients of the asymptotic expansion of* $\mathscr{I}_p^{\mathcal{V}} \in \mathrm{End}(E_G)$ *in* (0.20) *is*

$$\begin{aligned}\Phi_0 &= (\dim_{\mathbb{C}}\mathcal{V})^2,\\ \Phi_1 &= \frac{1}{8\pi}(\dim_{\mathbb{C}}\mathcal{V})^2\left(r_{x_0}^{X_G} + 6\Delta_{X_G}\log h + 4R_{x_0}^{E_G}(w_j^0,\overline{w}_j^0)\right)\\ &\qquad + \frac{1}{2\pi}(\dim_{\mathbb{C}}\mathcal{V})\ \mathrm{Tr}_{\mathcal{V}^*}\left[\rho_*^{\mathcal{V}^*}(\Theta)(w_j^0,\overline{w}_j^0)\right].\end{aligned} \tag{4.12}$$

### 4.3. Averaging the Bergman kernel: a direct proof of (0.15) and (0.16)

We use the same assumption and notation as in Theorem 0.2.

Let $P_p(x,x')$ be the smooth kernel of the orthogonal projection $P_p$ from $\Omega^{0,\bullet}(X, L^p\otimes E)$ onto $\mathrm{Ker}\, D_p$ with respect to $dv_X(x')$. Then $P_p(x,x')$ is the usual Bergman kernel associated to $D_p$.

Let $dg$ be a Haar measure on $G$. By Schur Lemma,

$$P_p^G(x,x') = \int_G ((g,1)\cdot P_p)(x,x')dg = \int_G (g,1)\cdot P_p(g^{-1}x,x')dg. \tag{4.13}$$

One possible way to get Theorem 0.2 is to apply the full off-diagonal expansion [**17**, Theorem 4.18′] to (4.13).

Unfortunately, we do not know how to get the full off-diagonal expansion, especially the fast decay along $N_G$ in (0.14) in this way.

However, it is easy to get (0.15) and (0.16) as direct consequences of [**17**, Theorem 4.18′] and (4.13).

As in Section 2.5, we denote by $TY$ the sub-bundle of $TX$ on a neighborhood of $P=\mu^{-1}(0)$ generated by the $G$-action and by $T^HP$ the orthogonal complement of $TY|_P$ in $(TP, g^{TP})$.

Take $y_0\in P$. Let $\{e_i\}_{i=1}^{2(n-n_0)}$, $\{f_l\}_{l=1}^{n_0}$ be orthonormal basis of $T_{y_0}^HP$, $T_{y_0}Y$. Then $\{e_i\}_{i=1}^{2(n-n_0)}\cup\{f_l, J_{y_0}f_l\}_{l=1}^{n_0}$ is an orthonormal basis of $T_{y_0}X$. We use this orthonormal basis to get a local coordinate of $X$ by using the exponential map $\exp_{y_0}^X$.

We identify $B^{T_{y_0}X}(0,\varepsilon)$ to $B^X(y_0,\varepsilon)$ by the exponential map $Z\to\exp_{y_0}^X(uZ)$.

Let $\nabla^{\mathrm{Cliff}\otimes E}$ be the connection on $\Lambda(T^{*(0,1)}X)\otimes E$ induced by $\nabla^{\mathrm{Cliff}}$ and $\nabla^E$.

For $Z \in B^{T_{y_0}X}(0,\varepsilon)$, we identify $L_Z$, $(\Lambda(T^{*(0,1)}X)\otimes E)_Z$, $(E_p)_Z$ to $L_{y_0}$, $(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}$, $(E_p)_{y_0}$ by parallel transport with respect to the connections $\nabla^L$, $\nabla^{\mathrm{Cliff}\otimes E}$, $\nabla^{E_p}$ along the curve $\gamma_Z : [0,1]\ni u \to uZ$.

Under this identification, for $Z, Z' \in B^{T_{y_0}X}(0,\varepsilon)$, one has

$$P_p(Z,Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}.$$

Let $\kappa_1(Z)$ be the function on $B^{T_{y_0}X}(0,\varepsilon)$ defined by

$$dv_X(Z) = \kappa_1(Z) dv_{T_{x_0}X}. \tag{4.14}$$

By [**17**, Theorem 4.18′] (i.e. Theorem 0.2 for $G=\{1\}$), there exist $J_r(Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}$, polynomials in $Z'$ with the same parity as $r$, such that for any $k, m' \in \mathbb{N}$, there exist $C, M>0$ such that for $Z' \in T_{y_0}X, |Z'|\leqslant \varepsilon$,

$$\begin{aligned}\Big|\frac{1}{p^n}P_p(Z',0) - \sum_{r=0}^{k} J_r(\sqrt{p}Z')\kappa_1^{-1}(Z')e^{-\frac{\pi}{2}p|Z'|^2}p^{-\frac{r}{2}}\Big|_{\mathscr{C}^{m'}(P)} \\ \leqslant Cp^{-(k+1)/2}(1+\sqrt{p}|Z'|)^M \exp(-\sqrt{C''\nu_0}\sqrt{p}|Z'|) + \mathscr{O}(p^{-\infty}),\end{aligned} \tag{4.15}$$

and

$$J_0(Z) = I_{\mathbb{C}\otimes E}. \tag{4.16}$$

For $K\in\mathfrak{g}$, $|K|$ small, $e^K$ maps $(\Lambda(T^{*(0,1)}X)\otimes E)_{e^{-K}y_0}$, $L_{e^{-K}y_0}$ to $(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}$, $L_{y_0}$, and under our identification, we denote these maps by

$$f^E(K) \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}, \quad f^L(K)\in \mathrm{End}(L_{y_0}) \simeq \mathbb{C}. \tag{4.17}$$

As the $G$-action preserves $h^L$ and $\nabla^L$, we know $|f^L(K)|=1$ and $f^E(K)$ is also an isometry.

For $K\in\mathfrak{g}$, let $\mathrm{ad}\,K$ be the adjoint representation defined by $(\mathrm{ad}\,K)K' = [K,K']$ for $K'\in\mathfrak{g}$. By [**1**, Prop. 5.1], if we denote by

$$j_{\mathfrak{g}}(K) = \det{}_{\mathfrak{g}}\Big(\frac{1-e^{-\mathrm{ad}\,K}}{\mathrm{ad}\,K}\Big) \tag{4.18}$$

for $K\in\mathfrak{g}$, then in exponential coordinates of $G$,

$$d(e^K) = j_{\mathfrak{g}}(K)dK. \tag{4.19}$$

As the $G$-action preserves all metrics and connections, thus for any smooth kernel $\Psi_p = \mathscr{O}(p^{-\infty})$, we have $(g,1)\cdot\Psi_p(g^{-1}x,x') = \mathscr{O}(p^{-\infty})$ for any $g\in G$.

By [**17**, Prop. 4.1] (i.e. Theorem 0.1 for $G=\{1\}$), (4.13), as $G$ acts freely on $P$, we know

$$P_p^G(y_0,y_0) = \int_{K\in\mathfrak{g},|K|\leqslant\varepsilon} f^E(K)(f^L(K))^p P_p(e^{-K}y_0,y_0)j_{\mathfrak{g}}(K)dK + \mathscr{O}(p^{-\infty}). \tag{4.20}$$

Let $S^L$ be the section of $L$ on $B^{T_{y_0}X}(0,\varepsilon)$ obtained by parallel transport of a unit vector of $L_{y_0}$ with respect to the connection $\nabla^L$ along the curve $\gamma_Z$. Let $\Gamma^L$ be the connection form of $L$ with respect to this trivialization.

Recall that for $K \in \mathfrak{g}$, the corresponding vector field $K^X$ on $X$ is defined in Section 1.1. Recall that $\{K_i\}$ is a basis of $\mathfrak{g}$.

By (2.104), for $K \in \mathfrak{g}$,

$$(4.21)\quad \begin{aligned} &(e^K \cdot S^L)(0) = e^K \cdot S^L(e^{-K} y_0) = f^L(K) S^L(0), \text{ with } f^L(0) = 1,\\ &\Gamma^L_Z(K^X) = \frac{1}{2} R^L_{y_0}(Z, K^X) + \mathscr{O}(|Z|^2). \end{aligned}$$

By (2.16), (2.17), (4.21) and $\mu = 0$ on $P$, we get

$$(4.22)\quad \begin{aligned} (L_{K_j}(L_{K_i} S^L))(0) &= (\nabla^L_{K_j^X}(\nabla^L_{K_i^X} S^L - 2\pi\sqrt{-1}\mu(K_i) S^L))(0)\\ &= \frac{1}{2} R^L_{y_0}(K_j^X, K_i^X) S^L(0) = \pi\sqrt{-1}\langle d\mu(K_i), K_j^X\rangle S^L(0) = 0. \end{aligned}$$

By (2.16), (4.21), (4.22), $\mu = 0$ on $P$ and $K^X \in TY$ on $P$, we get

$$(4.23)\quad \begin{aligned} &\frac{\partial f^L}{\partial K_i}(0) S^L(0) = (L_{K_i} S^L)(0) = (\nabla^L_{K_i^X} S^L)(0) = 0,\\ &\frac{\partial^2 f^L}{\partial K_i \partial K_j}(0) S^L(0) = \frac{\partial^2}{\partial t_1 \partial t_2}(e^{t_1 K_i + t_2 K_j} \cdot S^L)(0)|_{t_1 = t_2 = 0}\\ &\qquad = (L_{K_j}(L_{K_i} S^L) + L_{K_i}(L_{K_j} S^L))(0) = 0. \end{aligned}$$

Thus from (4.23),

$$(4.24)\quad (f^L(K))^p = (1 + \mathscr{O}(|K|^3))^p.$$

Moreover, from (2.95), (2.106), (2.108) (for $G = \{1\}$),

$$(4.25)\quad \begin{aligned} &f^E(K) = \mathrm{Id}_{(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}} + \mathscr{O}(|K|),\\ &\kappa_1(Z) = 1 + \mathscr{O}(|Z|^2). \end{aligned}$$

Let $dv_Y$ be the Riemannian volume form on $(TY, g^{TY})$. Observe also that if we denote by $i_{y_0} : G \to G y_0$ the map defined by $i_{y_0}(g) = g y_0$, then

$$(4.26)\quad \frac{1}{h^2(y)} dv_Y(y) = (i_{y_0}^{-1})^* dg,$$

which gives us a factor $\frac{1}{h^2(y_0)}$ when we take the integral on $\mathfrak{g}$ instead on the normal coordinates on $X$.

By (4.13), (4.15), (4.20), (4.24)-(4.26) and the Taylor expansion for $\kappa_1$, $f^E$, $f^L$, as in [**1**, Theorems 5.8, 5.9], we know that there exist $J'_r(Z)$ polynomials in $Z$ with same parity on $r$, and $J'_0 = I_{\mathbb{C}\otimes E}$, such that

$$(4.27)\quad P^G_p(y_0, y_0) \sim p^n \frac{1}{h^2(y_0)} \int_{K\in\mathfrak{g}, |K|\leqslant\varepsilon} e^{-\frac{\pi}{2}p|K|^2} \sum_{r=0}^{\infty} J'_r(\sqrt{p}K) p^{-r/2} dK.$$

Recall that

$$(4.28)\quad \int_{K\in\mathfrak{g}} e^{-\frac{\pi}{2}p|K|^2} dK = 2^{\frac{n_0}{2}} p^{-\frac{n_0}{2}}.$$

After taking the integral on $\mathfrak{g}$, from (4.27) and (4.28), we get (0.15) and (0.16).

By (4.7), (4.27) and (4.28), we get also the asymptotic expansion for $P_p^{\mathcal{V}}(y_0, y_0)$, $y_0 \in P$.

## 4.4. Berezin-Toeplitz quantization

Let $(X, \omega)$ be a compact symplectic manifold of real dimension $2n$. Let $(L, h^L)$ be a Hermitian line bundle over $X$ endowed with a Hermitian connection $\nabla^L$ such that (0.1) holds.

Let $(E, h^E)$ be a Hermitian vector bundle on $X$ with Hermitian connection $\nabla^E$.

Let $g^{TX}$ be a Riemannian metric on $X$ and let $J$ be an almost complex structure such that (0.3) holds and that $\omega(\cdot, J\cdot)$ defines a metric on $TX$.

Let $P_p(x, x')$ be the smooth kernel of the orthogonal projection $P_p$ from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto $\operatorname{Ker} D_p$ with respect to the Riemannian volume form $dv_X(x')$. Then $P_p(x, x')$ is the usual Bergman kernel associated to $D_p$.

***Definition 4.3***. — A family of operators $T_p : \operatorname{Ker} D_p \to \operatorname{Ker} D_p$ is a Toeplitz operator if there exists a sequence of smooth sections $g_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ with an asymptotic expansion $g(\cdot, p)$ of the form $\sum_{l=0}^\infty p^{-l} g_l(x)$ such that for any $k \geqslant 0$, there exists $C > 0$ such that for any $p \in \mathbb{N}$,

$$\|T_p - P_p \sum_{l=0}^{k} p^{-l} g_l(x) P_p\|^{0,0} \leqslant C p^{-k-1}. \tag{4.29}$$

Here $\| \quad \|^{0,0}$ is the operator norm with respect to the norm $\| \quad \|_{L^2}$. We call $g_0(x)$ the principal symbol of $T_p$. If $T_p$ is self-adjoint, then we call $T_p$ is a self-adjoint Toeplitz operator.

We express (4.29) symbolically by

$$T_p = P_p \Big( \sum_{l=0}^{k} p^{-l} g_l \Big) P_p + \mathcal{O}(p^{-k-1}). \tag{4.30}$$

If (4.29) holds for any $k \in \mathbb{N}$, then we write

$$T_p = P_p \Big( \sum_{l=0}^{\infty} p^{-l} g_l \Big) P_p + \mathcal{O}(p^{-\infty}). \tag{4.31}$$

The map which associates to a section $\mathbf{f} \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ the bounded operator

$$T_{\mathbf{f},p} = P_p \mathbf{f} P_p : L^2(X, E_p) \longrightarrow L^2(X, E_p), \text{ with } E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E, \tag{4.32}$$

is called the Berezin-Toeplitz quantization.

Recall that $a^X$ is the injectivity radius of $(X, g^{TX})$. In what follows, we fix $\varepsilon \in ]0, a^X/4[$. For $x \in X$, we identify $B^{T_xX}(0, 4\varepsilon)$ with $B^X(x, 4\varepsilon)$ by using the exponential map. Let $dv_{TX}$ be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$ for $x_0 \in X$. Let $\kappa_{x_0}$ be a smooth positive function on $T_{x_0}X$ with $\kappa_{x_0}(0) = 1$ defined by

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z). \tag{4.33}$$

We denote by $\det_{\mathbb{C}}$ for the determinant function on the complex bundle $T^{(1,0)}X$. Set $\mathbf{J} \in \mathscr{C}^\infty(X, \mathrm{End}(TX))$ as in (0.2), and $|\mathbf{J}_{x_0}| = (-\mathbf{J}_{x_0}^2)^{1/2}$. Set $\mathscr{P}(Z, Z')$, $(Z, Z' \in T_{x_0}X)$ be the analogue of $P_{\mathscr{L}}$ in (3.19),

$$\mathscr{P}(Z,Z') = \det{}_{\mathbb{C}}(|\mathbf{J}_{x_0}|) \exp\Big( -\frac{\pi}{2} \langle |\mathbf{J}_{x_0}|(Z-Z'), (Z-Z')\rangle - \pi\sqrt{-1}\, \langle \mathbf{J}_{x_0} Z, Z'\rangle \Big). \tag{4.34}$$

We trivialize $L$, $E$ and $E_p$ over $B^{T_xX}(0, 4\varepsilon)$ by using the parallel transport with respect to $\nabla^L$, $\nabla^E$ and $\nabla^{E_p}$ along the curves $\gamma_Z(u) = uZ$.

Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of $TX$ on $X$.

Let $\{\Xi_p\}_{p\in\mathbb{N}}$ be a sequence of linear operators $\Xi_p : L^2(X, E_p) \longrightarrow L^2(X, E_p)$ with smooth kernel $\Xi_p(x, y)$ with respect to $dv_X(y)$. Then under the above trivialization, $\Xi_p(x, y)$ induces a smooth section $\Xi_{p,\,x_0}(Z, Z')$ of $\pi^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E))$ over $TX \times_X TX$ with $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < 4\varepsilon$ which depends smoothly on $x_0$.

We will denote

$$p^{-n}\Xi_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}})\,, \tag{4.35}$$

if

$$\{Q_{r,\,x_0} \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}[Z, Z']\}_{0\leqslant r\leqslant k,\, x_0\in X}$$

is a smooth family of polynomials on $Z, Z'$ with respect to the parameter $x_0 \in X$, such that there exist constants $\varepsilon' \in ]0, 4\varepsilon]$ and $C_0 > 0$ with the following property: for every $l \in \mathbb{N}$, there exist $C_{k,\,l} > 0$, $M > 0$ such that for $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$ and $p \in \mathbb{N}$ the following estimate holds : [(1)]

$$\begin{aligned} &\Big| p^{-n}\Xi_{p,\,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} \Big|_{\mathscr{C}^l(X)} \\ &\qquad \leqslant C_{k,\,l}\, p^{-\frac{k+1}{2}} (1 + \sqrt{p}\,|Z| + \sqrt{p}\,|Z'|)^M \exp(-\sqrt{C_0 p}\,|Z-Z'|) + \mathscr{O}(p^{-\infty})\,. \end{aligned} \tag{4.36}$$

---

(1) By Theorems 0.1, 0.2 for $G = \{1\}$ (or [**31**, Theorem 4.2.1]) , if $\Xi_p = P_p \Xi_p P_p$, then (4.36) is equivalent to: for any $l, m \in \mathbb{N}$, there exist $C > 0$, $M > 0$ such that for $x_0 \in X$, $|Z|, |Z'| < \varepsilon'$, $|\alpha| + |\alpha'| \leqslant m$ and $p \in \mathbb{N}$, the following estimate holds :

$$\begin{aligned} &\Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big( p^{-n}\Xi_{p,\,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} \Big) \Big|_{\mathscr{C}^l(X)} \\ &\leqslant C\, p^{-\frac{k+1-m}{2}} (1 + \sqrt{p}\,|Z| + \sqrt{p}\,|Z'|)^M \exp(-\sqrt{C_0 p}\,|Z-Z'|) + \mathscr{O}(p^{-\infty})\,. \end{aligned}$$

Even (4.36) holds for any $l \in \mathbb{N}$, in the proof of Theorem 4.4 (i.e. [**30**, Theorem 4.9]), we only use $l = 0$.

In [**30**, Theorem 4.9] (cf. also [**31**, Theorem 7.3.1]), Ma and Marinescu established a useful criterion which ensures that a given family is a Toeplitz operator.

***Theorem 4.4***. — *Let* $\{T_p : L^2(X, E_p) \longrightarrow L^2(X, E_p)\}$ *be a family of bounded linear operators which satisfies the following three conditions:*

(i) *For any* $p \in \mathbb{N}$, $P_p\, T_p\, P_p = T_p$ .

(ii) *For any* $\varepsilon_0 > 0$ *and any* $l \in \mathbb{N}$, *there exists* $C_l > 0$ *such that for all* $p \geqslant 1$ *and all* $(x, x') \in X \times X$ *with* $d(x, x') > \varepsilon_0$,

$$|T_p(x, x')| \leqslant C_l p^{-l}. \tag{4.37}$$

(iii) *There exists a family of polynomials* $\{\mathcal{Q}_{r,x_0} \in \operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}[Z, Z']\}_{x_0\in X}$ *such that: (a) each* $\mathcal{Q}_{r,x_0}$ *has the same parity as* $r$,

*(b) the family is smooth in* $x_0 \in X$ *and*

*(c) there exists* $0 < \varepsilon' < \varepsilon$ *such that for any* $x_0 \in X$ *and* $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$, *in the sense of* (4.35) *and* (4.36), *we have*

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} P_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}). \tag{4.38}$$

*Then* $\{T_p\}$ *is a Toeplitz operator.*

By the asymptotic expansion of $P_p$ as $p \to \infty$ (Theorems 0.1, 0.2 for $G = \{1\}$), for any $\mathbf{f} \in \mathscr{C}^\infty(X, \operatorname{End}(E))$, the Toeplitz operator $T_{\mathbf{f},p}$ verifies the conditions in Theorem 4.4.

Moreover, from the proof of Theorem 4.4, in fact

$$\mathcal{Q}_{0,\, x_0}(Z, Z') = \mathcal{Q}_{0,\, x_0}(0, 0), \quad \text{for} \quad x_0 \in X, \tag{4.39}$$

and we set

$$g_0(x_0) = \mathcal{Q}_{0,\, x_0}(0, 0)|_{\mathbb{C}\otimes E} \in \operatorname{End}(E_{x_0}) , \tag{4.40}$$

then

$$p^{-n}(T_p - T_{g_0,\, p})_{x_0}(Z, Z') \cong \mathcal{O}(p^{-1}) , \tag{4.41}$$

which implies

$$T_p = P_p\, g_0\, P_p + \mathcal{O}(p^{-1}). \tag{4.42}$$

And by recurrence as in (4.40), we find $g_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ such that (4.29) holds.

The Poisson bracket $\{\, , \}$ on $(X, 2\pi\omega)$ is defined by: for $g_1, g_2 \in \mathscr{C}^\infty(X)$, if $\xi_{g_2}$ is the Hamiltonian vector field generated by $g_2$ which is defined by $2\pi i_{\xi_{g_2}}\omega = dg_2$, then

$$\{g_1, g_2\} = -\xi_{g_2}(dg_1). \tag{4.43}$$

As a corollary of Theorem 4.4, we get the following result [**30**, Theorem 1.1] (cf. also [**31**, Theorems 7.4.1 and 8.1.10]),

***Theorem 4.5***. — *Let $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Then the product of the Toeplitz operators $T_{f,\,p}$ and $T_{g,\,p}$ is a Toeplitz operator, more precisely, it admits an asymptotic expansion in the sense of* (4.31):

$$T_{f,\,p}\,T_{g,\,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{4.44}$$

*where $C_r$ are bidifferential operators such that $C_r(f,g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ and $C_0(f,g) = fg$.*

*If $f, g \in \mathscr{C}^\infty(X)$, we have*

$$C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}\,\mathrm{Id}_E, \tag{4.45}$$

*and therefore*

$$[T_{f,\,p}\,, T_{g,\,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}). \tag{4.46}$$

In conclusion, the set of Toeplitz operators forms an algebra. In particular, when $(X, J, \omega)$ is a compact Kähler manifold and $E = \mathbb{C}$, $g^{TX} = \omega(\cdot, J\cdot)$, Theorem 4.5 recovers the result in [**9**] (cf. also [**39, 23**], [**20**]) where the theory of Toeplitz structures by Boutet de Monvel and Guillemin [**11**] is used. Some related results were also announced in [**10**].

***Lemma 4.6***. — *Let*

$$T_p = \sum_{l=0}^{\infty} P_p g_l p^{-l} P_p + \mathscr{O}(p^{-\infty}) : \mathrm{Ker}\, D_p \to \mathrm{Ker}\, D_p$$

*be a Toeplitz operator with principal symbol $g_0 \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Then*

*i) If $g_0$ is invertible, then $T_p^{-1}$ is a Toeplitz operator with principal symbol $g_0^{-1}$.*

*ii) If $g_0 = g\,\mathrm{Id}_E$ with $g \in \mathscr{C}^\infty(X)$, $g > 0$, and $T_p$ is self-adjoint, then for any $q \in \mathbb{N}^*$, $T_p^{1/q}$ is a self-adjoint Toeplitz operator with principal symbol $g^{1/q}\,\mathrm{Id}_E$.*

*Proof*. — We only prove ii), the proof of i) is similar and simpler.

As $g > 0$, there exist $C_0, C_1 > 0$ such that $C_0 < g < C_1$. Thus for any $s \in \mathrm{Ker}\, D_p$,

$$\langle T_p s, s\rangle = \langle g_0 s, s\rangle + \mathscr{O}\left(\frac{1}{p}\right) \|s\|^2_{L^2} \geq \left(C_0 + \mathscr{O}\left(\frac{1}{p}\right)\right) \|s\|^2_{L^2}. \tag{4.47}$$

Thus for $p$ large enough, $T_p^{1/q} : \mathrm{Ker}\, D_p \to \mathrm{Ker}\, D_p$ is well defined. (In the case i), we get $T_p^{-1} : \mathrm{Ker}\, D_p \to \mathrm{Ker}\, D_p$ is well defined for $p$ large enough.)

Let $\delta_1$ be a smooth bounded closed counterclockwise oriented contour on $\{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0\}$ such that $[\frac{1}{2}C_0, 2C_1]$ is in the interior domain got by $\delta_1$.

As in the proof of Theorem 4.4, by recurrence, we will find $f_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that

$$p^{-n}(T_p - (T_{k,p})^q) = \mathcal{O}(p^{-k-1}) \quad \text{with} \quad T_{k,p} = \sum_{l=0}^{k} P_p f_l p^{-l} P_p. \tag{4.48}$$

Then for $p$ large enough,

$$T_p^{1/q} - T_{k,p} = \frac{1}{2\pi i}\int_{\lambda\in\delta_1} \lambda^{1/q}\Big[(\lambda - T_p)^{-1} - (\lambda - (T_{k,p})^q)^{-1}\Big] d\lambda \tag{4.49}$$
$$= \frac{1}{2\pi i}\int_{\lambda\in\delta_1} \lambda^{1/q}(\lambda - T_p)^{-1}(T_p - (T_{k,p})^q)(\lambda - (T_{k,p})^q)^{-1} d\lambda.$$

If (4.48) holds, then by (4.49) we know that in the sense of the operator norm,

$$T_p^{1/q} - T_{k,p} = \mathcal{O}(p^{-k-1}). \tag{4.50}$$

To complete the proof of Lemma 4.6, it remains to establish (4.48).

As explained after Theorem 4.4, there exist $Q_{0,r} \in \operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$ such that in the sense of (4.35),

$$p^{-n}T_p(Z,Z') \cong \sum_{r=0}^{\infty}(Q_{0,r}\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}(p^{-\infty}). \tag{4.51}$$

We will prove by recurrence that there exist $f_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ self-adjoint such that for any $k \in \mathbb{N}$,

$$\Big|p^{-n+n_0}(T_p - (T_{k,p})^q)(\sqrt{p}Z,\sqrt{p}Z')\Big| \tag{4.52}$$
$$\leqslant p^{-(2k+1)/2}(1+\sqrt{p}|Z| + \sqrt{p}|Z'|)^M \exp(-\sqrt{C''\nu_0}\sqrt{p}|Z-Z'|) + \mathscr{O}(p^{-\infty}).$$

Set $f_0 = g^{1/q}\operatorname{Id}_E$. Then

$$p^{-n}(T_p - (T_{0,p})^q)(Z,Z') \cong \sum_{r=0}^{\infty}((Q_{0,r} - \widetilde{Q}^0_{0,r})\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2}. \tag{4.53}$$

Now as $Q_{0,0} = \widetilde{Q}^0_{0,0} = g\operatorname{Id}_E$, by (4.41), we know

$$Q_{0,1} - \widetilde{Q}^0_{0,1} = 0. \tag{4.54}$$

Thus (4.48) is verified for $k = 0$.

Assume that for $k \leqslant m$, we have found $f_l$ such that (4.48) holds. If we denote the expansion of $(T_{m,p})^q$ in the sense of (4.35),

$$p^{-n}(T_{m,p})^q(Z,Z') \cong \sum_{r=0}^{\infty}(\widetilde{Q}^m_{0,r}\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-\frac{k+1}{2}}). \tag{4.55}$$

By (4.48),

$$p^{-n}(T_p - (T_{m,p})^q)(Z,Z') \cong \sum_{r=2m+2}^{\infty}((Q_{0,r} - \widetilde{Q}^m_{0,r})\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2}. \tag{4.56}$$

By (4.39), (4.40), we set

$$f_{m+1}(x_0) = \frac{1}{q}g^{-\frac{q-1}{q}}(Q_{0,2m+2} - \widetilde{Q}^m_{0,2m+2})(0,0). \tag{4.57}$$

Then by (4.56) and (4.57),

$$p^{-n}(T_p-(T_{m+1,p})^q)(Z,Z')\cong\sum_{r=2m+3}^{\infty}((Q_{0,r}-\widetilde{Q}^{m+1}_{0,r})\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2}. \tag{4.58}$$

By (4.40), (4.41) and (4.58), we know

$$(Q_{0,2m+3}-\widetilde{Q}^{m}_{0,2m+3})(0,0)=0. \tag{4.59}$$

Thus (4.48) holds for $k=m+1$.

By the above argument, we have established (4.48), thus Lemma 4.6. □

Assume now that $(X,\omega)$ is a compact symplectic orbifold and $L,E$ are proper orbifold vector bundles verifying the conditions of the beginning of this section. Otherwise, as explained in [31, Remark 5.4.5], we are working on the proper orbifold sub-bundle $E^{\rm pr}$ of $E$.

We can still define the spin$^c$ Dirac operator $D_p:\Omega^{0,\bullet}(X,L^p\otimes E)\longrightarrow\Omega^{0,\bullet}(X,L^p\otimes E)$. The orthogonal projection $P_p:L^2(X,E_p)\longrightarrow\operatorname{Ker}D_p$ with $E_p:=\Lambda(T^{*(0,1)}X)\otimes L^p\otimes E$ is called the Bergman projection. A Toeplitz operator is a family of linear operator $T_p:\operatorname{Ker}D_p\to\operatorname{Ker}D_p$ verifying (4.29).

We need to introduce the correct analogue of (4.35) in the orbifold case, in order to take into account the group action associated to an orbifold chart. Let $\{\Xi_p\}_{p\in\mathbb{N}}$ be a sequence of linear operators $\Xi_p:L^2(X,E_p)\longrightarrow L^2(X,E_p)$ with smooth kernel $\Xi_p(x,y)$ with respect to $dv_X(y)$.

Let $k\in\mathbb{N}$, we write

$$p^{-n}\,\Xi_{p,\,x_0}(Z,Z')\cong\sum_{r=0}^{k}(Q_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}+\mathcal{O}(p^{-\frac{k+1}{2}})\,, \tag{4.60}$$

if for every open set $U\in\mathcal{U}$ and every orbifold chart $(H_U,\widetilde{U})\xrightarrow{\tau_U}U$, there exists a sequence of kernels $\{\widetilde{\Xi}_{p,U}(\widetilde{x},\widetilde{x}')\}_{p\in\mathbb{N}}$ and a family

$$\{Q_{r,\,x_0}\}_{0\leqslant r\leqslant k,\,x_0\in X}\in\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}[\widetilde{Z},\widetilde{Z}']$$

smooth with respect to the parameter $x_0\in X$ such that for every fixed $\varepsilon''>0$ and every $\widetilde{x},\widetilde{x}'\in\widetilde{U}$ the following hold

$$\begin{aligned}&(g,1)\widetilde{\Xi}_{p,U}(g^{-1}\widetilde{x},\widetilde{x}')=(1,g^{-1})\widetilde{\Xi}_{p,U}(\widetilde{x},g\widetilde{x}')\quad\text{for any}\quad g\in H_U\\&\widetilde{\Xi}_{p,U}(\widetilde{x},\widetilde{x}')=\mathscr{O}(p^{-\infty})\qquad\text{for } d(x,x')>\varepsilon'',\\&\Xi_p(x,x')=\sum_{g\in H_U}(g,1)\widetilde{\Xi}_{p,U}(g^{-1}\widetilde{x},\widetilde{x}')+\mathscr{O}(p^{-\infty}),\end{aligned} \tag{4.61}$$

and moreover, for every relatively compact open subset $\widetilde{V}\subset\widetilde{U}$, the relation

$$p^{-n}\,\widetilde{\Xi}_{p,U,\widetilde{x}_0}(\widetilde{Z},\widetilde{Z}')\cong\sum_{r=0}^{k}(Q_{r,\,\widetilde{x}_0}\mathscr{P}_{\widetilde{x}_0})(\sqrt{p}\widetilde{Z},\sqrt{p}\widetilde{Z}')p^{-\frac{r}{2}}+\mathcal{O}(p^{-\frac{k+1}{2}})\qquad\text{for}\ \ \widetilde{x}_0\in\widetilde{V}, \tag{4.62}$$

holds in the sense of (4.35).

Note that although the notation (4.60) and (4.35) are formally similar, they have different meaning.

Then in [**30**, §6], we find the following analogue of Theorem 4.4.

***Theorem 4.7***. — *Let* $\{T_p : L^2(X, E_p) \longrightarrow L^2(X, E_p)\}$ *be a family of bounded linear operators which satisfies i), ii) of Theorem 4.4 and* (4.60). *Then* $\{T_p\}$ *is a Toeplitz operator.*

From Theorem 4.7, we extend also Theorem 4.5 to the orbifold case, for more details, see [**30**, §6].

## 4.5. Toeplitz operators on $X_G$

In this Section, we suppose that $(X, \omega)$ is a Kähler manifold, $\mathbf{J} = J$, and $L, E$ are holomorphic vector bundles with holomorphic Hermitian connections $\nabla^L, \nabla^E$. Let $G$ be a compact connected Lie group acting holomorphically on $X, L, E$ which preserves $h^L$ and $h^E$.

We suppose that $G$ acts freely on $P = \mu^{-1}(0)$. Then $(X_G, \omega_G)$ is Kähler and $L_G, E_G$ are holomorphic on $X_G$.

In this case, there exists a natural isomorphism from $(\operatorname{Ker} D_p)^G$ onto $\operatorname{Ker} D_{G,p}$.

At the end of this Section, we will explain the corresponding result in the symplectic case, especially, for $p \gg 1$, we construct a natural isomorphism from $(\operatorname{Ker} D_p)^G$ onto $\operatorname{Ker} D_{G,p}$.

In the current situation, the spin$^c$ Dirac operator $D_p$ was given by (0.21) and $D_p^2$ preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p \otimes E)$. Similar properties hold for $D_{G,p}$.

As in Section 2.3, let $P_{G,p}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$ onto $\operatorname{Ker} D_{G,p}$, and let $P_{G,p}(x, x')$ be the corresponding smooth kernel.

By the Kodaira vanishing theorem, for $p$ large enough,

$$(\operatorname{Ker} D_p)^G = H^0(X, L^p \otimes E)^G, \quad \operatorname{Ker} D_{G,p} = H^0(X_G, L_G^p \otimes E_G). \tag{4.63}$$

As $D_p^2, D_{G,p}^2$ preserve the $\mathbb{Z}$-gradings of $\Omega^{0,\bullet}(X, L^p \otimes E)$, $\Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$ respectively, we only need to take care of their restrictions on $\mathscr{C}^\infty(X, L^p \otimes E)$ and $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$. In this way,

$$\begin{aligned} &P_p^G(x, x') \in \mathscr{C}^\infty(X \times X, \mathrm{pr}_1^*(L^p \otimes E) \otimes \mathrm{pr}_2^*(L^p \otimes E)^*), \\ &P_{G,p}(x_0, x_0') \in \mathscr{C}^\infty(X_G \times X_G, \mathrm{pr}_1^*(L_G^p \otimes E_G) \otimes \mathrm{pr}_2^*(L_G^p \otimes E_G)^*). \end{aligned} \tag{4.64}$$

Recall that the morphism $\sigma_p : H^0(X, L^p \otimes E)^G \to H^0(X_G, L_G^p \otimes E_G)$ was defined in (0.27). Set

$$\sigma_p^G = \sigma_p \circ P_p^G : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X_G, L_G^p \otimes E_G). \tag{4.65}$$

Let $\sigma_p^{G*}$ be the adjoint of $\sigma_p^G$ with respect to the natural inner products (cf. (1.19)) on $\mathscr{C}^\infty(X, L^p \otimes E)$, $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$. Set

$$\mathcal{P}_p^{X_G} := p^{-\frac{n_0}{2}} \sigma_p^G \circ \sigma_p^{G*}. \tag{4.66}$$

Let $\{s_{p,i}\}_{i=1}^{d_p}$ be an orthonormal basis of $H^0(X, L^p \otimes E)^G$. For $y_0 \in X_G,\ x,\ x' \in X$, one verifies

$$\begin{aligned} P_p^G(x,x') &= \sum_{i=1}^{d_p} s_{p,i}(x) \otimes s_{p,i}(x')^*, \\ \sigma_p^G(y_0,x) &= P_p^G(y_0,x), \quad \sigma_p^{G*}(x,y_0) = P_p^G(x,y_0), \end{aligned} \tag{4.67}$$

where by $P_p^G(y_0,x)$ (resp. $P_p^G(x,y_0)$) we mean $P_p^G(y,x)$ (resp. $P_p^G(x,y)$) for any $y \in \pi_G^{-1}(y_0)$, which is well-defined by the $G$-invariance of $P_p^G$.

From (0.27), we know that $\mathcal{P}_p^{X_G}$ commutes with the operator $P_{G,p}$ and

$$\mathcal{P}_p^{X_G} = P_{G,p} \mathcal{P}_p^{X_G} P_{G,p}. \tag{4.68}$$

Let $P_p^G|_P$ be the restriction of the smooth kernel $P_p^G(x,x')$ on $P \times P$. Then

$$P_p^G|_P(x,x') \in \mathscr{C}^\infty(P \times P, \mathrm{pr}_1^*(L^p \otimes E) \otimes \mathrm{pr}_2^*(L^p \otimes E)^*)$$

is $G \times G$-invariant. By composing with $\pi_G$,

$$(\pi_G \circ P_p^G|_P)(x_0,x_0') \in \mathscr{C}^\infty(X_G \times X_G, \mathrm{pr}_1^*(L_G^p \otimes E_G) \otimes \mathrm{pr}_2^*(L_G^p \otimes E_G)^*).$$

We denote by $\pi_G \circ P_p^G|_P$ the operator defined by the smooth kernel $(\pi_G \circ P_p^G|_P)(x_0,x_0')$ and the Riemannian volume form $dv_{X_G}(x_0')$. Then from (4.67), we verify that

$$\mathcal{P}_p^{X_G}(x_0,x_0') = p^{-\frac{n_0}{2}} P_p^G(x_0,x_0') = p^{-\frac{n_0}{2}} \pi_G \circ P_p^G|_P(x_0,x_0'). \tag{4.69}$$

Recall that $h$ is the fiberwise volume function defined by (0.10).

Let $dg$ be a Haar measure on $G$.

The main result of this Section is the following result.

***Theorem 4.8***. — *Let $f$ be a smooth section of* $\mathrm{End}(E)$ *on $X$. Let $f^G \in \mathscr{C}^\infty(X_G,$ $\mathrm{End}(E_G))$ be the $G$-invariant part of $f$ on $P$ defined by $f^G(x) = \int_G g \cdot f(g^{-1}x)dg$. Then $\mathcal{T}_{f,p} = p^{-\frac{n_0}{2}} \sigma_p^G f \sigma_p^{G*}$ is a Toeplitz operator with principal symbol $2^{\frac{n_0}{2}} \frac{f^G}{h^2}(x)$. In particular $\mathcal{P}_p^{X_G}$ is a Toeplitz operator with principal symbol $2^{\frac{n_0}{2}}/h^2(x)$.*

*Proof*. — We need to find a family of sections $g_l \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$ such that for any $m \geqslant 1$,

$$\mathcal{T}_{f,p} = \sum_{l=0}^m P_{G,p}\, g_l\, p^{-l}\, P_{G,p} + \mathscr{O}(p^{-m-1}). \tag{4.70}$$

By Theorem 0.1, (4.65), (4.67), we know for $\varepsilon > 0$, and any $l \in \mathbb{N}$, there exists $C_l > 0$ such that for all $p \geqslant 1$ and all $(x,x') \in X_G \times X_G$ with $d(x,x') > \varepsilon_0$,

$$|\mathcal{T}_{f,p}(x,x')| \leqslant C_l p^{-l}. \tag{4.71}$$

We still need to verify the condition iii) of Theorem 4.4.

Let $U$ be a $G$-neighborhood of $P = \mu^{-1}(0)$ as in Theorem 0.2.

Let $\psi$ be a $G$-invariant function on $X$ such that $\psi = 1$ on an open neighborhood of $P$ and $\mathrm{supp}(\psi) \subset \{y \in X, d(y,P) < \varepsilon_0/2\} \cap U$.

Write

$$\sigma_p^G f \sigma_p^{G*} = \sigma_p^G \psi f \sigma_p^{G*} + \sigma_p^G (1-\psi) f \sigma_p^{G*}. \tag{4.72}$$

For $x_0, x_0' \in X_G$, let $x, x' \in P$ such that $\pi(x) = x_0$, $\pi(x') = x_0'$. By (4.67),

$$(\sigma_p^G((1-\psi)f)\sigma_p^{G*})(x_0,x_0') = \int_X P_p^G(x,y)((1-\psi)f)(y)P_p^G(y,x')dv_X(y). \tag{4.73}$$

From Theorem 0.1, (4.73) and $\mathrm{supp}((1-\psi)f) \cap P = \emptyset$, we know that for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for any $p \in \mathbb{N}, x_0, x_0' \in X_G$,

$$|(\sigma_p^G((1-\psi)f)\sigma_p^{G*})(x_0,x_0')|_{\mathscr{C}^m(X_G \times X_G)} \leqslant C_{l,m} p^{-l}. \tag{4.74}$$

We define $f_B \in \mathscr{C}^\infty(B, \mathrm{End}(E_B))$ by

$$f_B(x_0) = \int_G g \cdot (\psi f)(g^{-1}x) dg \tag{4.75}$$

for $x_0 \in B, x \in U$ such that $\pi(x) = x_0$. Clearly, if $x_0 \in P$, as $\psi|_P = 1$, one gets

$$f_B(x_0) = f^G(x_0). \tag{4.76}$$

From (4.75), for $x_0,\ x_0' \in B,\ x,\ x' \in U$ such that $\pi(x) = x_0,\ \pi(x') = x_0'$, one gets

$$\begin{aligned}\sigma_p^G \psi f \sigma_p^{G*}(x_0,x_0') &= \int_U P_p^G(x,y)(\psi f)(y) P_p^G(y,x') dv_X(y) \\ &= \int_B P_p^G(x_0,y_0) f_B(y_0) P_p^G(y_0,x_0') h^2(y_0) dv_B(y_0).\end{aligned} \tag{4.77}$$

For $x_0 \in X_G$, we will work on the normal coordinates of $X_G$ with center $x_0$ as in Theorem 0.2.

Recall that $P_{\mathscr{L}}(Z^0, Z'^0)$ was defined by (3.19) with $a_i = a_i^\perp = 2\pi$ therein.

By (4.72), (4.74) and (4.77), for $|Z^0|, |Z'^0| \leqslant \varepsilon_0/2$,

$$\mathcal{T}_{f,p}(Z^0,Z'^0) - p^{-n_0/2} \int_{\substack{|W| \leqslant \varepsilon_0,\\ W \in T_{x_0}B}} P_p^G(Z^0,W)(f_B h^2)(W) P_p^G(W,Z'^0) dv_B(W) = \mathscr{O}(p^{-\infty}). \tag{4.78}$$

By Theorem 0.2, (4.78) and the Taylor expansion of $f_B$, there exist $Q_{0,r} \in \mathrm{End}(E_{G,x_0})$ polynomials on $Z^0, Z'^0$ with same parity on $r$ such that the following formula, obtained through compositions, holds,

(4.79)
$$\left|p^{-n+n_0}\mathcal{T}_{f,p}(Z^0,Z'^0)\kappa^{\frac12}(x_0,Z^0)\kappa^{\frac12}(x_0,Z'^0)-\sum_{r=0}^{k}(Q_{0,r}P_{\mathscr{L}})(\sqrt{p}Z^0,\sqrt{p}Z'^0)p^{-\frac{r}{2}}\right|_{\mathscr{C}^{m'}(X_G)}$$
$$\leqslant Cp^{-(k+1)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^M\exp(-\sqrt{C''\nu}\sqrt{p}|Z^0-Z'^0|)+\mathscr{O}(p^{-\infty}).$$

Moreover, by (0.13), (4.75) and (4.78),

$$(4.80)\quad (Q_{0,0}P_{\mathscr{L}})(Z^0,Z'^0)=P_{\mathscr{L}}(Z^0,Z'^0)\frac{f^G}{h^2}(x_0)2^{n_0}\int_{\mathbb{R}^{n_0}}\exp(-2\pi|W^{\perp}|^2)dW^{\perp}$$
$$=\frac{f^G}{h^2}(x_0)2^{n_0/2}P_{\mathscr{L}}(Z^0,Z'^0).$$

By Theorem 4.4, (4.71) and (4.79), there exist $g_l\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$ such that (4.70) holds, and by (4.40) and (4.42),

$$(4.81)\qquad \mathcal{T}_{f,p}=2^{n_0/2}P_{G,p}\frac{f^G}{h^2}P_{G,p}+\mathcal{O}(p^{-1}).$$

The proof of Theorem 4.8 is complete. □

***Corollary 4.9***. — *For $f_1,f_2\in\mathscr{C}^\infty(X)$, we have*

$$(4.82)\qquad [\mathcal{T}_{f_1,p},\mathcal{T}_{f_2,p}]=\frac{2^{n_0}\sqrt{-1}}{p}P_{G,p}\left\{\frac{f_1^G}{h^2},\frac{f_2^G}{h^2}\right\}P_{G,p}+\mathscr{O}(p^{-2}).$$

*Here $\{\ ,\ \}$ is the Poisson bracket on $(X_G,2\pi\omega_G)$.*

*Proof*. — By Theorems 4.5, 4.8, we get immediately (4.82). □

Since the isomorphism $\sigma_p:H^0(X,L^p\otimes E)^G\to H^0(X_G,L^p_G\otimes E_G)$ is not an isometry, we define the associated unitary operator,

$$(4.83)\qquad \Sigma_p=\sigma_p^{G*}(\sigma_p^G\circ\sigma_p^{G*})^{-1/2}:H^0(X_G,L^p_G\otimes E_G)\to H^0(X,L^p\otimes E)^G.$$

***Theorem 4.10***. — *Let $f$ be a $\mathscr{C}^\infty$ section of* $\mathrm{End}(E)$ *on $X$. Then*

$$(4.84)\qquad T^G_{f,p}=\Sigma_p^*f\Sigma_p:H^0(X_G,L^p_G\otimes E_G)\to H^0(X_G,L^p_G\otimes E_G)$$

*is a Toeplitz operator on $X_G$. Its principal symbol is $f^G\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$.*

*Proof*. — By (4.68) and (4.83),

$$(4.85)\qquad T^G_{f,p}=(\mathcal{P}_p^{X_G})^{-\frac12}\mathcal{T}_{f,p}(\mathcal{P}_p^{X_G})^{-\frac12}.$$

By Theorem 4.8, (4.66), $\mathcal{P}_p^{X_G}=p^{-\frac{n_0}{2}}\sigma_p^G\circ\sigma_p^{G*}$, $\mathcal{T}_{f,p}$ are Toeplitz operators on $X_G$ with principal symbols $2^{n_0/2}/h^2(x)$, $2^{n_0/2}\frac{f^G}{h^2}(x)$ respectively.

By Lemma 4.6, we know that $(\mathcal{P}_p^{X_G})^{-\frac12}$ is a Toeplitz operator on $X_G$ with principal symbol $2^{-n_0/4}h(x)$.

By Theorem 4.5, we then know that $T^G_{f,p}$ is a Toeplitz operator and its principal symbol is $f^G(x)$. □

***Remark 4.11.*** — i) When $E = \mathbb{C}$, and $f = 1$, from Theorem 4.8, $\mathcal{P}_p^{X_G}$ is an elliptic (i.e. its principal symbol is invertible) Toeplitz operator. This is the analytic core result claimed in [**37**, §8].

ii) When $E = \mathbb{C}$ and $G$ is the torus $\mathbb{T}^{n_0}$, Theorem 4.10 is one of the main results of Charles [**15**, Theorem 1.2], and in [**15**, §5.6], he knew also that $\mathcal{P}_p^{X_G}$ is an elliptic Toeplitz operator. Moreover, he established the corresponding version when $X_G$ is an orbifold.

If $X$ is only symplectic and $\mathbf{J} = J$, then as the argument in [**44**, §3e)], $J$ induces an almost complex structure $J_G$ on $(TX)_B$, and $J_G$ preserves $N_{G,J} = N_G \oplus J_G N_G$ and $TX_G$. Thus one can construct canonically the Hermitian vector bundles $N^{(1,0)}_{G,J}$ etc, which further give the canonical identification of Hermitian vector bundles

$$\Lambda(T^{*(0,1)}X)_B|_{X_G} = \Lambda(N^{*(0,1)}_{G,J})\widehat{\otimes}\Lambda(T^{*(0,1)}X_G). \tag{4.86}$$

Let $q$ be the canonical orthogonal projection

$$q : \Lambda(N^{*(0,1)}_{G,J})\widehat{\otimes}\Lambda(T^{*(0,1)}X_G) \otimes L^p_G \otimes E_G \to \Lambda(T^{*(0,1)}X_G) \otimes L^p_G \otimes E_G \tag{4.87}$$

which acts as identity on $\Lambda(T^{*(0,1)}X_G) \otimes L^p_G \otimes E_G$ and maps each

$$\Lambda^i(N^{*(0,1)}_{G,J})\widehat{\otimes}\Lambda(T^{*(0,1)}X_G) \otimes L^p_G \otimes E_G, \quad i \geqslant 1, \text{ to zero.}$$

We define

$$\sigma_p := P_{G,p} q \pi_G i^* P^G_p : (\operatorname{Ker} D_p)^G \to \operatorname{Ker} D_{G,p}. \tag{4.88}$$

Certainly in the Kähler case, $\sigma_p$ coincides with (0.27).

By using Theorems 0.1, 0.2 as in the proof of Theorem 4.8, we get

***Theorem 4.12.*** — *Let $f$ be a smooth section of* $\operatorname{End}(E)$ *on $X$, then $\mathcal{T}_{f,p} = p^{-n_0/2}\sigma_p f \sigma_p^* : \operatorname{Ker} D_{G,p} \to \operatorname{Ker} D_{G,p}$ is a Toeplitz operator with principal symbol $2^{n_0/2}\frac{f^G}{h^2}(x) \in \operatorname{End}(E_G)$.*

***Corollary 4.13.*** — *For $p$ large enough, $\sigma_p$ in* (4.88) *is an isomorphism. Thus $\sigma_p$ defines a natural identification for 'quantization commutes with reduction' in the (asymptotic) symplectic case.*

*Proof.* — From Theorem 4.12 for $f = 1$, we get

$$p^{-n_0/2}\sigma_p\sigma_p^* = 2^{n_0/2} P_{G,p} h^{-2} P_{G,p} + \mathcal{O}(\frac{1}{p}). \tag{4.89}$$

Thus for $p$ large enough, $\sigma_p\sigma_p^*$ is an isomorphism. Thus $\sigma_p$ is surjective.

In view of (0.6), $\sigma_p$ in (4.88) is an isomorphism. □

***Remark 4.14***. — If we replace the condition $\mathbf{J} = J$ by (3.2), then the canonical map $\sigma_p$ in (4.88) is still well defined. From the argument here, we still know that $\sigma_p$ is an isomorphism for $p$ large enough.

Now, we relax further our condition. As in Section 4.1, we only suppose that $0 \in \mathfrak{g}^*$ is a regular value of $\mu$, then the symplectic reduction $X_G$ is a compact symplectic orbifold. Then (4.86)-(4.88) are still well defined.

As explained in Theorem 4.1, Theorem 0.1 still holds. From Theorem 4.7, (4.1) and the proof of Theorem 4.8, we get

***Theorem 4.15***. — *If* $f \in \mathscr{C}^\infty(X, \operatorname{End}(E))$, *then* $T_{f,p} = p^{-n_0/2}\sigma_p f \sigma_p^* : \operatorname{Ker} D_{G,p} \to \operatorname{Ker} D_{G,p}$ *is a Toeplitz operator with principal symbol* $2^{n_0/2}\frac{f^G}{\hat{h}^2}(x) \in \operatorname{End}(E_G)$.

*For* $p$ *large enough,* $\sigma_p$ *in* (4.88) *is an isomorphism.*

## 4.6. Generalization to non-compact manifolds

In this Section, let $(X, \omega)$ be a symplectic manifold, and $(L, \nabla^L)$ (resp. $(E, \nabla^E)$) be Hermitian line (vector) bundle, with Hermitian connections, on $X$, and the compact connected Lie group $G$ acts on $X$ as in Introduction, especially, $\omega = \frac{\sqrt{-1}}{2\pi} R^L$. But we only suppose that $(X, g^{TX})$ is a complete Riemannian manifold.

If $G = 1$, these kind results were studied in [**28**, §3.5].

By the argument in Section 2.3, if the square of the spin$^c$ Dirac operator $D_p^2$ has a spectral gap as in (2.15), then we can localize our problem and get a version of Theorems 0.1, 0.2 from Section 2.6. In particular, if the geometric data on $X$ verify the bounded geometry, then $D_p^2$ verify the spectral gap (2.15).

We explain in more details now.

We suppose

i) The tensors $R^E, r^X, \operatorname{Tr}[R^{T^{(1,0)}X}]$ are uniformly bounded with respect on $(X, g^{TX})$.

ii) There exists $c > 0$ such that

$$\sqrt{-1}R^L(\cdot, J\cdot) \geqslant c g^{TX}(\cdot, \cdot). \tag{4.90}$$

***Remark 4.16***. — For the operator $D_p = \sqrt{2}(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*})$ in the holomorphic case, the above condition i) can be replaced by [**28**, (3.39)]:

$$\sqrt{-1}(R^{\det} + R^E) \geqslant -C\Theta \operatorname{Id}_E, \quad |\partial\Theta|_{g^{TX}} < C. \tag{4.91}$$

Here $R^{\det}$ is the curvature of the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$, $\Theta = g^{TX}(J\cdot, \cdot)$. For two $(1,1)$-forms $\Omega$ and $\Omega'$ we write $\Omega \geqslant \Omega'$ if $(\Omega - \Omega')(\cdot, J\cdot) \geqslant 0$.

Then by the argument in [**27**, p. 656] (cf. [**28**, §3.5]), we know that Theorem 2.2 still holds. Thus Theorem 2.5 still holds.

Let $P_p^G$ be the orthogonal projection from $L^2(X, E_p)$ onto $(\operatorname{Ker} D_p)^G$, and $P_p^G(x, x')$ $(x, x' \in X)$ be its kernel as in Def. 2.3.

Note that $\operatorname{Ker} D_p$ and $(\operatorname{Ker} D_p)^G$ need not be finite dimensional.

By the proof of Prop. 2.6, we know that for any compact subset $K \subset X$, $l, m \in \mathbb{N}$, there exists $C_{l,m}(K) > 0$ such that for $p \geqslant C_L/\nu$,

$$|\widetilde{F}(\mathcal{L}_p)(x, x') - P_p^G(x, x')|_{\mathscr{C}^m(K \times K)} \leqslant C_{l,m}(K) p^{-l}. \tag{4.92}$$

By the proof of Theorem 0.1, we get

***Theorem 4.17.*** — *For any compact subset $K \subset X$, $0 < \varepsilon_0 \leqslant \delta_0$, $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ (depending on $K$, $\varepsilon$) such that for $p \geqslant 1$, $x, x' \in K, d^X(Gx, x') \geqslant \varepsilon_0$ or $x, x' \in (X \setminus X_{2\varepsilon_0}) \cap K$,*

$$|P_p^G(x, x')|_{\mathscr{C}^m} \leqslant C_{l,m} p^{-l}. \tag{4.93}$$

From Section 2.6, we get Theorem 0.2, but now the norm $\mathscr{C}^{m'}(X_G)$ in (0.14) should be replaced by $\mathscr{C}^{m'}(K)$ for the compact subset $K \subset X_G$.

One interesting case of the above discussion is when $P = \mu^{-1}(0)$ is compact, by the same argument as in Theorems 4.8, 4.12, we can prove a version of Section 4.5. Especially, the map $\sigma_p : (\operatorname{Ker} D_p)^G \to \operatorname{Ker} D_{G,p}$ in (0.27), (4.88) is still well defined. Thus we get the following extension of Theorems 4.8, 4.12, 4.15:

***Theorem 4.18.*** — *Under the assumption i), ii), if $P = \mu^{-1}(0)$ is compact and $0 \in \mathfrak{g}^*$ is a regular value of $\mu$, then for $f \in \mathscr{C}^\infty_{Const}(X, \operatorname{End}(E))$, the algebra of smooth sections of $X$ which are a constant map (i.e. $C\operatorname{Id}_E$) outside a compact set, then $\mathcal{T}_{f,p} = p^{-n_0/2}\sigma_p f \sigma_p^* : \operatorname{Ker} D_{G,p} \to \operatorname{Ker} D_{G,p}$ is a Toeplitz operator with principal symbol $2^{n_0/2}\frac{f^G}{\hat{h}^2}(x) \in \operatorname{End}(E_G)$.*

In fact, when $X = \mathbb{C}^n, G = \mathbb{T}^{n_0}$, the torus, $L$ is the trivial line bundle with the metric $|1|_{h^L}(Z) = e^{-|z|^2}$, the Toeplitz operator type properties was studied by Charles [**15**].

Another interesting case is a version of Theorem 0.2 for covering manifolds.

Let $\widetilde{X}$ be a para-compact smooth manifold, such that there is a discrete group $\Gamma$ acting freely on $\widetilde{X}$ with a compact quotient $X = \widetilde{X}/\Gamma$.

Let $\pi_\Gamma : \widetilde{X} \longrightarrow X$ be the projection. Assume that all the above geometric data on $X$ can be lift on $\widetilde{X}$. We denote by $\widetilde{\mathbf{J}}$, $g^{T\widetilde{X}}$, $\widetilde{\omega}$, $\widetilde{J}$, $\widetilde{L}$, $\widetilde{E}$ the pull-back of the corresponding objects in Introduction by the projection $\pi_\Gamma : \widetilde{X} \to X$, moreover, we assume that the $G$-action and the $\Gamma$-action on them commute.

By the above arguments (cf. [**27**, Theorems 4.4 and 4.6]), there exists a spectral gap for the square of the spin$^c$ Dirac operator $\widetilde{D}_p$ on $\widetilde{X}$.

By the finite propagation speed of solutions of hyperbolic equations (2.66), we get an extension of [**28**, Theorem 3.14] where $G = 1$.

***Theorem 4.19***. — *We fix* $0 < \varepsilon_0 < \inf_{x\in X}\{$*injectivity radius of* $x\}$. *For any* $k, l \in \mathbb{N}$, *there exists* $C_{k,l} > 0$ *such that for* $x, x' \in \widetilde{X}$, $p \in \mathbb{N}$,

$$\begin{aligned} &\Big|\widetilde{P}^G_p(x,x') - P^G_p(\pi_\Gamma(x), \pi_\Gamma(x'))\Big|_{\mathscr{C}^l} \leqslant C_{k,l}\, p^{-k-1}\,, \quad \text{if } d^{\widetilde{X}}(x,x') < \varepsilon_0, \\ &\Big|\widetilde{P}^G_p(x,x')\Big|_{\mathscr{C}^l} \leqslant C_{k,l}\, p^{-k-1}\,, \quad \text{if } d^{\widetilde{X}}(x,x') \geqslant \varepsilon_0. \end{aligned} \tag{4.94}$$

*Especially,* $\widetilde{P}^G_p(x,x)$ *has the same asymptotic expansion as* $P^G_p(\pi_\Gamma(x), \pi_\Gamma(x))$ *in Corollary 0.4 on* $\widetilde{X}$.

## 4.7. Relation on the Bergman kernel on $X_G$

From (2.62), if the operator $\Phi\mathcal{L}_p\Phi^{-1}$ has the form $D^2_{G,p} + \Delta_N + 4\pi|\mu|^2p^2 - 2\pi n_0 p$ under the splitting (4.86), then we will find the full asymptotic expansion of the Bergman kernel on $X_G$ from $P^G_p(x,x')$.

In this Section, we suppose that $X$ is compact and $G$ is a torus $\mathbb{T}^{n_0} = \mathbb{R}^{n_0}/\mathbb{Z}^{n_0}$.

Let $\theta : TP \to \mathfrak{g}$ be a connection form for the $G$-principal bundle $\pi : P = \mu^{-1}(0) \to X_G$ with curvature $\Theta$. Let $T^HP = \operatorname{Ker}\theta \subset TP$.

Set $M = P \times \mathfrak{g}^*$, $\mathbf{q} : M \to \mathfrak{g}^*$ be the natural projection and

$$\omega^M = \pi^*\omega_G + d\langle \mathbf{q}, \theta\rangle = \pi^*\omega_G + \langle \mathbf{q}, \Theta\rangle + \langle d\mathbf{q}, \theta\rangle. \tag{4.95}$$

By the normal crossing formula [**22**, Prop. 40.1], we know there exists a symplectic diffeomorphism such that on a neighborhood $U$ of $P$,

$$\Psi_{sym} : (X,\omega) \simeq (M, \omega^M), \tag{4.96}$$

and under this identification, the moment map $\mu$ (cf. (2.16)) is defined by $-\mathbf{q}$.

From now on, we use this neighborhood of $P$ and we will choose metrics and connections.

Let $g^{\mathfrak{g}}$ be the metric on $\mathfrak{g}$ induced by the canonical flat metric on $\mathbb{R}^{n_0}$, and $\{K_i\}$ be the canonical unitary basis of $\mathbb{R}^{n_0}$.

Now we choose $J$ an almost -complex structure on $TX$ compatible with $\omega$ such that on $T^HP$ on $U$, $J$ is induced by an almost-complex structure on $TX_G$ which is compatible with $\omega_G$, and on $\mathfrak{g}\oplus\mathfrak{g}^*$, for $K \in \mathfrak{g}$, $JK \in \mathfrak{g}^*$ is defined by $(JK, K') = \langle K, K'\rangle_{\mathfrak{g}}$ for $K' \in \mathfrak{g}$.

We also suppose $\Theta$ is $J$-invariant.

Let $g^{TX}$ be a $J$-invariant metric on $TX$ such that

$$g^{TX} = \pi^* g^{TX_G} \oplus g^{\mathfrak{g}} \oplus g^{\mathfrak{g}^*} \quad \text{on } U. \tag{4.97}$$

As $g^{\mathfrak{g}}$ is a constant metric on $TY = \mathfrak{g}$, $\nabla^{TY}$ is the trivial connection on $TY$. By (1.3), on $U$,

$$\nabla^{TP}_{U_1^H} = \nabla^{TX_G}_{U_1^H} + \nabla^{TY}_{U_1^H} + S(U_1^H). \tag{4.98}$$

Let $\nabla^{\Lambda(N^{*(0,1)}_{G,J})}$ be the trivial connection on the trivial bundle $\Lambda(N^{*(0,1)}_{G,J})$ (cf. (4.86)) on $U$, and $\nabla^{\mathrm{Cliff}_{X_G}}$ be the Clifford connection on $\Lambda(T^{*(0,1)}X_G)$.

By (1.7), (4.98), under the identification (4.86), on $U$, we have

$$\begin{aligned}\nabla^{\mathrm{Cliff}}_{e_i^H} &= \nabla^{\mathrm{Cliff}_{X_G}}_{e_i^H}\otimes \mathrm{Id} + \mathrm{Id}\otimes \nabla^{\Lambda(N^{*(0,1)}_{G,J})}_{e_i^H} + \frac{1}{2}\langle S(e_i^H)e_j^H, K_l\rangle c(e_j^H)c(K_l)\\ &= \nabla^{\mathrm{Cliff}_{X_G}}_{e_i^H}\otimes \mathrm{Id} + \mathrm{Id}\otimes \nabla^{\Lambda(N^{*(0,1)}_{G,J})}_{e_i^H} + \frac{1}{4}\langle \Theta(e_i,e_j), K_l\rangle c(e_j^H)c(K_l).\end{aligned} \tag{4.99}$$

However, the last term does not preserve $\Lambda(T^{*(0,1)}X_G)$ and $\Lambda(N^{*(0,1)}_{G,J})$.

From (2.62) and (4.99), in general, $\Phi\mathcal{L}_p\Phi^{-1}$ will not preserve $\Lambda(T^{*(0,1)}X_G)$ and $\Lambda(N^{*(0,1)}_{G,J})$ if $\Theta$ is not null.

Now, we suppose that $\Theta = 0$ on $X_G$.

In this situation, on $B = U/G \subset X_G \times \mathfrak{g}^*$, by (2.62), we have

$$\Phi\mathcal{L}_p\Phi^{-1} = D^2_{G,p} - \sum_l (\nabla^{\Lambda(N^{*(0,1)}_{G,J})}_{K_l})^2 + 4\pi^2|\mathbf{q}|^2p^2 - 2n_0\pi p. \tag{4.100}$$

By Theorem 0.2, Section 3.2 and (3.19), we know that the asymptotic expansion of the Bergman kernel has the following relation for $(x, Z^\perp) \in N_{G,x}$, $(x', Z'^\perp) \in N_{G,x'}$,

$$P^G_p((x,Z^\perp),(x',Z'^\perp)) = P_{G,p}(x,x')p^{n_0/2}P_{\mathscr{L}^\perp}(\sqrt{p}Z^\perp, \sqrt{p}Z'^\perp) + \mathcal{O}(p^{-\infty}). \tag{4.101}$$

# CHAPTER 5

## COMPUTING THE COEFFICIENT $\Phi_1$

In this Chapter, $(X, \omega, J)$ is a compact Kähler manifold, $g^{TX}$ is a $G$-invariant Riemannian metric on $TX$ which is compatible with $J$. $(E, h^E)$, $(L, h^L)$ are holomorphic Hermitian vector bundles on $X$, and $\nabla^E, \nabla^L$ are the holomorphic Hermitian connections on $(E, h^E)$, $(L, h^L)$. Moreover,

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega.$$

The action of $G$ is holomorphic and $G$ acts freely on $P = \mu^{-1}(0)$. Thus $(X_G, \omega_G, J_G)$ is a compact Kähler manifold.

In Sections 5.1-5.4, we suppose that in (0.2), $\mathbf{J} = J$ on a $G$-neighborhood $U$ of $P = \mu^{-1}(0)$.

The main purpose here is to compute the coefficient $\Phi_1$ in (0.20).

By (0.19) (cf. also Theorem 2.23),

$$\Phi_1(x_0) = \int_{Z \in N_{G,x_0}} P^{(2)}_{x_0}(Z, Z) dv_{N_G}(Z). \tag{5.1}$$

We will first compute explicitly the terms $\mathcal{O}_1$ and $\mathcal{O}_2$ involved in $P^{(2)}$ in (3.32), (3.62), and then compute the integration of $P^{(2)}$ along the normal spaces to $X_G$.

Sometimes the computations seem to be long and tedious, involving many subtle relations between metrics, connections and curvatures near $X_G$, but fortunately the final result on $\Phi_1$ is still of a simple form, as expected.

Throughout the computations below, a key idea is to rewrite all operators by using the creation and annihilation operators $b_i, b_i^+$, $b_j^\perp, b_j^{\perp +}$, then under the help of (3.9) and Theorem 3.1, we can do the operations and obtain the crucial Lemmas 5.9, 5.11.

To get the final simple formula (0.25), we still need to prove a highly non-trivial identity (5.131).

In the usual case, i.e. $G = \{1\}$, Ma-Marinescu have used the similar formula (3.62) to compute the coefficients in varies generalities. In the Kähler case (cf. [**31**, §4.1.8]), the computation is quite easy as $\mathcal{O}_1 = 0$. In the symplectic case [**28**, §2], $\mathcal{O}_1 \neq 0$, but the contribution from $\mathcal{O}_1$ is zero at $(0,0)$ and in the spin$^c$ Dirac operator case [**29**, §2], $\mathcal{O}_1 \neq 0$, and the contribution from $\mathcal{O}_1$ is non zero at $(0,0)$.

This Chapter is organized as follows. In Section 5.1, we explain various relations of the curvature of the fibration $P \to X_G$ and the second fundamental form of $P$. In Section 5.2, we obtain the explicit formulas for the operators $\mathcal{O}_1$, $\mathcal{O}_2$. In Section 5.3, we apply the formulas in Section 5.2 and (5.1) to (3.62), and we get a formula for the coefficient $\Phi_1$. In Section 5.4, we compute finally $\Phi_1$, thus proving Theorem 0.6. In Section 5.5, we explain how to reduce the general case to the case $\mathbf{J} = J$ which has been worked out in Sections 5.1-5.4.

In the whole Chapter, if there is no other specific notification, when we meet the operation $|\quad|^2$, we will first do this operation, then take the sum of the indices.

## 5.1. The second fundamental form of $P$

We use the notations in Sections 2.2, 2.3. Then the normal bundle $N_G$ of $X_G$ in $U/G$ is $(JTY)_G$.

Let $\iota : X_G \to U/G$ be the natural embedding.

We will apply the notation in Section 1.1 to $B = U/G$.

Let $\nabla^{TX_G}$, $\nabla^{N_G}$ be connections on $TX_G, N_G$ induced by projections of the Levi-Civita connection $\nabla^{TB}$ on $TB$. Then $\nabla^{TX_G}$ is the Levi-Civita connection on $(TX_G, g^{TX_G})$.

Let

$$
{}^0\nabla^{TB} = \nabla^{TX_G} \oplus \nabla^{N_G} \tag{5.2}
$$

be the connection on $TB$ on $X_G$ induced by $\nabla^{TX_G}$, $\nabla^{N_G}$ with curvature ${}^0R^{TB}$.

Set

$$
A = \nabla^{TB}|_{X_G} - {}^0\nabla^{TB}. \tag{5.3}
$$

Then $A$ is a 1-form on $X_G$ taking values in the skew-adjoint endomorphisms of $(TB)|_{X_G}$ which exchange $TX_G$ and $N_G$.

We recall the following properties of $R^{TB}$, the curvature of $\nabla^{TB}$: for $U, V, W, W_2 \in TB$,

$$
\begin{aligned}
&\left\langle R^{TB}(U,V)W, W_2\right\rangle = \left\langle R^{TB}(W,W_2)U, V\right\rangle, \\
&R^{TB}(U,V)W + R^{TB}(V,W)U + R^{TB}(W,U)V = 0.
\end{aligned} \tag{5.4}
$$

On $X_G$, let $\{e_i^0\}$ be an orthonormal frame of $TX_G$, let $\{e_j^\perp\}$ be an orthonormal frame of $N_G$, then $\{e_i\} = \{e_i^0, e_j^\perp\}$ is an orthonormal frame of $TB$.

The following result gives detail informations on the torsion $T$ of the fibration, as well as the second fundamental form $A$.

***Theorem 5.1***. — *On $P$, the restriction of the tensor $\langle JT(\cdot, J\cdot), \cdot\rangle$ on $(N_G)^{\otimes 3}$ is symmetric, and*

$$(A(e_i^0)e_j^0)^H = \frac{1}{2} JT(e_i^{0,H}, Je_j^{0,H}), \tag{5.5a}$$

$$T(e_i^{0,H}, e_j^{0,H}) = T((J_G e_i^0)^H, (J_G e_j^0)^H), \tag{5.5b}$$

$$T(e_i^{0,H}, e_j^{\perp,H}) = 2T((J_G e_i^0)^H, Je_j^{\perp,H}), \tag{5.5c}$$

$$\left\langle T(e_i^{0,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = \left\langle T(e_i^{0,H}, e_k^{\perp,H}), Je_j^{\perp,H}\right\rangle, \tag{5.5d}$$

$$\sum_k \left\langle T(e_k^{\perp,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = 0. \tag{5.5e}$$

*Proof*. — Observe first that we have

$$\nabla^{TX} J = 0; \tag{5.6a}$$

$$(J_G e_i^0)^H = Je_i^{0,H} \quad \text{on} \quad P. \tag{5.6b}$$

Let $Z$ be a smooth section of $TY$, then by (3.1), $JZ \in JTY \subset T^H X$ on $P$, by (1.3), (1.7), (3.1) and (5.6a), on $P$, we have

$$\begin{aligned}\langle J(A(e_i^0)e_j^0)^H, Z\rangle &= -\left\langle \nabla^{T^H X}_{e_i^{0,H}} e_j^{0,H}, JZ\right\rangle = -\left\langle \nabla^{TX}_{e_i^{0,H}} e_j^{0,H}, JZ\right\rangle \\ &= \left\langle \nabla^{TX}_{e_i^{0,H}} (Je_j^{0,H}), Z\right\rangle = \left\langle S(e_i^{0,H}) Je_j^{0,H}, Z\right\rangle = -\frac{1}{2}\left\langle T(e_i^{0,H}, Je_j^{0,H}), Z\right\rangle.\end{aligned} \tag{5.7}$$

Thus we get (5.5a), as $A(e_i^0)e_j^0 \in N_G = (JTY)_G$ on $X_G$.

Note that $[Z, e_i^H] \in TY$, by (1.3), (1.7) and (5.6a),

$$\langle T(e_i^H, e_j^H), Z\rangle = 2\left\langle \nabla^{TX}_{e_i^H} Z, e_j^H\right\rangle = 2\left\langle \nabla^{TX}_Z e_i^H, e_j^H\right\rangle = 2\left\langle \nabla^{TX}_Z (Je_i^H), Je_j^H\right\rangle. \tag{5.8}$$

From (5.6b) and (5.8), we get (5.5b).

From (1.3), (1.7), (5.8) and $Je_j^{\perp,H}, Je_k^{\perp,H} \in TY$ on $P$, we get

$$\left\langle T(e_i^{0,H}, e_j^{\perp,H}), Z\right\rangle = 2\left\langle S(Z)(Je_i^{0,H}), Je_j^{\perp,H}\right\rangle = 2\left\langle T(Je_i^{0,H}, Je_j^{\perp,H}), Z\right\rangle. \tag{5.9}$$

Thus we get (5.5c). By (1.6), (5.9), we get

$$\left\langle T(e_i^{0,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = 2\left\langle T(Je_i^{0,H}, Je_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = \left\langle T(e_i^{0,H}, e_k^{\perp,H}), Je_j^{\perp,H}\right\rangle. \tag{5.10}$$

Thus we get (5.5d). By (1.3), (1.7), (5.6a) and $Je_j^{\perp,H} \in TY$ on $P$,

$$\begin{aligned}&\left\langle T(e_i^{\perp,H}, Je_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = \left\langle \nabla^{TX}_{Je_k^{\perp,H}} e_i^{\perp,H}, Je_j^{\perp,H}\right\rangle \\ &= -\left\langle \nabla^{TX}_{Je_k^{\perp,H}} (Je_i^{\perp,H}), e_j^{\perp,H}\right\rangle = \left\langle \nabla^{TX}_{Je_k^{\perp,H}} e_j^{\perp,H}, Je_i^{\perp,H}\right\rangle = \left\langle T(e_j^{\perp,H}, Je_i^{\perp,H}), Je_k^{\perp,H}\right\rangle.\end{aligned} \tag{5.11}$$

By (1.7) and (5.11), $\langle JT(\cdot, J\cdot), \cdot\rangle$ is symmetric on the horizontal lift of $N_G^{\otimes 3}$.

Note that $\{Je_k^{\perp,H}\}$ is a $G$-invariant orthonormal frame of $TY$ on $P$, by (5.8),

$$\left\langle T(e_i^{\perp,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = 2\left\langle \nabla^{TY}_{Je_k^{\perp,H}}(Je_i^{\perp,H}), Je_j^{\perp,H}\right\rangle. \tag{5.12}$$

By (1.9) and (5.12), we get (5.5e). The proof of Theorem 5.1 is complete. □

***Remark 5.2***. — From (1.6) and (5.5b), $\Theta|_{X_G}$ is a $(1,1)$-form on $X_G$. Especially, for any complex representation $F$ of $G$, $P\times_G F$ is a holomorphic vector bundle on $X_G$. Moreover, by (5.5a), for $U\in TX_G, V\in N_G$, we have at $x_0$,

$$A(U)V = \langle A(U)V, e_j^0\rangle\, e_j^0 = -\langle V, A(U)e_j^0\rangle\, e_j^0 = \frac{1}{2}\langle T(U, Je_j^0), JV\rangle\, e_j^0. \tag{5.13}$$

For $x_0\in X_G$, if $\{e_j^{\perp}\}$ is a fixed orthonormal basis of $N_{G,x_0}$ as above, then for $U\in T_{x_0}X_G$, we will denote by

$$\begin{aligned} &\mathcal{T}_{ijk} = \langle JT(e_i^{\perp}, Je_j^{\perp}), e_k^{\perp}\rangle\,,\ \widetilde{\mathcal{T}}_{ijk} = \langle JT(e_i^{\perp}, e_j^{\perp}), e_k^{\perp}\rangle\,,\\ &\mathcal{T}_{jk}(U) = \langle JT(U, e_j^{\perp}), e_k^{\perp}\rangle\,. \end{aligned} \tag{5.14}$$

By Theorem 5.1, $\mathcal{T}_{ijk}$ is symmetric on $i,j,k$ and $\mathcal{T}_{jk}\in T^*_{x_0}X_G$ is symmetric on $j,k$, $\widetilde{\mathcal{T}}_{ijk}$ is anti-symmetric on $i,j$. Moreover, as functions along the fiber $Gx_0$, $\mathcal{T}_{ijk}$, $\mathcal{T}_{jk}$, $\widetilde{\mathcal{T}}_{ijk}$ are constant.

***Remark 5.3***. — From Remark 1.2 and (5.12), we know that $\langle JT(\cdot,\cdot),\cdot\rangle$ is anti-symmetric on $(N_G)^{\otimes 3}$ if $g^{TY}$ is induced by a family of Ad-invariant metric on $\mathfrak{g}$. If $G$ is abelian, then by (1.12), (5.12), $T(\cdot,\cdot)=0$ on $(N_G)^{\otimes 2}$, thus $\widetilde{\mathcal{T}}_{ijk}=0$.

## 5.2. The operators $\mathscr{O}_1$, $\mathscr{O}_2$ in (2.102)

We use the notations in Sections 2.6, 3.1, and all tensors will be evaluated at $x_0\in X_G$.

Recall that $(X,\omega)$ is Kähler and $\mathbf{J}=J$ on a $G$-neighborhood $U$ of $P=\mu^{-1}(0)$, then in (3.5)

$$a_i = a_j^{\perp} = 2\pi. \tag{5.15}$$

Clearly, on $U$, the Levi-Civita connection $\nabla^{TX}$ preserves $T^{(1,0)}X$ and $T^{(0,1)}X$, and $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X}\nabla^{TX}P^{T^{(1,0)}X}$ is the holomorphic Hermitian connection on $T^{(1,0)}X$, while the Clifford connection $\nabla^{\mathrm{Cliff}}$ on $\Lambda(T^{*(0,1)}X)$ is $\nabla^{\Lambda(T^{*(0,1)}X)}$, the natural connection induced by $\nabla^{T^{(1,0)}X}$.

Let $\overline{\partial}^{L^p\otimes E,*}$ be the canonical formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on $\Omega^{0,\bullet}(U, L^p\otimes E)$. Then the operator $D_p$ in (2.14) is

$$D_p = \sqrt{2}\left(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\right). \tag{5.16}$$

Note that $D_p^2$ preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(U, L^p\otimes E)$.

Set

$$D^2_{p,i} = D^2_p|_{\Omega^{0,i}(U,L^p\otimes E)}. \tag{5.17}$$

Let $\Delta^{L^p\otimes E}$ be the Laplacian on $L^p \otimes E$ associated to $\nabla^{L^p\otimes E}$. Then by (2.51) (cf. also [**31**, (1.4.31)]) , as $\mathbf{J} = J$ on $U$, we have

$$D^2_{p,0} = \Delta^{L^p\otimes E} - R^E_\tau - 2\pi np \quad \text{on } U. \tag{5.18}$$

Since $\nabla^{\text{Cliff}}$ preserves the $\mathbb{Z}$-grading of $\Lambda(T^{*(0,1)}X)$, the operator $\mathscr{L}^t_2$ in (2.100) also preserves the $\mathbb{Z}$-grading on $\Lambda(T^{*(0,1)}X_0)$. Moreover, $\mathscr{L}^t_2$ is invertible on $\oplus^n_{q=1}\Omega^{0,q}(X_0, L^p_0 \otimes E_0)$ for $t$ small enough (cf. Theorem 2.2 or [**31**, Theorem 1.5.5]).

From Section 3.2, for $P^{(r)}$ in (0.12),

$$P^{(r)} = I_{\mathbb{C}\otimes E_G} P^{(r)} I_{\mathbb{C}\otimes E_G}. \tag{5.19}$$

Thus we only need to do the computation for $D^2_{p,0}$.

In what follows, we compute everything on $\mathscr{C}^\infty(U, L^p \otimes E)$.

Take $x_0 \in X_G$.

If $Z \in T_{x_0}B$, $Z = Z^0 + Z^\perp$, $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$, $|Z^0|, |Z^\perp| \leqslant \varepsilon$, as in Section 2.6, we identify $Z$ with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)} \tau_{Z^0}(Z^\perp)$. This identification is a diffeomorphism from $B^{TX_G}_{x_0}(0,\varepsilon) \times B^{N_G}_{x_0}(0,\varepsilon)$ into an open neighborhood $\mathscr{U}(x_0)$ of $x_0$ in $B$, we denote it by $\Psi$. Then $\mathscr{U}(x_0) \cap X_G = B^{TX_G}_{x_0}(0,\varepsilon) \times \{0\}$.

In what follows, we use indifferently the notation $B^{TX_G}_{x_0}(0,\varepsilon) \times B^{N_G}_{x_0}(0,\varepsilon)$ or $\mathscr{U}(x_0)$, $x_0$ or $0, \cdots$.

From now on, we replace $U/G$ by $\mathbb{R}^{2n-n_0} \simeq T_{x_0}B$ as in Section 2.6, and we use the notation therein. Especially,

$$\nabla_t = tS^{-1}_t \kappa^{1/2} \nabla^{(L^p\otimes E)_B} \kappa^{-1/2} S_t, \tag{5.20}$$

and $\mathscr{O}_r$ in (2.102) takes value in $\text{End}(E_B)$.

Let $\{e^0_i\}$, $\{e^\perp_j\}$ be orthonormal basis of $T_{x_0}X_G$, $N_{G,x_0}$ respectively. We will also denote $\Psi_*(e^0_i)$, $\Psi_*(e^\perp_j)$ by $e^0_i$, $e^\perp_j$.

Let $\{e_i\}$ denote the basis $\{e^0_i, e^\perp_j\}$. Thus in our coordinates,

$$\frac{\partial}{\partial Z^0_i} = e^0_i, \qquad \frac{\partial}{\partial Z^\perp_j} = e^\perp_j. \tag{5.21}$$

We denote by $(g^{ij}(Z))$ the inverse of the matrix $(g_{ij}(Z)) = (g^{TB}_{ij}(Z))$ (cf. (2.106)).

Recall that $\Gamma^l_{ij}$ is the connection form of $\nabla^{TB}$, with respect to the frame $\{e_i\}$, defined in (2.106). Also recall that $\mathcal{R}$, $\mathcal{R}^0$ and $\mathcal{R}^\perp$ are defined in (2.72).

As in (1.14), the moment map $\mu$ induces a $G$-invariant $\mathscr{C}^\infty$ section $\widetilde{\mu}$ of $TY$ on $U$.

Note also that by (2.50), $R^E_\tau \in \text{End}(E)$ defines a section of $\text{End}(E_B)$ on $B = U/G$. Recall that $h(x) = \sqrt{\text{vol}(Gx)}$ is defined in (0.10).

Set

$$
\begin{aligned}
\mathscr{L}_3^t(Z) = -g^{ij}(tZ)\Big(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma_{ij}^k(tZ)\nabla_{t,e_k}\Big) \\
+ t^2\Big(\frac{1}{h}g^{ij}(\nabla_{e_i}\nabla_{e_j}h - \Gamma_{ij}^k\nabla_{e_k}h)\Big)(tZ) - t^2R_\tau^E(tZ) - 2\pi n.
\end{aligned}
\tag{5.22}
$$

By (2.62), (2.100) and (5.22), we can reformulate (2.101), (2.109), in using the notations in (3.10), as follows,

$$
\begin{aligned}
&\nabla_{0,\cdot} = \nabla_\cdot + \frac{1}{2}R_{x_0}^{L_B}(\mathcal{R},\cdot) = \nabla_\cdot - \pi\sqrt{-1}\left\langle J_{x_0}Z^0,\cdot\right\rangle_{x_0},\\
&\mathscr{L}_2^0 = \sum_{j=1}^{n-n_0} b_jb_j^+ + \sum_{j=1}^{n_0} b_j^\perp b_j^{\perp+} = -\sum_j(\nabla_{0,e_j})^2 + 4\pi^2|Z^\perp|^2 - 2\pi n,\\
&\mathscr{L}_2^t(Z) = \mathscr{L}_3^t(Z) + 4\pi^2\Big|\frac{1}{t}\widetilde{\mu}\Big|_{g^{TY}}^2(tZ) - \left\langle 4\pi\sqrt{-1}\widetilde{\mu} + t^2\widetilde{\mu}^E, \widetilde{\mu}^E\right\rangle_{g^{TY}}(tZ).
\end{aligned}
\tag{5.23}
$$

If there is no other specification, we will evaluate our tensors at $x_0$, and most of time, we will omit the subscript $x_0$.

Set $h_0 = h_{x_0} := h(x_0)$, and for $U \in T_{x_0}B$, set

$$
\begin{aligned}
&B(Z,U) = \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^{L_B})_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},U),\\
&I_1 = -B(Z,e_i)\nabla_{0,e_i} - \frac{1}{2}\nabla_{e_i}(B(Z,e_i)),\\
&I_2 = \Big(\Big\langle\frac{1}{3}R^{TX_G}(\mathcal{R}^0,e_i^0)\mathcal{R}^0 + \nabla_{\mathcal{R}^0}^{TX_G}(A(e_i^0)\mathcal{R}^\perp), e_j^0\Big\rangle\\
&\quad + \Big\langle e_i^0, \nabla_{\mathcal{R}^0}^{TX_G}(A(e_j^0)\mathcal{R}^\perp)\Big\rangle - 3\left\langle A(e_i^0)\mathcal{R}^\perp, A(e_j^0)\mathcal{R}^\perp\right\rangle\\
&\qquad + \left\langle R^{TB}(\mathcal{R}^\perp,e_i^0)\mathcal{R}^\perp, e_j^0\right\rangle\Big)\nabla_{0,e_i^0}\nabla_{0,e_j^0}\\
&\quad + \Big(\left\langle R^{N_G}(\mathcal{R}^0,e_j^0)\mathcal{R}^\perp, e_i^\perp\right\rangle + \frac{4}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_j^0)\mathcal{R}^\perp, e_i^\perp\right\rangle\Big)\nabla_{0,e_i^\perp}\nabla_{0,e_j^0}\\
&\quad + \frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^\perp)\mathcal{R}^\perp, e_j^\perp\right\rangle\nabla_{0,e_i^\perp}\nabla_{0,e_j^\perp}.
\end{aligned}
\tag{5.24}
$$

Recall that the operator $\mathscr{L}$ has been defined in (3.10).

Set also

$$
\begin{aligned}
\Gamma_{ii}(\mathcal{R}) =& \frac{2}{3} R^{TX_G}_{x_0}(\mathcal{R}^0, e^0_i)e^0_i + \nabla^{TB}_{\mathcal{R}^0}(A(e^0_i)e^0_i) + R^{TB}(\mathcal{R}^\perp, e^0_i)e^0_i \\
& + A(e^0_i)A(e^0_i)\mathcal{R}^\perp + \nabla^{TX_G}_{e^0_i}(A(e^0_i)\mathcal{R}^\perp) - A(\mathcal{R}^0)A(e^0_i)e^0_i, \\
K_2(\mathcal{R}) =& \frac{1}{3}\left\langle R^{TX_G}(\mathcal{R}^0, e^0_i)\mathcal{R}^0, e^0_i\right\rangle + \left\langle R^{TB}(\mathcal{R}^\perp, e^0_i)\mathcal{R}^\perp, e^0_i\right\rangle \\
& + \frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp, e^\perp_i)\mathcal{R}^\perp, e^\perp_i\right\rangle + 2\Big(\sum_i \left\langle A(e^0_i)e^0_i, \mathcal{R}^\perp\right\rangle\Big)^2 \\
& - |A(e^0_i)\mathcal{R}^\perp|^2 + 2\left\langle \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_i)\mathcal{R}^\perp), e^0_i\right\rangle.
\end{aligned} \tag{5.25}
$$

***Lemma 5.4.*** — *There exist second order differential operators $\mathcal{O}'_r$ as in Theorem 2.11 such that for $|t| \leqslant 1$,*

$$
\mathscr{L}^t_3 = \mathscr{L}^0_3 + \sum_{r=1}^m t^r \mathcal{O}'_r + \mathscr{O}(t^{m+1}), \tag{5.26}
$$

*with*

(5.27)

$$
\begin{aligned}
\mathscr{L}^0_3 =& \mathscr{L} - \sum_{j=1}^{n_0} (\nabla_{e^\perp_j})^2 - 2\pi n_0 = \mathscr{L}^0_2 - 4\pi^2 |Z^\perp|^2, \\
\mathcal{O}'_1 =& -\frac{2}{3}(\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i) Z_j \nabla_{0,e_i} - \frac{1}{3}(\partial_i R^{L_B})_{x_0}(\mathcal{R}, e_i) \\
& - 2\left\langle A(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle \nabla_{0,e^0_i}\nabla_{0,e^0_j}, \\
\mathcal{O}'_2 =& I_1 + I_2 + \Big[\frac{1}{4}K_2(\mathcal{R}) - \frac{3}{8}\Big(\sum_l \left\langle A(e^0_l)e^0_l, \mathcal{R}^\perp\right\rangle\Big)^2, \mathscr{L}^0_2\Big] \\
& - 2\left\langle A(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle \Big(\frac{2}{3}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e^0_j) Z_k \nabla_{0,e^0_i} + \frac{1}{3}(\partial^0_j R^{L_B})_{x_0}(\mathcal{R}, e^0_i)\Big) \\
& + \left\langle \Gamma_{ii}(\mathcal{R}), e_j\right\rangle \nabla_{0,e_j} - \frac{1}{2}\left\langle A(e^0_l)e^0_l, \mathcal{R}^\perp\right\rangle \nabla_{A(e^0_k)e^0_k} + 2\left\langle A(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle \nabla_{A(e^0_i)e^0_j} \\
& + \frac{2}{3}\left\langle R^{TB}(\mathcal{R}^\perp, e^\perp_i)e^\perp_i, e_j\right\rangle \nabla_{0,e_j} - R^{E_B}_{x_0}(\mathcal{R}, e_i)\nabla_{0,e_i} - R^{E_B}_{\tau,x_0} \\
& - \frac{1}{9}\sum_i \Big[\sum_j (\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i) Z_j\Big]^2 + \frac{1}{h_0}\Big(\nabla_{e_j}\nabla_{e_j} h - \nabla_{A(e^0_i)e^0_i} h\Big)_{x_0}.
\end{aligned}
$$

*Proof.* — By (2.103) and (5.20),

$$
\begin{aligned}
\nabla_{t,e_i} = \kappa^{1/2}(tZ)\Big(\nabla_{e_i} + \Big(\frac{1}{2}R^{L_B}_{x_0} + \frac{t}{3}(\partial_k R^{L_B})_{x_0} Z_k \\
+ \frac{t^2}{4}\sum_{|\alpha|=2} (\partial^\alpha R^{L_B})_{x_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}R^{E_B}_{x_0}\Big)(\mathcal{R}, e_i) + \mathscr{O}(t^3)\Big)\kappa^{-1/2}(tZ).
\end{aligned} \tag{5.28}
$$

To get (5.27), we could use (2.92)-(2.96), while here we will get it directly from the local computation.

By [**1**, Prop. 1.28] (cf. [**28**, (1.31)]) and (2.103),

$$\begin{aligned}&\left\langle e_i^0, e_j^0\right\rangle_{Z^0} = \delta_{ij} + \frac{1}{3}\left\langle R^{TX_G}_{x_0}(\mathcal{R}^0, e_i^0)\mathcal{R}^0, e_j^0\right\rangle_{x_0} + \mathscr{O}(|Z^0|^3),\\ &(\nabla^{N_G}_{e_k^0}\nabla^{N_G}_{e_i^0}e_j^\perp)_{x_0} = \frac{1}{2}R^{N_G}_{x_0}(e_k^0, e_i^0)e_j^\perp.\end{aligned} \tag{5.29}$$

Moreover, for $W, V \in N_{G,x_0}$, $\gamma_s(t) = (Z^0, t(W+sV))$ is a family of geodesics from $(Z^0, 0)$ in $B$. Set $Y = \frac{\partial}{\partial t}\gamma_s(t)$, $X(\gamma_s(t)) = \frac{\partial}{\partial s}\gamma_s(t) = tV$.

Since $\nabla^{TB}_Y Y = 0$, $\nabla^{TB}_Y X - \nabla^{TB}_X Y = [Y, X] = \gamma_*[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$, we get

$$0 = \nabla^{TB}_X\nabla^{TB}_Y Y = \nabla^{TB}_Y\nabla^{TB}_Y X - R^{TB}(Y, X)Y. \tag{5.30}$$

Take $V = e_i^\perp$, we get at $s = t = 0$,

$$(\nabla^{TB}_W\nabla^{TB}_W e_i^\perp)_{Z^0} = \frac{1}{3}\nabla^{TB}_Y\nabla^{TB}_Y\nabla^{TB}_Y X = \frac{1}{3}R^{TB}(W, e_i^\perp)W. \tag{5.31}$$

Under our coordinates, we have

$$\begin{aligned}&(\nabla^{TB}_{e_j^\perp}e_i^\perp)_{x_0} = (\nabla^{TX_G}_{e_i^0}e_j^0)_{x_0} = (\nabla^{N_G}_{e_j^0}e_i^\perp)_{x_0} = 0, \quad (\nabla^{TB}_{e_i^0}e_j^0)_{x_0} = A_{x_0}(e_i^0)e_j^0,\\ &(\nabla^{TB}_{e_j^\perp}e_i^0)_{x_0} = (\nabla^{TB}_{e_i^0}e_j^\perp)_{x_0} = A_{x_0}(e_i^0)e_j^\perp,\\ &(\nabla^{TB}_{\mathcal{R}^\perp}e_i^\perp)_Z = 0,\\ &(\nabla^{TB}_{e_j^\perp}e_i^\perp)_{Z^0} = (\nabla^{N_G}_{\mathcal{R}^0}e_i^\perp)_{Z^0} = 0.\end{aligned} \tag{5.32}$$

Moreover, by (5.4), (5.29), (5.31) and (5.32) (comparing with [**28**, (1.31)]), as $[e_i, e_j] = 0$ by (5.21), we have at $x_0$ that

$$\begin{aligned}&\nabla^{TB}_{e_k^\perp}\nabla^{TB}_{e_j^\perp}e_i^\perp = \frac{1}{3}R^{TB}(e_k^\perp, e_j^\perp)e_i^\perp + \frac{1}{3}R^{TB}(e_k^\perp, e_i^\perp)e_j^\perp,\\ &\nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^\perp}e_i^\perp = 0,\\ &\nabla^{TB}_{e_k^\perp}\nabla^{TB}_{e_j^\perp}e_i^0 = \nabla^{TB}_{e_k^\perp}\nabla^{TB}_{e_i^0}e_j^\perp = R^{TB}(e_k^\perp, e_i^0)e_j^\perp,\\ &\nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^\perp}e_i^0 = \nabla^{TB}_{e_k^0}\nabla^{TB}_{e_i^0}e_j^\perp\\ &\qquad = \nabla^{N_G}_{e_k^0}\nabla^{N_G}_{e_i^0}e_j^\perp + A(e_k^0)A(e_i^0)e_j^\perp + \nabla^{TX_G}_{e_k^0}(A(e_i^0)e_j^\perp)\\ &\qquad = \frac{1}{2}R^{N_G}(e_k^0, e_i^0)e_j^\perp + A(e_k^0)A(e_i^0)e_j^\perp + \nabla^{TX_G}_{e_k^0}(A(e_i^0)e_j^\perp),\\ &\nabla^{TB}_{e_j^\perp}\nabla^{TB}_{e_k^0}e_i^0 = R^{TB}(e_j^\perp, e_k^0)e_i^0 + \nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^\perp}e_i^0,\\ &\nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^0}e_i^0 = \nabla^{TX_G}_{e_k^0}\nabla^{TX_G}_{e_j^0}e_i^0 + \nabla^{TB}_{e_k^0}(A(e_j^0)e_i^0)\\ &\qquad = \frac{1}{3}R^{TX_G}(e_k^0, e_j^0)e_i^0 + \frac{1}{3}R^{TX_G}(e_k^0, e_i^0)e_j^0 + \nabla^{TB}_{e_k^0}(A(e_j^0)e_i^0).\end{aligned} \tag{5.33}$$

In the following, for a tensor $\psi$ and the covariant derivative $\nabla^B$ acting on $\psi$ induced by $\nabla^{TB}$, we denote by

$$(\nabla^B\nabla^B\psi)_{(c_je_j,c'_ke_k)} = c_jc'_k(\nabla^B_{e_j}\nabla^B_{e_k}\psi)_{x_0}.$$

From (5.33), we get at $x_0$ the following formula which will be used in (5.38), (5.39), (5.56), (5.57) and (6.26),

(5.34)
$$\begin{aligned}
&(\nabla^{TB}\nabla^{TB}e^0_i)_{(\mathcal{R}^0,\mathcal{R}^0)} = \frac{1}{3}R^{TX_G}(\mathcal{R}^0,e^0_i)\mathcal{R}^0 + \nabla^{TB}_{\mathcal{R}^0}(A(e^0_j)e^0_i)Z^0_j,\\
&(\nabla^{TB}\nabla^{TB}e^0_i)_{(\mathcal{R}^0,\mathcal{R}^\perp)} = \frac{1}{2}R^{N_G}(\mathcal{R}^0,e^0_i)\mathcal{R}^\perp + A(\mathcal{R}^0)A(e^0_i)\mathcal{R}^\perp + \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_i)\mathcal{R}^\perp),\\
&(\nabla^{TB}\nabla^{TB}e^0_i)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)} = R^{TB}(\mathcal{R}^\perp,e^0_i)\mathcal{R}^\perp,\\
&(\nabla^{TB}\nabla^{TB}e^\perp_j)_{(\mathcal{R}^0,\mathcal{R}^0)} = A(\mathcal{R}^0)A(\mathcal{R}^0)e^\perp_j + \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_k)e^\perp_j)Z^0_k,\\
&(\nabla^{TB}\nabla^{TB}e^\perp_j)_{(\mathcal{R}^0,\mathcal{R}^\perp)} = 0,\\
&(\nabla^{TB}\nabla^{TB}e^\perp_j)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)} = \frac{1}{3}R^{TB}(\mathcal{R}^\perp,e^\perp_j)\mathcal{R}^\perp,\\
&(\nabla^{TB}\nabla^{TB}e_j)_{(\mathcal{R}^\perp,\mathcal{R}^0)} = (\nabla^{TB}\nabla^{TB}e_j)_{(\mathcal{R}^0,\mathcal{R}^\perp)} + R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e_j.
\end{aligned}$$

Note that by (5.32), $\nabla^{TB}_{\mathcal{R}}(A_{x_0}(e^0_i)e^0_i) = A(\mathcal{R}^0)A_{x_0}(e^0_i)e^0_i$. From (5.32), (5.33), we get

$$\begin{aligned}
(\nabla^{TB}_{e^\perp_i}e^\perp_i)_Z &= \frac{2}{3}R^{TB}(\mathcal{R}^\perp,e^\perp_i)e^\perp_i + \mathscr{O}(|Z|^2),\\
(\nabla^{TB}_{e^0_i}e^0_i)_Z &= A_{x_0}(e^0_i)e^0_i + \nabla^{TB}_{\mathcal{R}}(\nabla^{TB}_{e^0_i}e^0_i - A_{x_0}(e^0_i)e^0_i) + \mathscr{O}(|Z|^2)\\
&= A_{x_0}(e^0_i)e^0_i - \nabla^{TB}_{\mathcal{R}}(A_{x_0}(e^0_i)e^0_i) + \frac{2}{3}R^{TX_G}(\mathcal{R}^0,e^0_i)e^0_i\\
&\quad + \nabla^{TB}_{\mathcal{R}^0}(A(e^0_i)e^0_i) + A(e^0_i)A(e^0_i)\mathcal{R}^\perp\\
&\quad + \nabla^{TX_G}_{e^0_i}(A(e^0_i)\mathcal{R}^\perp) + R^{TB}(\mathcal{R}^\perp,e^0_i)e^0_i + \mathscr{O}(|Z|^2)\\
&= A_{x_0}(e^0_i)e^0_i + \Gamma_{ii}(\mathcal{R}) + \mathscr{O}(|Z|^2),
\end{aligned} \tag{5.35}$$

Thus by (5.32), (5.33) and (5.34), at $x_0$,

$$\begin{aligned}
\nabla_{\mathcal{R}^0}\nabla_{\mathcal{R}^\perp}\left\langle e^\perp_j, e^0_i\right\rangle &= \left\langle \nabla^{TB}_{\mathcal{R}^0}e^\perp_j, \nabla^{TB}_{\mathcal{R}^\perp}e^0_i\right\rangle + \left\langle e^\perp_j, \nabla^{TB}_{\mathcal{R}^0}\nabla^{TB}_{\mathcal{R}^\perp}e^0_i\right\rangle\\
&= \frac{1}{2}\left\langle R^{N_G}(\mathcal{R}^0,e^0_i)\mathcal{R}^\perp, e^\perp_j\right\rangle.
\end{aligned} \tag{5.36}$$

On the other hand, we have the following expansion for $\langle e_j, e_i\rangle_Z$,

(5.37)
$$\begin{aligned}\langle e_i, e_j\rangle_Z &= \langle e_i, e_j\rangle_{Z^0} + (\nabla_{\mathcal{R}^\perp}\langle e_i, e_j\rangle)_{Z^0} + \frac{1}{2}(\nabla\nabla\langle e_i, e_j\rangle)_{(\mathcal{R}^\perp,\mathcal{R}^\perp),x_0} + \mathscr{O}(|Z|^3)\\ &= \langle e_i, e_j\rangle_{Z^0} + (\nabla_{\mathcal{R}^\perp}\langle e_i, e_j\rangle)_{x_0} + (\nabla_{\mathcal{R}^0}\nabla_{\mathcal{R}^\perp}\langle e_i, e_j\rangle)_{x_0} + \left\langle \nabla^{TB}_{\mathcal{R}^\perp} e_i, \nabla^{TB}_{\mathcal{R}^\perp} e_j\right\rangle_{x_0}\\ &\quad + \frac{1}{2}\left\langle (\nabla^{TB}\nabla^{TB} e_i)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)}, e_j\right\rangle + \frac{1}{2}\left\langle e_i, (\nabla^{TB}\nabla^{TB} e_j)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)}\right\rangle + \mathscr{O}(|Z|^3).\end{aligned}$$

Thus by (5.4), (5.29), (5.32), (5.34) and (5.36)-(5.37),

$$\begin{aligned}\left\langle e^0_i, e^0_j\right\rangle_Z &= \delta_{ij} - 2\left\langle A_{x_0}(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle + \frac{1}{3}\left\langle R^{TX_G}(\mathcal{R}^0, e^0_i)\mathcal{R}^0, e^0_j\right\rangle\\ &\quad + \left\langle \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_i)\mathcal{R}^\perp), e^0_j\right\rangle + \left\langle e^0_i, \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_j)\mathcal{R}^\perp)\right\rangle\\ &\quad + \left\langle A(e^0_i)\mathcal{R}^\perp, A(e^0_j)\mathcal{R}^\perp\right\rangle + \left\langle R^{TB}(\mathcal{R}^\perp, e^0_i)\mathcal{R}^\perp, e^0_j\right\rangle + \mathscr{O}(|Z|^3),\end{aligned} \tag{5.38}$$

and

$$\begin{aligned}\left\langle e^0_i, e^\perp_j\right\rangle_Z &= \frac{1}{2}\left\langle R^{N_G}(\mathcal{R}^0, e^0_i)\mathcal{R}^\perp, e^\perp_j\right\rangle + \frac{2}{3}\left\langle R^{TB}(\mathcal{R}^\perp, e^0_i)\mathcal{R}^\perp, e^\perp_j\right\rangle + \mathscr{O}(|Z|^3),\\ \left\langle e^\perp_i, e^\perp_j\right\rangle_Z &= \delta_{ij} + \frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp, e^\perp_i)\mathcal{R}^\perp, e^\perp_j\right\rangle + \mathscr{O}(|Z|^3).\end{aligned} \tag{5.39}$$

Note that $\det(\delta_{ij} + a_{ij}) = 1 + \sum_i a_{ii} + \sum_{i<j}(a_{ii}a_{jj} - a_{ij}a_{ji}) + \cdots$. From (5.25), (5.38) and (5.39), we get

(5.40)
$$\begin{aligned}&\det g_{ij}(Z) = 1 - 2\left\langle A_{x_0}(e^0_i)e^0_i, \mathcal{R}^\perp\right\rangle + K_2(\mathcal{R}) + \mathscr{O}(|Z|^3),\\ &\kappa^{\frac{1}{2}}(tZ) = (\det g_{ij})^{1/4}(tZ)\\ &\quad = 1 - \frac{t}{2}\left\langle A(e^0_i)e^0_i, \mathcal{R}^\perp\right\rangle - \frac{3t^2}{8}\Big(\sum_i \left\langle A(e^0_i)e^0_i, \mathcal{R}^\perp\right\rangle\Big)^2 + \frac{t^2}{4}K_2(\mathcal{R}) + \mathscr{O}(t^3),\\ &\kappa^{-\frac{1}{2}}(tZ) = 1 + \frac{t}{2}\left\langle A(e^0_i)e^0_i, \mathcal{R}^\perp\right\rangle + \frac{5t^2}{8}\Big(\sum_i \left\langle A(e^0_i)e^0_i, \mathcal{R}^\perp\right\rangle\Big)^2 - \frac{t^2}{4}K_2(\mathcal{R}) + \mathscr{O}(t^3).\end{aligned}$$

Moreover, as a $2(n-n_0)\times 2(n-n_0)$-matrix, we have

$$\begin{aligned}\Big((\delta_{ij} - 2\left\langle A_{x_0}(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle)\Big)^{-1} &= \big(\delta_{ij} + 2\left\langle A_{x_0}(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle\big)\\ &\quad + 4\Big(\left\langle A_{x_0}(e^0_i)\mathcal{R}^\perp, A_{x_0}(e^0_j)\mathcal{R}^\perp\right\rangle\Big) + \mathscr{O}(|Z|^3).\end{aligned} \tag{5.41}$$

Note that from (3.9), (5.23),

$$[\left\langle A(e^0_i)e^0_i, \mathcal{R}^\perp\right\rangle, \mathscr{L}^0_2] = 2\left\langle A(e^0_i)e^0_i, e^\perp_k\right\rangle \nabla_{0,e^\perp_k}. \tag{5.42}$$

Thus from (5.25), (5.28), (5.35), (5.38)-(5.40), the coefficients of $t$, $t^2$ in the expansion of $g^{ij}(tZ)t\Gamma_{ij}^k(tZ)\nabla_{t,e_k} = tg^{ij}(tZ)\nabla_{t,(\nabla^{TB}_{e_i}e_j)(tZ)}$ are

$$\begin{aligned}(5.43)\quad & \left\langle A(e_i^0)e_i^0, e_k^\perp\right\rangle \nabla_{0,e_k^\perp};\\ & 2\left\langle A(e_i^0)e_j^0, \mathcal{R}^\perp\right\rangle \nabla_{A(e_i^0)e_j^0} + \langle\Gamma_{ii}(\mathcal{R}), e_j\rangle \nabla_{0,e_j} + \frac{2}{3}\left\langle R^{TB}(\mathcal{R}^\perp, e_i^\perp)e_i^\perp, e_j\right\rangle \nabla_{0,e_j}\\ & \quad - \left[\frac{1}{2}\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, \nabla_{A(e_i^0)e_i^0}\right] + \frac{1}{3}(\partial_k R^{L_B})_{x_0} Z_k(\mathcal{R}, A(e_i^0)e_i^0).\end{aligned}$$

By (5.22), (5.28) and (5.38)-(5.43), the coefficient of $t$ in the expansion of $\mathscr{L}_3^t$ is $\mathscr{O}_1'$ in (5.27).

We denote by $[A,B]_+ = AB + BA$.

By (5.22), (5.28), (5.35) and (5.38)-(5.41), the coefficient of $t^2$ in the expansion of $\mathscr{L}_3^t - (g^{ij}t\Gamma_{ij}^k)(tZ)\nabla_{t,e_k}$ is

$$\begin{aligned}(5.44)\quad & I_2 - 2\left\langle A(e_i^0)e_j^0, \mathcal{R}^\perp\right\rangle \Big[\frac{1}{3}\nabla_{0,e_i^0}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_j^0)Z_k\\ & \quad + \frac{1}{3}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i^0)Z_k\nabla_{0,e_j^0} - \frac{1}{2}[\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, \nabla_{0,e_i^0}\nabla_{0,e_j^0}]\Big]\\ & \quad + I_1 + \Big[\frac{1}{2}\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, [\frac{1}{3}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i)Z_k, \nabla_{0,e_i}]_+\Big]\\ & \quad + \Big[\frac{1}{4}K_2(\mathcal{R}) - \frac{3}{8}\Big(\sum_l \left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle\Big)^2, \mathscr{L}_2^0\Big]\\ & \quad - \frac{1}{4}[\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, \mathscr{L}_2^0]\left\langle A(e_k^0)e_k^0, \mathcal{R}^\perp\right\rangle - R^{E_B}_{x_0}(\mathcal{R}, e_i)\nabla_{0,e_i}\\ & \quad - \frac{1}{9}\sum_i\Big[\sum_j(\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i)Z_j\Big]^2 - R^E_{\tau,x_0} + \frac{1}{h_0}(\nabla_{e_j}\nabla_{e_j}h - \nabla_{A(e_i^0)e_i^0}h)_{x_0}.\end{aligned}$$

Here $I_2$ is from the coefficient of $t^2$ in the expansion of $g^{ij}$, the second term is the product of the coefficients of $t^1$ in the expansion of $g^{ij}$ and $\nabla_{t,e_i}\nabla_{t,e_j}$; $I_1$ is from the coefficient of $t^2$ in the expansion of $R^{L_B}$, the fourth term is from the product of the coefficients of $t^1$ in $\kappa^{1/2}, \kappa^{-1/2}$ and in $\kappa^{-1/2}\nabla_{t,e_i}\nabla_{t,e_i}\kappa^{1/2}$ (cf. (5.28)), the fifth and sixth terms are from the coefficients of $t^2$ in the expansions of $\kappa^{1/2}, \kappa^{-1/2}$ and the product of the coefficients of $t^1$ in the expansions of $\kappa^{1/2}$ and $\kappa^{-1/2}$; the seventh term is from $R^{E_B}$, and the eighth term is from the product of the coefficients of $t^1$ in the expansion of $R^{L_B}$.

Certainly,

$$\begin{aligned}(5.45)\quad & \frac{1}{6}\Big[\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, [(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i)Z_k, \nabla_{0,e_i}]_+\Big]\\ & \qquad = -\frac{1}{3}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, A(e_l^0)e_l^0)Z_k.\end{aligned}$$

By (5.42)-(5.45) and by the fact that $A(e_i^0)e_j^0$ is symmetric on $i, j$, we see that the coefficient of $t^2$ in the expansion of $\mathscr{L}_3^t$ is $\mathscr{O}_2'$ in (5.27). ∎

To simplify the notation, we will often denote by $e_i$ the lift $e_i^H$ of $e_i$.

***Lemma 5.5***. — *The following identities hold,*

$$(\partial_i R^{L_B})_{x_0}(\mathcal{R}, e_l) Z_i = -3\sqrt{-1}\pi \left\langle JT(\mathcal{R}, e_l) - JT(\mathcal{R}^0, P^{TX_G} e_l), \mathcal{R}^\perp \right\rangle, \tag{5.46a}$$

$$\begin{aligned}\frac{\sqrt{-1}}{\pi} B(Z, e_l^0) &= \frac{1}{6}\left\langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0)\mathcal{R}^0, e_l^0\right\rangle - \frac{5}{4}\left\langle J\mathcal{R}^\perp, \nabla^{TY}_{\mathcal{R}}(T(e_i, e_l^0))Z_i\right\rangle \\ &+ \frac{1}{2}\left\langle 2\nabla^{TX_G}_{\mathcal{R}^0}(A(e_l^0)e_j^\perp)Z_j^\perp + R^{TB}(\mathcal{R}^\perp, e_l^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)e_l^0, J\mathcal{R}^0\right\rangle \\ &- \frac{1}{2}\left\langle 3\nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)e_j^\perp)Z_i^0 Z_j^\perp + 2R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp, Je_l^0\right\rangle \\ &- \frac{1}{2}\left\langle R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0, Je_l^0\right\rangle \\ &+ \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0 - \frac{1}{4}\mathcal{R}, e_i^0)\right\rangle\left\langle J\mathcal{R}^\perp, T(e_i^0, Je_l^0)\right\rangle \\ &+ \frac{1}{8}\left\langle T(\mathcal{R}^0, \mathcal{R}^\perp), T(e_l^0, J\mathcal{R}^0)\right\rangle + \frac{1}{8}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^\perp, e_l^0)\right\rangle \\ &- \frac{1}{8}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^0), T(\mathcal{R}, e_l^0)\right\rangle + \frac{1}{2}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}, e_l^0)\right\rangle \\ &- \frac{1}{8}\left\langle JT(e_l^0, J\mathcal{R}^0), e_j^\perp\right\rangle\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e_j^\perp)\right\rangle.\end{aligned} \tag{5.46b}$$

*Proof.* — By (1.6), (1.14), (1.18) and (2.16),

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi} R^{L_B}(e_k, e_l) &= \left\langle Je_k^H, e_l^H\right\rangle + \mu(\Theta)(e_k, e_l) \\ &= \left\langle Je_k^H, e_l^H\right\rangle + \langle \widetilde{\mu}, T(e_k, e_l)\rangle.\end{aligned} \tag{5.47}$$

Thus by (3.33), (5.5a), (5.6a) and $\mathbf{J} = J$, we get at $x_0$ the following formulas which will be used in (5.62),

$$\widetilde{\mu}_{x_0} = 0, \quad (\nabla^{TY}_{\mathcal{R}}\widetilde{\mu})_{x_0} = -J\mathcal{R}^\perp, \quad (\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})} = T(\mathcal{R}^\perp, J\mathcal{R}^\perp). \tag{5.48}$$

By (3.36) and $\mu = 0$ on $P$, we have at $x_0$,

$$\begin{aligned}(\nabla_{e_i}\langle\widetilde{\mu}, T(e_k, e_l)\rangle)_{x_0} &= \left\langle \nabla^{TY}_{e_i^H}\widetilde{\mu}, T(e_k, e_l)\right\rangle + \left\langle \widetilde{\mu}, \nabla^{TY}_{e_i^H}(T(e_k, e_l))\right\rangle \\ &= \langle JT(e_k, e_l), e_i\rangle.\end{aligned} \tag{5.49}$$

By (3.40), (5.6a) and (5.32), we have

$$\begin{aligned}(\nabla_{e_i^H}\left\langle Je_k^H, e_l^H\right\rangle)_{x_0} &= \left\langle J\nabla^{TX}_{e_i^H} e_k^H, e_l^H\right\rangle_{x_0} + \left\langle Je_k^H, \nabla^{TX}_{e_i^H} e_l^H\right\rangle_{x_0} \\ &= -\frac{1}{2}\langle JT(e_i, e_k), e_l\rangle - \frac{1}{2}\langle Je_k, T(e_i, e_l)\rangle \\ &\quad + \left\langle JA(P^{TX_G}e_i)P^{N_G}e_k + JA(P^{TX_G}e_k)P^{N_G}e_i, P^{TX_G}e_l\right\rangle \\ &\quad + \left\langle JP^{TX_G}e_k, A(P^{TX_G}e_i)P^{N_G}e_l + A(P^{TX_G}e_l)P^{N_G}e_i\right\rangle.\end{aligned} \tag{5.50}$$

By (5.5a), (5.47), (5.49) and (5.50), for $U \in T_{x_0}B$,

$$
\begin{aligned}
\frac{\sqrt{-1}}{2\pi}(\partial_U R^{L_B})_{x_0}(U, e_l) &= \frac{3}{2}\langle JT(U,e_l), U\rangle - 2\left\langle A(P^{TX_G}U)P^{N_G}U, JP^{TX_G}e_l\right\rangle \\
&\quad + \left\langle JP^{TX_G}U, A(P^{TX_G}U)P^{N_G}e_l + A(P^{TX_G}e_l)P^{N_G}U\right\rangle \\
&= \frac{3}{2}\left\langle JT(U,e_l) - JT(P^{TX_G}U, P^{TX_G}e_l), U\right\rangle .
\end{aligned} \tag{5.51}
$$

Note that $(JTY)_G = N_G$ on $X_G$, by (5.51), we get (5.46a).

By (5.24) and (5.47), one gets at $x_0$,

$$
\frac{\sqrt{-1}}{\pi}B(Z, e_l) = \frac{1}{2}\Big(\nabla\nabla\langle Je_k, e_l\rangle + \nabla\nabla\langle\widetilde{\mu}, T(e_k, e_l)\rangle\Big)_{(\mathcal{R},\mathcal{R})} Z_k. \tag{5.52}
$$

From (5.6a) we have

$$
\begin{aligned}
\Big(\nabla\nabla\left\langle Je_k^H, e_l^H\right\rangle\Big)_{(\mathcal{R},\mathcal{R})} Z_k &= \langle J\mathcal{R}, (\nabla^{TX}\nabla^{TX}e_l^H)_{(\mathcal{R},\mathcal{R})}\rangle \\
&\quad + \langle J(\nabla^{TX}\nabla^{TX}e_k^H)_{(\mathcal{R},\mathcal{R})}, e_l^H\rangle Z_k + 2\langle J\nabla^{TX}_{\mathcal{R}}e_k^H, \nabla^{TX}_{\mathcal{R}}e_l^H\rangle Z_k.
\end{aligned} \tag{5.53}
$$

From (1.2), (5.32), one finds at $x_0$ that

$$
\begin{aligned}
& J\mathcal{R}^\perp \in TY, \quad J\mathcal{R}^0 \in TX_G, \\
& \nabla^{TB}_{\mathcal{R}} e_i^0 = A(e_i^0)\mathcal{R}, \quad \nabla^{TB}_{\mathcal{R}} e_i^\perp = A(\mathcal{R}^0)e_i^\perp, \\
& (\nabla^{T^HX}_{e_j^H} e_i^H) Z_i Z_j = (\nabla^{TB}_{e_j} e_i)^H Z_i Z_j = 2A(\mathcal{R}^0)\mathcal{R}^\perp + A(\mathcal{R}^0)\mathcal{R}^0.
\end{aligned} \tag{5.54}
$$

Now by (3.40),

$$
(\nabla^{TX}_{e_j^H}\nabla^{TX}_{e_i^H}e_k^H)_{x_0} = \nabla^{T^HX}_{e_j^H}\nabla^{T^HX}_{e_i^H}e_k^H - \frac{1}{2}T(e_j^H, \nabla^{T^HX}_{e_i^H}e_k^H) - \frac{1}{2}\nabla^{TX}_{e_j^H}(T(e_i^H, e_k^H)). \tag{5.55}
$$

By (5.34), we get

$$
\begin{aligned}
(\nabla^{TB}\nabla^{TB}e_k)_{(\mathcal{R},\mathcal{R})}Z_k &= \nabla^{TB}_{\mathcal{R}^0}(A(e_j^0)e_i^0)Z_j^0Z_i^0 + 3A(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^\perp \\
&\quad + 3\nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)\mathcal{R}^\perp)Z_i^0 + 2R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0.
\end{aligned} \tag{5.56}
$$

From (5.34), (5.54), (5.55), (5.56), the anti-symmetric property of the torsion tensor $T$ and the fact that $A$ exchanges $TX_G$ and $N_G$, we get

$$
\begin{aligned}
\langle J\mathcal{R}, (\nabla^{TX}\nabla^{TX}e_l^{0,H})_{(\mathcal{R},\mathcal{R})}\rangle &= \left\langle \frac{1}{3}R^{TX_G}(\mathcal{R}^0, e_l^0)\mathcal{R}^0 + \nabla^{TB}_{\mathcal{R}^0}(A(e_j^0)e_l^0)Z_j^0, J\mathcal{R}^0\right\rangle \\
&\quad + \left\langle 2\nabla^{TX_G}_{\mathcal{R}^0}(A(e_l^0)e_j^\perp)Z_j^\perp + R^{TB}(\mathcal{R}^\perp, e_l^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)e_l^0, J\mathcal{R}^0\right\rangle \\
&\quad - \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e_l^0)\mathcal{R})\right\rangle - \frac{1}{2}\left\langle J\mathcal{R}, \nabla^{TX}_{\mathcal{R}}(T(e_i, e_l^0))Z_i\right\rangle, \\
\langle J(\nabla^{TX}\nabla^{TX}e_k^H)_{(\mathcal{R},\mathcal{R})}, e_l^{0,H}\rangle Z_k &= \left\langle 2JR^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + JR^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0, e_l^0\right\rangle \\
&\quad + \left\langle J\nabla^{TB}_{\mathcal{R}^0}(A(e_j^0)e_i^0)Z_j^0Z_i^0 + 3J\nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)e_j^\perp)Z_i^0Z_j^\perp, e_l^0\right\rangle .
\end{aligned} \tag{5.57}
$$

Note that from (1.8), (5.3), (5.5a), (5.54) and $A$ exchanges $TX_G$ and $N_G$,

$$\begin{aligned}(5.58)\qquad \left\langle J\mathcal{R}, \nabla^{TX}_{\mathcal{R}}(T(e_i,e^0_l))Z_i\right\rangle &= \left\langle J\mathcal{R}^\perp, \nabla^{TY}_{\mathcal{R}}(T(e_i,e^0_l))Z_i\right\rangle \\ &\quad + \frac{1}{2}\left\langle T(\mathcal{R}, J\mathcal{R}^0), T(\mathcal{R}, e^0_l)\right\rangle, \\ \left\langle J\nabla^{TB}_{\mathcal{R}^0}(A(e^0_j)e^0_i)Z^0_jZ^0_i, e^0_l\right\rangle &= -\left\langle A(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^0, Je^0_l\right\rangle \\ &= -\frac{1}{4}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^0, e^0_l)\right\rangle, \\ \left\langle \nabla^{TB}_{\mathcal{R}^0}(A(e^0_j)e^0_l), J\mathcal{R}^0\right\rangle &= -\left\langle A(e^0_j)e^0_l, A(\mathcal{R}^0)J\mathcal{R}^0\right\rangle = 0.\end{aligned}$$

By (3.40), (5.6a), (5.13), (5.54) and the fact that $A$ exchanges $TX_G$ and $N_G$, at $x_0$,

$$\begin{aligned}(5.59)\qquad \left\langle J\nabla^{TX}_{\mathcal{R}}e^H_k, \nabla^{TX}_{\mathcal{R}}e^{0,H}_l\right\rangle Z_k &= \left\langle J\nabla^{TB}_{\mathcal{R}}e_k, A(e^0_l)\mathcal{R} - \frac{1}{2}T(\mathcal{R}, e^0_l)\right\rangle Z_k \\ &= \left\langle JA(\mathcal{R}^0)\mathcal{R}^0, -\frac{1}{2}T(\mathcal{R}, e^0_l)\right\rangle + 2\left\langle JA(\mathcal{R}^0)\mathcal{R}^\perp, A(e^0_l)\mathcal{R}^\perp\right\rangle \\ &= \frac{1}{4}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e^0_l)\right\rangle + \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0, e^0_j)\right\rangle\left\langle J\mathcal{R}^\perp, T(e^0_l, Je^0_j)\right\rangle.\end{aligned}$$

By (5.53), (5.57)-(5.59), at $x_0$,

$$\begin{aligned}(5.60)\qquad &\left(\nabla\nabla\left\langle Je^H_k, e^{0,H}_l\right\rangle\right)_{(\mathcal{R},\mathcal{R})} Z_k = \frac{1}{3}\left\langle R^{TX_G}(\mathcal{R}^0, e^0_l)\mathcal{R}^0, J\mathcal{R}^0\right\rangle \\ &+ \left\langle 2\nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_l)e^\perp_j)Z^\perp_j + R^{TB}(\mathcal{R}^\perp, e^0_l)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)e^0_l, J\mathcal{R}^0\right\rangle \\ &- \left\langle 2R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0 + 3\nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_i)e^\perp_j)Z^0_iZ^\perp_j, Je^0_l\right\rangle \\ &- \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e^0_l)\mathcal{R}) + \nabla^{TY}_{\mathcal{R}}(T(e_i, e^0_l))Z_i\right\rangle + \frac{1}{4}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^\perp, e^0_l)\right\rangle \\ &- \frac{1}{4}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^0), T(\mathcal{R}, e^0_l)\right\rangle + \left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0, e^0_j)\right\rangle\left\langle J\mathcal{R}^\perp, T(e^0_l, Je^0_j)\right\rangle.\end{aligned}$$

Observe that $A(e^0_i)\mathcal{R}^0 \in N_G$, $A(e^0_i)\mathcal{R}^\perp \in TX_G$. By (5.5a), (5.5b), (5.5d) and (5.13),

$$\begin{aligned}(5.61)\qquad \left\langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e^0_l)\mathcal{R})\right\rangle &= \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e^0_l)\mathcal{R}^0)\rangle + \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e^0_l)\mathcal{R}^\perp)\rangle \\ &= \frac{1}{2}\left\langle JT(e^0_l, J\mathcal{R}^0), e^\perp_j\right\rangle\left\langle J\mathcal{R}^\perp, T(\mathcal{R}, e^\perp_j)\right\rangle + \left\langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e^0_l)\mathcal{R}^\perp)\right\rangle \\ &= -\frac{1}{2}\left\langle T(e^0_l, J\mathcal{R}^0), T(\mathcal{R}^0, \mathcal{R}^\perp)\right\rangle + \frac{1}{2}\left\langle JT(e^0_l, J\mathcal{R}^0), e^\perp_j\right\rangle\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e^\perp_j)\right\rangle \\ &\quad + \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}, e^0_j)\right\rangle\left\langle J\mathcal{R}^\perp, T(e^0_l, Je^0_j)\right\rangle.\end{aligned}$$

From (5.48), at $x_0$,

$$\begin{aligned}(5.62)\quad &(\nabla\nabla\langle\widetilde{\mu}, T(e_k,e_l)\rangle)_{(\mathcal{R},\mathcal{R})}\\ &= \left\langle(\nabla_{\cdot}^{TY}\nabla_{\cdot}^{TY}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})}, T(e_k,e_l)\right\rangle + 2\left\langle\nabla_{\mathcal{R}}^{TY}\widetilde{\mu}, \nabla_{\mathcal{R}}^{TY}(T(e_k,e_l))\right\rangle\\ &= \left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(e_k,e_l)\right\rangle - 2\left\langle\nabla_{\mathcal{R}}^{TY}(T(e_k,e_l)), J\mathcal{R}^\perp\right\rangle.\end{aligned}$$

Finally, by (5.4), (5.52), (5.60), (5.61) and (5.62), we get (5.46b). □

We now examine the coefficients in the expansion of terms involving the moment map $\widetilde{\mu}$.

Set

$$\begin{aligned}(5.63)\quad \mathscr{O}_2'' = &-\frac{1}{3}\left\langle(\nabla_{\cdot}^{TY}\dot{g}_{\cdot}^{TY})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^\perp, J\mathcal{R}^\perp\right\rangle + \frac{1}{6}\left\langle\nabla_{\mathcal{R}}^{TY}(T(e_j, J_{x_0}e_i^0)), J\mathcal{R}^\perp\right\rangle Z_jZ_i^0\\ &+\frac{1}{3}\left\langle\nabla_{\mathcal{R}^0}^{N_G}(A(e_j^0)e_i^0)Z_j^0Z_i^0 + R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0, \mathcal{R}^\perp\right\rangle\\ &-\frac{1}{12}\sum_l\left\langle T(\mathcal{R},e_l), J\mathcal{R}^\perp\right\rangle^2 + \frac{1}{4}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e_l^0)\right\rangle\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0, e_l^0)\right\rangle\\ &+\frac{7}{12}|T(\mathcal{R}^\perp, J\mathcal{R}^\perp)|^2 + \frac{1}{3}\left\langle T(\mathcal{R}^0, J\mathcal{R}^\perp), T(\mathcal{R}^\perp, J\mathcal{R}^\perp)\right\rangle.\end{aligned}$$

***Lemma 5.6***. — *For $|t|\leqslant 1$, we have*

$$\begin{aligned}(5.64)\quad &|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ) = |Z^\perp|^2 - t\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), J\mathcal{R}^\perp\right\rangle + t^2\mathscr{O}_2'' + \mathscr{O}(t^3),\\ &\left\langle\widetilde{\mu},\widetilde{\mu}^E\right\rangle_{g^{TY}}(tZ) = -t\left\langle J\mathcal{R}^\perp, \widetilde{\mu}^E_{x_0}\right\rangle\\ &\qquad + t^2\Big(\frac{1}{2}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), \widetilde{\mu}^E_{x_0}\right\rangle - \left\langle J\mathcal{R}^\perp, \nabla_{\mathcal{R}}^{TY}\widetilde{\mu}^E\right\rangle_{x_0}\Big) + \mathscr{O}(t^3).\end{aligned}$$

*Proof*. — By (3.36), (3.38), (3.39), (5.6a), (5.54), $\mathbf{J} = J$ and $\widetilde{\mu} = 0$ on $P$, we get, at $x_0$,

$$\begin{aligned}(5.65)\quad (\nabla_{e_k^H}^{TY}\nabla_{e_j^H}^{TY}\nabla_{e_i^H}^{TY}\widetilde{\mu})_{x_0} = &-P^{TY}J\nabla_{e_k^H}^{TX}\nabla_{e_j^H}^{TX}e_i^H - \frac{1}{2}T(e_k^H, P^{T^HX}J\nabla_{e_j^H}^{TX}e_i^H)\\ &-\frac{1}{2}\nabla_{e_k^H}^{TY}(T(e_j^H, P^{T^HX}Je_i^H)) - \frac{1}{2}(\nabla_{e_j^H}^{TY}\dot{g}_{e_i^H}^{TY})(\nabla_{e_k^H}^{TY}\widetilde{\mu})\\ &-\frac{1}{2}(\nabla_{e_k^H}^{TY}\dot{g}_{e_i^H}^{TY})(\nabla_{e_j^H}^{TY}\widetilde{\mu}) - \frac{1}{2}\dot{g}_{e_i^H}^{TY}(\nabla_{e_k^H}^{TY}\nabla_{e_j^H}^{TY}\widetilde{\mu}).\end{aligned}$$

From (3.40), (5.48), (5.54), (5.55), (5.56) and (5.65), we have

$$\begin{aligned}(5.66)\quad &(\nabla_{\cdot}^{TY}\nabla_{\cdot}^{TY}\nabla_{\cdot}^{TY}\widetilde{\mu})_{(\mathcal{R},\mathcal{R},\mathcal{R})} := (\nabla_{e_k^H}^{TY}\nabla_{e_j^H}^{TY}\nabla_{e_i^H}^{TY}\widetilde{\mu})_{x_0}Z_kZ_jZ_i\\ &= -J\nabla_{\mathcal{R}^0}^{N_G}(A(e_j^0)e_i^0)Z_j^0Z_i^0 - 3JA(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^\perp - 2P^{TY}JR^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^\perp\\ &\quad - P^{TY}JR^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0 - T(\mathcal{R}, JA(\mathcal{R}^0)\mathcal{R}^\perp)\\ &\quad -\frac{1}{2}\nabla_{\mathcal{R}}^{TY}(T(e_j^H, P^{T^HX}Je_i^H))Z_jZ_i + (\nabla_{\cdot}^{TY}\dot{g}_{\cdot}^{TY})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^\perp - \frac{1}{2}\dot{g}_{\mathcal{R}}^{TY}(T(\mathcal{R}^\perp, J\mathcal{R}^\perp)).\end{aligned}$$

Now by (3.50), (5.48), and $\widetilde{\mu}=0$ on $P$, we have

$$(5.67)\quad |\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ)=\sum_{k=2}^{4}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big(|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big)|_{t=0}\,t^{k-2}+\mathscr{O}(t^3)$$
$$=|\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}|^2_{x_0}+t\left\langle(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})},\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}\right\rangle_{x_0}$$
$$+\frac{t^2}{4!}\Big(8\left\langle(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R},\mathcal{R})},\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}\right\rangle_{x_0}+6|(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})}|^2_{x_0}\Big)+\mathscr{O}(t^3).$$

By (5.5c),

$$(5.68)\qquad T(\mathcal{R}^0,J\mathcal{R}^\perp)=\frac{1}{2}T(\mathcal{R}^\perp,J\mathcal{R}^0).$$

From (1.6), (5.13), (5.48), (5.66), (5.67) and (5.68), we get the coefficients of $t^0,t^1$ in the expansion of $|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ)$ in (5.64), and the coefficient of $t^2$ is

$$(5.69)\quad \frac{1}{3}\Big\langle J\nabla^{N_G}_{\mathcal{R}^0}(A(e^0_j)e^0_i)Z^0_jZ^0_i+3JA(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^\perp+JR^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0,J\mathcal{R}^\perp\Big\rangle$$
$$+\frac{1}{3}\left\langle 2JR^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^\perp+T(\mathcal{R},JA(\mathcal{R}^0)\mathcal{R}^\perp),J\mathcal{R}^\perp\right\rangle$$
$$-\frac{1}{3}\left\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^\perp,J\mathcal{R}^\perp\right\rangle+\frac{1}{6}\Big\langle\nabla^{TY}_{\mathcal{R}}(T(e^H_j,P^{T^HX}Je^H_i))Z_jZ_i,J\mathcal{R}^\perp\Big\rangle$$
$$+\frac{1}{3}\left\langle T(\mathcal{R},J\mathcal{R}^\perp),T(\mathcal{R}^\perp,J\mathcal{R}^\perp)\right\rangle+\frac{1}{4}\Big|T(\mathcal{R}^\perp,J\mathcal{R}^\perp)\Big|^2$$
$$=-\frac{1}{3}\left\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^\perp,J\mathcal{R}^\perp\right\rangle+\frac{1}{6}\Big\langle\nabla^{TY}_{\mathcal{R}}(T(e^H_j,P^{T^HX}Je^H_i))Z_jZ_i,J\mathcal{R}^\perp\Big\rangle$$
$$+\frac{1}{3}\Big\langle\nabla^{N_G}_{\mathcal{R}^0}(A(e^0_j)e^0_i)Z^0_jZ^0_i+R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0,\mathcal{R}^\perp\Big\rangle$$
$$-\frac{1}{4}\sum_j\left\langle T(\mathcal{R}^0,e^0_j),J\mathcal{R}^\perp\right\rangle^2+\frac{1}{6}\left\langle T(\mathcal{R},e^0_j),J\mathcal{R}^\perp\right\rangle\left\langle T(\mathcal{R}^0,e^0_j),J\mathcal{R}^\perp\right\rangle$$
$$+\frac{7}{12}\Big|T(\mathcal{R}^\perp,J\mathcal{R}^\perp)\Big|^2+\frac{1}{3}\left\langle T(\mathcal{R}^0,J\mathcal{R}^\perp),T(\mathcal{R}^\perp,J\mathcal{R}^\perp)\right\rangle.$$

To get (5.64) from (5.69), we need to compute $\nabla^{TY}_{e^H_k}(T(e^H_j,P^{T^HX}Je^H_i))$.

For $W$ a section of $TX$, $U$ a section of $TB$, we have by (1.7),

$$(5.70)\quad \Big\langle\nabla^{T^HX}_{e^H_k}P^{T^HX}W,U^H\Big\rangle=e^H_k\left\langle W,U^H\right\rangle-\Big\langle P^{T^HX}W,\nabla^{TX}_{e^H_k}U^H\Big\rangle$$
$$=\Big\langle P^{T^HX}\nabla^{TX}_{e^H_k}W,U^H\Big\rangle+\Big\langle P^{TY}W,\nabla^{TX}_{e^H_k}U^H\Big\rangle.$$

From (1.7), (5.70), we get at $x_0$,

$$(5.71)\qquad \nabla^{T^HX}_{e^H_k}P^{T^HX}W=P^{T^HX}\nabla^{TX}_{e^H_k}W-\frac{1}{2}\left\langle T(e^H_k,e^H_l),P^{TY}W\right\rangle e^H_l.$$

Remark that $Je^{\perp,H}_i\in TY, Je^0_i\in T^HX$ only hold on $P$.

From (3.40), (5.5b), (5.6a), (5.13), (5.32) and (5.71),

(5.72)
$$\begin{aligned}(\nabla^{T^HX}_{e_k^H} P^{T^HX} J e_i^{\perp,H})_{x_0} &= JA(P^{TX_G} e_k) e_i^{\perp} - \frac{1}{2} JT(e_k, e_i^{\perp}) - \frac{1}{2}\left\langle T(e_k, e_l), J e_i^{\perp}\right\rangle e_l \\ &= -\frac{1}{2} JT(e_k, e_i^{\perp}) - \frac{1}{2}\left\langle T(e_k, e_l) - T(P^{TX_G} e_k, P^{TX_G} e_l), J e_i^{\perp}\right\rangle e_l,\\ (\nabla^{T^HX}_{e_k} P^{T^HX} J e_i^0)_{x_0} &= P^{T^HX} J \nabla^{TX}_{e_k^H} e_i^{0,H} = JA(e_i^0) P^{N_G} e_k - \frac{1}{2} JT(e_k, e_i^0)\\ &= -\frac{1}{2} JT(e_k, e_i^0) + \frac{1}{2}\left\langle J P^{N_G} e_k, T(e_i^0, e_l^0)\right\rangle e_l^0,\\ (\nabla^{TB}_{e_k} J_{x_0} e_i^0)_{x_0} &= A(J_{x_0} e_i^0) e_k = -\frac{1}{2} JT(P^{TX_G} e_k, e_i^0) + \frac{1}{2}\left\langle J P^{N_G} e_k, T(e_i^0, e_l^0)\right\rangle e_l^0.\end{aligned}$$

From (5.72), we get at $x_0$ that

$$\begin{aligned}&\left\langle \nabla^{TY}_{\mathcal{R}} (T(e_j^H, P^{T^HX} J e_i^H)) Z_j Z_i, J\mathcal{R}^{\perp}\right\rangle - \left\langle \nabla^{TY}_{\mathcal{R}} (T(e_j, J_{x_0} e_i^0)) Z_j Z_i^0, J\mathcal{R}^{\perp}\right\rangle \\ &\quad = \left\langle T(e_j, \nabla^{T^HX}_{\mathcal{R}} P^{T^HX} J e_i^H - \nabla^{T^HX}_{\mathcal{R}} (J_{x_0} P^{TX_G} e_i)^H) Z_j Z_i, J\mathcal{R}^{\perp}\right\rangle \\ &= \left\langle T\left(\mathcal{R}, -\frac{1}{2} JT(\mathcal{R}, \mathcal{R}^{\perp}) - \frac{1}{2}\left\langle T(\mathcal{R}, e_l) - T(\mathcal{R}^0, P^{TX_G} e_l), J\mathcal{R}^{\perp}\right\rangle e_l\right), J\mathcal{R}^{\perp}\right\rangle \\ &\quad + \left\langle T\left(e_j, -\frac{1}{2} JT(e_k, e_i^0) + \frac{1}{2} JT(P^{TX_G} e_k, e_i^0)\right) Z_k Z_j Z_i^0, J\mathcal{R}^{\perp}\right\rangle \\ &\quad = -\frac{1}{2}\left\langle T\left(\mathcal{R}, \left\langle T(\mathcal{R}, e_l) - T(\mathcal{R}^0, P^{TX_G} e_l), J\mathcal{R}^{\perp}\right\rangle e_l\right), J\mathcal{R}^{\perp}\right\rangle \\ &\quad = -\frac{1}{2}\sum_l \left\langle T(\mathcal{R}, e_l), J\mathcal{R}^{\perp}\right\rangle^2 + \frac{1}{2}\left\langle T(\mathcal{R}, e_l^0), J\mathcal{R}^{\perp}\right\rangle \left\langle T(\mathcal{R}^0, e_l^0), J\mathcal{R}^{\perp}\right\rangle. \end{aligned} \tag{5.73}$$

From (5.69) and (5.73), $\mathcal{O}_2''$ is the coefficient of $t^2$ in the expansion of $|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ)$. By (5.48), we get also the second equation of (5.64).
The proof of Lemma 5.6 is complete. □

The following is the main result of this Section.

***Theorem 5.7.*** — *The following identities hold,*

$$\begin{aligned}\mathcal{O}_1 =& 2\pi\sqrt{-1}\left\langle JT(\mathcal{R}^{\perp}, e_i^0), \mathcal{R}^{\perp}\right\rangle \nabla_{0,e_i^0} + 2\pi\sqrt{-1}\left\langle JT(\mathcal{R}, e_i^{\perp}), \mathcal{R}^{\perp}\right\rangle \nabla_{0,e_i^{\perp}} \\ &+ \pi\sqrt{-1}\left\langle JT(\mathcal{R}^0, e_i^{\perp}), e_i^{\perp}\right\rangle - \left\langle JT(e_i^0, Je_j^0), \mathcal{R}^{\perp}\right\rangle \nabla_{0,e_i^0}\nabla_{0,e_j^0} \\ &+ 4\pi^2\left\langle JT(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}), \mathcal{R}^{\perp}\right\rangle + 4\pi\sqrt{-1}\left\langle J\mathcal{R}^{\perp}, \widetilde{\mu}^E_{x_0}\right\rangle, \\ \mathcal{O}_2 =& \mathcal{O}_2' + 4\pi^2\mathcal{O}_2'' - 4\pi\sqrt{-1}\Big(\frac{1}{2}\left\langle T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}), \widetilde{\mu}^E_{x_0}\right\rangle - \left\langle J\mathcal{R}^{\perp}, \nabla^{TY}_{\mathcal{R}}\widetilde{\mu}^E\right\rangle\Big) \\ &- \left\langle \widetilde{\mu}^E_{x_0}, \widetilde{\mu}^E_{x_0}\right\rangle_{g^{TY}}.\end{aligned} \tag{5.74}$$

*Proof.* — By (5.5e), at $x_0$

$$\left\langle JT(\mathcal{R}, e_i), e_i\right\rangle = \left\langle JT(\mathcal{R}^0, e_i^{\perp}), e_i^{\perp}\right\rangle. \tag{5.75}$$

By (5.46a), (5.51) and (5.75),

$$\begin{aligned}
&-\frac{2}{3}(\partial_{\mathcal{R}}R^{L_B})_{x_0}(\mathcal{R},e_i)\nabla_{0,e_i}\\
&\qquad = 2\pi\sqrt{-1}\Big(\left\langle JT(\mathcal{R}^\perp,e_i^0),\mathcal{R}^\perp\right\rangle\nabla_{0,e_i^0}+\left\langle JT(\mathcal{R},e_i^\perp),\mathcal{R}^\perp\right\rangle\nabla_{0,e_i^\perp}\Big),\\
&-\frac{1}{3}(\partial_iR^{L_B})_{x_0}(\mathcal{R},e_i)=\pi\sqrt{-1}\left\langle JT(\mathcal{R}^0,e_i^\perp),e_i^\perp\right\rangle.
\end{aligned}\tag{5.76}$$

From (5.5a), (5.23), (5.27), (5.64) and (5.76), we get (5.74). □

## 5.3. Computation of the coefficient $\Phi_1$

Recall that the operator $\mathscr{L}_2^0$ is defined in (5.23), $P_{\mathscr{L}^\perp}$ is the orthogonal projection from $L^2(\mathbb{R}^{n_0})$ onto $\operatorname{Ker}\mathscr{L}^\perp$ and $P_{\mathscr{L}}$ is the orthogonal projection from $L^2(\mathbb{R}^{2n-2n_0})$ onto $\operatorname{Ker}\mathscr{L}$ as in (3.19).

For $Z^\perp\in\mathbb{R}^{n_0}$, set

$$\begin{aligned}
\Psi_{1,1}(Z^\perp)&=\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1P^N\Big)\left((0,Z^\perp),(0,Z^\perp)\right),\\
\Psi_{1,2}(Z^\perp)&=-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_2P^N\Big)\left((0,Z^\perp),(0,Z^\perp)\right),\\
\Psi_{1,3}(Z^\perp)&=\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1P^N\mathscr{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big)\left((0,Z^\perp),(0,Z^\perp)\right),\\
\Psi_{1,4}(Z^\perp)&=\Big(P^N\mathscr{O}_1(\mathscr{L}_2^0)^{-2}P^{N^\perp}\mathscr{O}_1P^N\Big)\left((0,Z^\perp),(0,Z^\perp)\right),\\
\widetilde{\Psi}_{1,1}(Z^\perp)&=\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\mathscr{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1P^N\Big)\left((0,Z^\perp),(0,Z^\perp)\right),\\
\widetilde{\Psi}_{1,2}(Z^\perp)&=-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\mathscr{O}_2P^N\Big)\left((0,Z^\perp),(0,Z^\perp)\right),\\
\Phi_{1,i}&=\int_{\mathbb{R}^{n_0}}\Psi_{1,i}(Z^\perp)dv_{N_G}(Z^\perp),\quad\text{for } i=1,2,3,4.
\end{aligned}\tag{5.77}$$

***Proposition 5.8***. — *The following two identities hold for $i=1,2$,*

$$\int_{\mathbb{R}^{n_0}}\widetilde{\Psi}_{1,i}(Z^\perp)dv_{N_G}(Z^\perp)=\Phi_{1,i}.\tag{5.78}$$

*Proof.* — In fact, in our case, by (3.21), $P^N=P_{\mathscr{L}}\otimes P_{\mathscr{L}^\perp}\otimes\operatorname{Id}_E$.

By (3.18) and (3.19),

$$\begin{aligned}
&\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_2P^N\Big)(Z,(0,Z'^\perp))\\
&\qquad=\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_2P_{\mathscr{L}}(\cdot,0)G^\perp\Big)(Z)G^\perp(Z'^\perp).
\end{aligned}\tag{5.79}$$

From Theorem 3.1 and (5.79),

$$\begin{aligned}\Phi_{1,2} &= \Big\langle \Big(-(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_2 P_{\mathscr{L}}(\cdot,0)G^\perp\Big)(0,Z^\perp), G^\perp(Z^\perp)\Big\rangle_{L^2(\mathbb{R}^{n_0})} \\ &= \Big\langle \Big(-(\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\mathscr{O}_2 P_{\mathscr{L}}(\cdot,0)G^\perp\Big)(0,Z^\perp), G^\perp(Z^\perp)\Big\rangle_{L^2(\mathbb{R}^{n_0})} \\ &= \int_{\mathbb{R}^{n_0}} \widetilde{\Psi}_{1,2}(Z^\perp)dv_{N_G}(Z^\perp).\end{aligned} \tag{5.80}$$

In the same way, we get (5.78) for $i=1$. □

Note that the restriction of $\|\cdot\|_{t,0}$ in (2.114) on $\mathscr{C}^\infty(\mathbb{R}^{2n-n_0}, E_{G,x_0})$ does not depend on $t$ and we denote it by $\|\cdot\|_0$.

Since $\mathscr{L}_2^t$ in (5.23) is a self-adjoint elliptic operator with respect to $\|\cdot\|_0$ as we conjugated the operator with $\kappa^{1/2}$, $\mathscr{L}_2^0$ and $\mathscr{O}_r$ are also formally self-adjoint with respect to $\|\cdot\|_0$. Thus in the right hand side of (3.62), the third and fourth terms are the adjoints of the first two terms.

From (3.62), (5.1) and (5.77), we get

$$\Phi_1 = \Phi_{1,1} + \Phi_{1,2} + (\Phi_{1,1}+\Phi_{1,2})^* + \Phi_{1,3} - \Phi_{1,4}. \tag{5.81}$$

From (5.77), (5.78), (5.81), we learn that in order to compute $\Phi_1$, we only need to evaluate $\widetilde{\Psi}_{1,1}$, $\widetilde{\Psi}_{1,2}$, $\Phi_{1,3}$ and $\Phi_{1,4}$.

***Lemma 5.9***. — *The following identity holds,*

$$\widetilde{\Psi}_{1,1}(Z^\perp) = -\frac{1}{8\pi}\Big|T(\tfrac{\partial}{\partial \overline{z}_j^0}, e_k^\perp)\Big|^2 P_{\mathscr{L}^\perp}(Z^\perp, Z^\perp). \tag{5.82}$$

*Proof*. — Recall that the operators $b_i$, $b_i^+$, $b_j^\perp$ and $b_j^{\perp +}$ have been defined in (3.8). In particular, by (5.15), one has for $f \in T_{x_0}^* X_G$,

$$\begin{aligned}&4\pi Z_j^\perp = b_j^\perp + b_j^{\perp+}, \quad \nabla_{0,e_j^\perp} = \tfrac{\partial}{\partial Z_j^\perp} = \tfrac{1}{2}(b_j^{\perp+} - b_j^\perp), \\ &f(e_i^0)\nabla_{0,e_i^0} = -f(\tfrac{\partial}{\partial \overline{z}_i^0})b_i + f(\tfrac{\partial}{\partial z_i^0})b_i^+.\end{aligned} \tag{5.83}$$

By (3.8), (3.9) and (5.83), set

$$\begin{aligned}B_{jk}^\perp &= (4\pi)^2 Z_j^\perp Z_k^\perp = b_j^{\perp+}b_k^{\perp+} + b_k^\perp b_j^{\perp+} + b_j^\perp b_k^{\perp+} + b_j^\perp b_k^\perp + 4\pi\delta_{jk}, \\ B_{ijk}^\perp &= b_i^\perp b_j^\perp b_k^\perp + 3b_i^\perp b_j^\perp b_k^{\perp+} + 3b_i^\perp b_j^{\perp+} b_k^{\perp+} + b_i^{\perp+} b_j^{\perp+} b_k^{\perp+}.\end{aligned} \tag{5.84}$$

If $a_{ijk}$ is symmetric on $i,j,k$, then by (3.8), (3.9), (5.83) and (5.84), one verifies

$$a_{ijk}(4\pi)^3 Z_i^\perp Z_j^\perp Z_k^\perp = a_{ijk}B_{ijk}^\perp + 12\pi a_{ijj}(b_i^\perp + b_i^{\perp+}). \tag{5.85}$$

By (3.9), (5.5e), (5.14), (5.83), (5.84) and the fact that $T(\,,\,)$ is anti-symmetric, we get

$$(5.86)\quad 2\pi\left\langle JT(\mathcal{R}^\perp, e_i^\perp), \mathcal{R}^\perp\right\rangle \nabla_{0,e_i^\perp} = \frac{1}{16\pi}\widetilde{\mathcal{T}}_{jik}B_{jk}^\perp(b_i^{\perp+} - b_i^\perp)$$
$$= \frac{1}{16\pi}\widetilde{\mathcal{T}}_{jik}\left[(b_j^\perp b_k^{\perp+} + b_j^\perp b_k^\perp)b_i^{\perp+} - (b_j^{\perp+}b_k^{\perp+} + b_k^\perp b_j^{\perp+} + b_j^\perp b_k^{\perp+})b_i^\perp\right]$$
$$= -\frac{1}{8\pi}\widetilde{\mathcal{T}}_{ijk}(b_j^\perp b_k^{\perp+} + b_j^\perp b_k^\perp)b_i^{\perp+}.$$

By Theorem 5.1, Remark 5.2, (3.9), (3.12), (5.14), (5.74), (5.84)-(5.86), we can reformulate $\mathcal{O}_1$ as follows by using the creation and annihilation operators introduced in (3.8),

$$(5.87)\quad \mathcal{O}_1 = -\frac{\sqrt{-1}}{8\pi}\left\langle JT(\tfrac{\partial}{\partial z_i^0}, e_j^\perp), e_k^\perp\right\rangle B_{jk}^\perp b_i^+ + b_i\frac{\sqrt{-1}}{8\pi}\left\langle JT(\tfrac{\partial}{\partial \overline{z}_i^0}, e_j^\perp), e_k^\perp\right\rangle B_{jk}^\perp$$
$$+ \frac{\sqrt{-1}}{4}\left\langle JT(\mathcal{R}^0, e_i^\perp), e_j^\perp\right\rangle (b_i^{\perp+}b_j^{\perp+} - b_i^\perp b_j^\perp) - \frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijk}(b_j^\perp b_k^{\perp+} + b_j^\perp b_k^\perp)b_i^{\perp+}$$
$$- \frac{\sqrt{-1}}{4\pi}\left\langle JT(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}), e_k^\perp\right\rangle (b_k^{\perp+} + b_k^\perp)(2b_jb_i^+ + 4\pi\delta_{ij}) + \sqrt{-1}\left\langle Je_j^\perp, \widetilde{\mu}_{x_0}^E\right\rangle (b_j^{\perp+} + b_j^\perp)$$
$$+ \frac{1}{16\pi}\left\langle JT(e_i^\perp, Je_j^\perp), e_k^\perp\right\rangle [B_{ijk}^\perp + 12\pi\delta_{ik}(b_j^{\perp+} + b_j^\perp)]$$
$$= -\frac{\sqrt{-1}}{8\pi}\mathcal{T}_{jk}(\tfrac{\partial}{\partial z_i^0})B_{jk}^\perp b_i^+ + \frac{\sqrt{-1}}{8\pi}\mathcal{T}_{jk}(\tfrac{\partial}{\partial \overline{z}_i^0})b_iB_{jk}^\perp + \frac{\sqrt{-1}}{4}\mathcal{T}_{ij}(\mathcal{R}^0)(b_i^{\perp+}b_j^{\perp+} - b_i^\perp b_j^\perp)$$
$$+ \sqrt{-1}\left\langle Je_j^\perp, \widetilde{\mu}_{x_0}^E\right\rangle (b_j^{\perp+} + b_j^\perp) - \frac{\sqrt{-1}}{4\pi}\left\langle JT(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}), e_k^\perp\right\rangle (b_k^{\perp+} + b_k^\perp)(2b_jb_i^+ + 4\pi\delta_{ij})$$
$$- \frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijk}(b_j^\perp b_k^{\perp+} + b_j^\perp b_k^\perp)b_i^{\perp+} + \frac{1}{16\pi}\mathcal{T}_{ijk}[B_{ijk}^\perp + 12\pi\delta_{ik}(b_j^{\perp+} + b_j^\perp)].$$

From Theorem 3.1, (3.54), (5.84), (5.87) and $a_i = a_i^+ = 2\pi$, we get

$$(5.88)\quad \left((\mathscr{L}_2^0)^{-1}\mathcal{O}_1P^N\right)(Z, Z') = \sqrt{-1}\Big\{\frac{b_l}{8\pi}\mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_l^0}) + \left\langle Je_k^\perp, \widetilde{\mu}_{x_0}^E\right\rangle \frac{b_k^\perp}{4\pi}$$
$$- \left\langle JT(\tfrac{\partial}{\partial z_l^0}, \tfrac{\partial}{\partial \overline{z}_l^0}), e_k^\perp\right\rangle \frac{b_k^\perp}{4\pi} - \frac{b_l^\perp b_k^\perp}{32\pi}\mathcal{T}_{kl}(z^0 + \overline{z}'^0)$$
$$- \frac{\sqrt{-1}}{16\pi}\mathcal{T}_{klm}\Big[\frac{b_m^\perp b_l^\perp b_k^\perp}{12\pi} + 3b_k^\perp\delta_{lm}\Big]\Big\}P^N(Z, Z').$$

By Theorem 3.1, (3.55), (5.84) and (5.87),

$$
\begin{aligned}
(5.89)\quad P^{N^\perp} P_{\mathscr{L}^\perp}\mathcal{O}_1 = \sqrt{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\Big\{ & -\frac{1}{2}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_i^0})b_i^+ + \frac{1}{2}\mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0})b_i \\
& + \langle Je_j^\perp, \widetilde{\mu}_{x_0}^E\rangle b_j^{\perp+} - \frac{1}{4\pi}\Big\langle JT(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}), e_{j'}^\perp\Big\rangle b_{j'}^{\perp+}(2b_jb_i^+ + 4\pi\delta_{ij}) \\
& + \frac{1}{4}\Big(\mathcal{T}_{jj'}(\mathcal{R}^0) - \mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_i^0})\frac{b_i^+}{2\pi} + \mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})\frac{b_i}{2\pi}\Big)b_j^{\perp+}b_{j'}^{\perp+} \\
& - \frac{\sqrt{-1}}{16\pi}\mathcal{T}_{ijj'}[b_i^{\perp+}b_j^{\perp+}b_{j'}^{\perp+} + 12\pi\delta_{ij'}b_j^{\perp+}].
\end{aligned}
$$

In the following equation, by (3.9), (3.54), (3.55), we only need to pair the terms in (5.88) and (5.89) which have the same length on $b_j^{\perp+}$ and $b_j^\perp$, and the total degree on $b_i, b_i^+, z^0, \overline{z}^0$ should not be zero. Thus by (3.9), (3.54), (5.88) and (5.89),

(5.90)

$$
\begin{aligned}
\Big(P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}\mathcal{O}_1P^N\Big)(Z,(0,Z'^\perp)) = \Big\{ & P^{N^\perp}\Big[-\frac{1}{16\pi}\Big(\sum_{ij} b_i\mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big)^2 \\
& + \frac{1}{128\pi}\Big(\mathcal{T}_{jj'}(\mathcal{R}^0) + \frac{b_i}{2\pi}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big)b_j^{\perp+}b_{j'}^{\perp+}\cdot b_l^\perp b_k^\perp\mathcal{T}_{kl}(z^0)\Big]P^N\Big\}(Z,(0,Z'^\perp)).
\end{aligned}
$$

From (3.9), (3.54), (5.5d), (5.14), (5.90) and $a_i = a_i^+ = 2\pi$, one gets

(5.91)

$$
\begin{aligned}
\Big(P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}\mathcal{O}_1P^N\Big)(Z,(0,Z'^\perp)) = \Big\{ & P^{N^\perp}\Big[-\frac{1}{16\pi}\Big(\sum_{ij} b_i\mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big)^2 \\
& + \frac{1}{8}\Big\langle 2\pi JT(\mathcal{R}^0, e_l^\perp) + b_iJT(\tfrac{\partial}{\partial \overline{z}_i^0}, e_l^\perp), JT(z^0, e_l^\perp)\Big\rangle\Big]P^N\Big\}(Z,(0,Z'^\perp)).
\end{aligned}
$$

Set $P_{\mathscr{L}}^\perp = \mathrm{Id}_{L^2(\mathbb{R}^{2n-2n_0})} - P_{\mathscr{L}}$.

Let $h_i(Z^0)$ (resp. $F(Z^0)$) be polynomials in $Z^0$ with degree 1 (resp. 2) and $a_{ij} \in \mathbb{C}$.

By Theorem 3.1, (3.9) and (3.54),

$$
\begin{aligned}
(5.92)\quad & \big(F(Z^0)P_{\mathscr{L}}\big)(Z^0,0) \\
& = \Big(\frac{1}{2}\frac{\partial^2 F}{\partial z_i^0\partial z_j^0}z_i^0z_j^0 + \frac{\partial^2 F}{\partial z_i^0\partial \overline{z}_j^0}z_i^0\frac{b_j}{a_j} + \frac{1}{2}\frac{\partial^2 F}{\partial \overline{z}_i^0\partial \overline{z}_j^0}\frac{b_ib_j}{a_ia_j}\Big)P_{\mathscr{L}}(Z^0,0).
\end{aligned}
$$

By Theorem 3.1, (3.8), (3.9), (3.19), (3.54), (5.92) and $a_j = 2\pi$, we have

$$
\begin{aligned}
&(P^{\perp}_{\mathscr{L}} F P_{\mathscr{L}})(0,0) = -\frac{1}{\pi}\frac{\partial^2 F}{\partial z_i^0 \partial \overline{z}_i^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} a_{ij} b_i b_j P_{\mathscr{L}}\right)(0,0) = \left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} h_i P_{\mathscr{L}}\right)(0,0) = 0,\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} h_i b_i P_{\mathscr{L}}\right)(0,0) = \left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} b_i h_i P_{\mathscr{L}}\right)(0,0) = -\frac{1}{2\pi}\frac{\partial h_i}{\partial z_i^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} F P_{\mathscr{L}}\right)(0,0) = -\frac{1}{4\pi^2}\frac{\partial^2 F}{\partial z_i^0 \partial \overline{z}_i^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} b_i F b_j P_{\mathscr{L}}\right)(0,0) = -\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} b_i b_j F P_{\mathscr{L}}\right)(0,0) = -\frac{1}{2\pi}\frac{\partial^2 F}{\partial z_i^0 \partial z_j^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} F b_i b_j P_{\mathscr{L}}\right)(0,0) = -\frac{3}{2\pi}\frac{\partial^2 F}{\partial z_i^0 \partial z_j^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} \Big(\sum_i b_i h_i\Big)^2 P_{\mathscr{L}}\right)(0,0) = -\frac{1}{2\pi}\Big(\frac{\partial h_i}{\partial z_j^0}\frac{\partial h_j}{\partial z_i^0} - \Big(\sum_i \frac{\partial h_i}{\partial z_i^0}\Big)^2\Big).
\end{aligned}
\tag{5.93}
$$

Finally by (5.78), (5.91), (5.93) and $\mathscr{L}_2^0 = \mathscr{L} + \mathscr{L}^{\perp}$ , we get (5.82). ∎

***Lemma 5.10***. — *The following identity holds,*

$$
\Phi_{1,3} = \Phi_{1,4}. \tag{5.94}
$$

*Proof.* — Let $\mathcal{F}_2 \in T^*_{x_0} X_G$ with values in real polynomials on $Z^{\perp}$ with even degree, $\mathcal{F}_1 \in N^*_{G,x_0} \otimes \operatorname{End}(E_{G,x_0})$, $\mathcal{F}_3(Z^{\perp})$ a polynomial on $Z^{\perp}$ with odd degree, be defined by

$$
\begin{aligned}
&\mathcal{F}_1(e_k^{\perp}) = \sqrt{-1}\left\langle J e_k^{\perp}, \widetilde{\mu}^E_{x_0}\right\rangle - \sqrt{-1}\left\langle JT(\tfrac{\partial}{\partial z_l^0}, \tfrac{\partial}{\partial \overline{z}_l^0}), e_k^{\perp}\right\rangle + \frac{3}{4}\mathcal{T}_{llk},\\
&\mathcal{F}_2(\cdot, Z^{\perp}) P^N(Z,Z') = \Big(\mathcal{T}_{kl}(\cdot)\frac{b_l^{\perp} b_k^{\perp}}{32\pi} P^N\Big)(Z,Z'),\\
&\mathcal{F}_3(Z^{\perp}) P^N(Z,Z') = \frac{1}{16\pi}\Big(\mathcal{T}_{klm}\frac{b_m^{\perp} b_l^{\perp} b_k^{\perp}}{12\pi} P^N\Big)(Z,Z').
\end{aligned}
\tag{5.95}
$$

Then from (3.54), (5.88) and (5.95),

$$
\begin{aligned}
\left((\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N\right)(Z,Z') = \Big(&\frac{\sqrt{-1}}{4}\mathcal{T}_{kk}(\overline{z}^0 - \overline{z}'^0) - \sqrt{-1}\mathcal{F}_2(z^0 + \overline{z}'^0, Z^{\perp})\\
&+ \Big(\mathcal{F}_1 + \mathcal{F}_3\Big)(Z^{\perp})\Big) P^N(Z,Z').
\end{aligned}
\tag{5.96}
$$

Observe that $\mathcal{F}_i(Z^{\perp})^* = \mathcal{F}_i(Z^{\perp})$ for $i = 1, 3$, thus from (5.96),

$$
\begin{aligned}
\left(P^N \mathcal{O}_1 (\mathscr{L}_2^0)^{-1}\right)(Z',Z) &= \left(\left((\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N\right)(Z,Z')\right)^*\\
= \Big(-\frac{\sqrt{-1}}{4}\mathcal{T}_{kk}(z^0 - z'^0) &+ \sqrt{-1}\mathcal{F}_2(\overline{z}^0 + z'^0, Z^{\perp}) + \Big(\mathcal{F}_1 + \mathcal{F}_3\Big)(Z^{\perp})\Big) P^N(Z',Z).
\end{aligned}
\tag{5.97}
$$

For $h_1(z^0), h_2(\overline{z}^0)$ two linear functions on $z^0, \overline{z}^0$, by Theorem 3.1, (3.9) and (3.54),

$$(P_{\mathscr{L}} h_1(z^0)h_2(\overline{z}^0)P_{\mathscr{L}})(0,0) = \Big(P_{\mathscr{L}} h_1(z^0)\frac{\partial h_2}{\partial \overline{z}_i^0}\frac{b_i}{2\pi}P_{\mathscr{L}}\Big)(0,0) = \frac{1}{\pi}\frac{\partial h_1}{\partial z_i^0}\frac{\partial h_2}{\partial \overline{z}_i^0}. \tag{5.98}$$

From (3.19), (5.77) and (5.96)-(5.98),

$$\Psi_{1,3}(Z^\perp) = \Big[\Big((\mathcal{F}_1+\mathcal{F}_3)(Z^\perp)\Big)^2 + \frac{1}{\pi}\Big|\frac{1}{4}\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_i^0}) + \mathcal{F}_2(\tfrac{\partial}{\partial \overline{z}_i^0}, Z^\perp)\Big|^2\Big] G^\perp(Z^\perp)^2. \tag{5.99}$$

By Theorem 3.1, (3.18), (5.95), $\mathcal{F}_j G^\perp$ $(j=1,3)$, $\mathcal{F}_2(\frac{\partial}{\partial \overline{z}_i^0}, \cdot)G^\perp$ are eigenfunctions of $\mathscr{L}^\perp$ with eigenvalues $4\pi j$, $8\pi$, thus they are orthogonal to each other.

From (5.77), (5.96)-(5.98), we have

$$\begin{aligned}\Psi_{1,4}(Z^\perp) = G^\perp(Z^\perp)^2\int_{\mathbb{R}^{n_0}} \Big\{\Big((\mathcal{F}_1 G^\perp)(Z'^\perp)\Big)^2 + \big((\mathcal{F}_3 G^\perp)(Z'^\perp)\big)^2 \\ + \frac{1}{16\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_i^0})G^\perp\Big|^2(Z'^\perp) + \frac{1}{\pi}\Big|\mathcal{F}_2(\tfrac{\partial}{\partial \overline{z}_i^0}, \cdot)G^\perp\Big|^2(Z'^\perp)\Big\} dv_{N_G}(Z'^\perp).\end{aligned} \tag{5.100}$$

From (3.18), (5.77), (5.99), (5.100) and the above discussion, we get (5.94). □

Now we need to compute the contribution from $-(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2P^N$.

Recall that we denote by $\langle\ \ \rangle$ the $\mathbb{C}$-bilinear form on $TB\otimes_{\mathbb{R}}\mathbb{C}$ induced by $g^{TB}$.

***Lemma 5.11***. — *The following identity holds,*

$$\begin{aligned}\widetilde{\Psi}_{1,2}(Z^\perp) = \Big\{\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{1}{48\pi}\Big\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0})e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle \\ + \frac{1}{96\pi}\Big|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 - \frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp, Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle + \frac{13}{192\pi}\Big|T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 \\ + \frac{\sqrt{-1}}{96\pi}\Big\langle 11\,\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})) + 4\,\nabla^{TY}_{\frac{\partial}{\partial \overline{z}_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial z_j^0})) + 7\,\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\Big\rangle \\ - \frac{2}{3\pi}\nabla_{\frac{\partial}{\partial z_j^0}}\nabla_{\frac{\partial}{\partial \overline{z}_j^0}}\log h + \frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big\} P_{\mathscr{L}^\perp}(Z^\perp, Z^\perp).\end{aligned} \tag{5.101}$$

*Proof*. — By (3.9), (3.12), (3.54), (5.24) and (5.83),

$$I_1P^N = \Big\{\frac{1}{2}b_i^\perp B(Z, \tfrac{\partial}{\partial Z_i^\perp}) + b_j B(Z, \tfrac{\partial}{\partial \overline{z}_j^0}) + \tfrac{\partial}{\partial z_j^0}\Big(B(Z, \tfrac{\partial}{\partial \overline{z}_j^0})\Big) - \tfrac{\partial}{\partial \overline{z}_j^0}\Big(B(Z, \tfrac{\partial}{\partial z_j^0})\Big)\Big\}P^N. \tag{5.102}$$

By (3.55) and (5.102),

$$P_{\mathscr{L}^\perp} I_1 P^N = P_{\mathscr{L}^\perp}\Big\{b_j B(Z, \tfrac{\partial}{\partial \overline{z}_j^0}) + \tfrac{\partial}{\partial z_j^0}\Big(B(Z, \tfrac{\partial}{\partial \overline{z}_j^0})\Big) - \tfrac{\partial}{\partial \overline{z}_j^0}\Big(B(Z, \tfrac{\partial}{\partial z_j^0})\Big)\Big\}P^N. \tag{5.103}$$

By (5.46b), and observe that from Theorem 3.1, only the monomials which have even degree on $Z^\perp$ and $\nabla_{e_j^\perp}$, and which have also strictly positive degree on $Z^0$ and $\nabla_{0,e_j^0}$, have contributions in $P^{N^\perp}P_{\mathscr{L}^\perp}I_1P^N$.

By Remark 5.2, (3.55) and (5.46b),

$$(5.104)\quad P^{N^\perp}P_{\mathscr{L}^\perp}\Big(\tfrac{\partial}{\partial z_j^0}\Big(B(Z,\tfrac{\partial}{\partial \overline{z}_j^0})\Big)-\tfrac{\partial}{\partial \overline{z}_j^0}\Big(B(Z,\tfrac{\partial}{\partial z_j^0})\Big)\Big)P^N=-\pi\sqrt{-1}P^{N^\perp}P_{\mathscr{L}^\perp}$$
$$\frac{1}{6}\Big\{\tfrac{\partial}{\partial z_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle-\tfrac{\partial}{\partial \overline{z}_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial z_j^0}\Big\rangle\Big\}P^N$$
$$=-\frac{\pi}{3}P^{N^\perp}\Big\langle 2R^{TX_G}(z^0,\overline{z}^0)\tfrac{\partial}{\partial z_j^0}+R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\mathcal{R}^0)z^0+R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\overline{z}^0)\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle P^N.$$

By (5.23), (5.93) and (5.104),

$$(5.105)$$
$$-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\Big(\tfrac{\partial}{\partial z_j^0}\Big(B(Z,\tfrac{\partial}{\partial \overline{z}_j^0})\Big)-\tfrac{\partial}{\partial \overline{z}_j^0}\Big(B(Z,\tfrac{\partial}{\partial z_j^0})\Big)\Big)P^N\Big)((0,Z^\perp),(0,Z^\perp))$$
$$=-\frac{1}{6\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_j^0}+R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp).$$

Observe that if $Q$ is an odd degree monomial on $b_j, b_j^+, z_j^0, \overline{z}_j^0$, then

$$(5.106)\qquad \big(QP^N\big)\big(\big(0,Z^\perp\big),\big(0,Z'^\perp\big)\big)=0.$$

By using this observation, (5.4) and (5.46b), we get

$$(5.107)\quad -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}b_jB(Z,\tfrac{\partial}{\partial \overline{z}_j^0})P^N\Big)((0,Z^\perp),(0,Z'^\perp))$$
$$=\pi\sqrt{-1}\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}b_j\Big[\frac{1}{6}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle$$
$$-\frac{5}{4}\Big\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}))Z_k^\perp+\nabla^{TY}_{\mathcal{R}^\perp}(T(e_k^0,\tfrac{\partial}{\partial \overline{z}_j^0}))Z_k^0,J\mathcal{R}^\perp\Big\rangle$$
$$+\Big\langle\frac{1}{2}R^{TB}(\mathcal{R}^\perp,J\mathcal{R}^0)\mathcal{R}^\perp+\sqrt{-1}R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle$$
$$-\frac{3}{8}\sqrt{-1}\big\langle J\mathcal{R}^\perp,T(\mathcal{R}^0,e_i^0)\big\rangle\Big\langle J\mathcal{R}^\perp,T(e_i^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle$$
$$-\frac{1}{8}\Big\langle T(\mathcal{R}^\perp,J\mathcal{R}^0),T(\mathcal{R}^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle+\frac{1}{2}\Big\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp),T(\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle$$
$$-\frac{1}{8}\Big\langle JT(\tfrac{\partial}{\partial \overline{z}_j^0},J\mathcal{R}^0),e_j^\perp\Big\rangle\big\langle J\mathcal{R}^\perp,T(\mathcal{R}^\perp,e_j^\perp)\big\rangle\Big]P^N\Big\}((0,Z^\perp),(0,Z'^\perp)).$$

From (3.6), (3.54), (5.5b) and (5.84), we have

$$(5.108a)\qquad \Big\langle T(\tfrac{\partial}{\partial z_j^0},e_i^0),T(e_i^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle=-2\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2,$$

$$(5.108b)\qquad P_{\mathscr{L}^\perp}Z_k^\perp Z_l^\perp P_{\mathscr{L}^\perp}=\frac{\delta_{kl}}{4\pi}P_{\mathscr{L}^\perp}.$$

By (3.54), (5.5e), (5.93), (5.107), (5.108a) and (5.108b),

$$
\begin{aligned}
(5.109)\quad & -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}b_jB(Z,\tfrac{\partial}{\partial \overline{z}_j^0})P^N\Big)((0,Z^\perp),(0,Z^\perp))\\
&= \Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}b_j\Big[\frac{\pi}{3}\Big\langle R^{TX_G}(z^0,\overline{z}^0)\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\\
&\quad -\frac{5\sqrt{-1}}{16}\Big\langle \nabla^{TY}_{\mathcal{R}^0}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}))+\nabla^{TY}_{e_k^\perp}(T(e_i^0,\tfrac{\partial}{\partial \overline{z}_j^0}))Z_i^0, Je_k^\perp\Big\rangle\\
&\quad +\frac{1}{8}\Big\langle \sqrt{-1}R^{TB}(e_k^\perp,J\mathcal{R}^0)e_k^\perp-2R^{TB}(e_k^\perp,\mathcal{R}^0)e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\\
&\quad +\frac{3}{32}\Big\langle T(\mathcal{R}^0,e_i^0),T(e_i^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle-\frac{\sqrt{-1}}{32}\Big\langle T(e_k^\perp,J\mathcal{R}^0),T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\\
&\quad +\frac{\sqrt{-1}}{8}\Big\langle T(e_k^\perp,Je_k^\perp),T(\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\Big]P^N\Big\}((0,Z^\perp),(0,Z^\perp))\\
&= \Big\{-\frac{1}{12\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}+R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\\
&\quad +\frac{5\sqrt{-1}}{32\pi}\Big\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}))+\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})),Je_k^\perp\Big\rangle\\
&\quad +\frac{3}{16\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle+\frac{3}{32\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2\\
&\quad -\frac{1}{64\pi}\Big|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2-\frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\Big\}P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp).
\end{aligned}
$$

For $G_1(Z)$ (resp. $G_2(Z)$) polynomials on $Z$ with degree 1 (resp. 2) and $F \in T^*_{x_0}X_G\otimes T^*_{x_0}X_G$, by Theorem 3.1, (3.9), (3.12), (3.19), (3.54) and (3.55), for any $k,l$, $k',l'$,

$$
\begin{aligned}
& \nabla_{0,e_j^\perp}P^N = -2\pi Z_j^\perp P^N,\\
& P^{N^\perp}P_{\mathscr{L}^\perp}(G_1(Z)b_k^\perp+G_2(Z)b_k^\perp b_l^\perp+Z_{k'}^\perp b_{l'})P^N=0,\\
(5.110)\quad & \frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^\perp)\mathcal{R}^\perp,e_j^\perp\right\rangle \nabla_{0,e_i^\perp}\nabla_{0,e_j^\perp}P^N\\
&\qquad = -\frac{2\pi}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_j^\perp)\mathcal{R}^\perp,e_j^\perp\right\rangle P^N,\\
& F(e_i^0,e_j^0)\nabla_{0,e_i^0}\nabla_{0,e_j^0}P^N=\Big[F(\tfrac{\partial}{\partial \overline{z}_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})b_ib_j-4\pi F(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big]P^N.
\end{aligned}
$$

By (5.24) and (5.110), we get

(5.111)
$$\begin{aligned}I_2P^N = \Big\{ \Big( \Big\langle \frac{1}{3} R^{TX_G}(\mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_i^0})\mathcal{R}^0 + R^{TB}(\mathcal{R}^\perp, \tfrac{\partial}{\partial \overline{z}_i^0})\mathcal{R}^\perp + \nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial \overline{z}_i^0})\mathcal{R}^\perp), \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle \\ - 3 \Big\langle A(\tfrac{\partial}{\partial \overline{z}_i^0})\mathcal{R}^\perp, A(\tfrac{\partial}{\partial \overline{z}_j^0})\mathcal{R}^\perp \Big\rangle + \Big\langle \tfrac{\partial}{\partial \overline{z}_i^0}, \nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial \overline{z}_j^0})\mathcal{R}^\perp) \Big\rangle \Big) b_i b_j \\ - 4\pi \Big\langle \frac{1}{3} R^{TX_G}(\mathcal{R}^0, \tfrac{\partial}{\partial z_j^0})\mathcal{R}^0 + R^{TB}(\mathcal{R}^\perp, \tfrac{\partial}{\partial z_j^0})\mathcal{R}^\perp + \nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial z_j^0})\mathcal{R}^\perp), \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle \\ + 12\pi |A(\tfrac{\partial}{\partial \overline{z}_j^0})\mathcal{R}^\perp|^2 - 4\pi \Big\langle \tfrac{\partial}{\partial z_j^0}, \nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial \overline{z}_j^0})\mathcal{R}^\perp) \Big\rangle - \frac{2\pi}{3} \left\langle R^{TB}(\mathcal{R}^\perp, e_j^\perp)\mathcal{R}^\perp, e_j^\perp \right\rangle \Big\} P^N.\end{aligned}$$

Observe that as $A(e_i^0)e_i^0 \in N_G$, we have at $x_0$,

$$\left\langle \nabla^{TB}_{\mathcal{R}^0}(A(e_i^0)e_i^0), e_j^0 \right\rangle = \left\langle A(\mathcal{R}^0)A(e_i^0)e_i^0, e_j^0 \right\rangle. \tag{5.112}$$

Thus by (3.12), (3.54), (3.55), (5.25), (5.108b), (5.110)-(5.112), $a_j = a_j^\perp = 2\pi$, and the arguments above (5.104),

$$P^{N^\perp} P_{\mathscr{L}^\perp} \left\langle \Gamma_{ii}(\mathcal{R}), e_l \right\rangle \nabla_{0,e_l} P^N = -\frac{2}{3} P^{N^\perp} \Big\langle R^{TX_G}(\mathcal{R}^0, e_i^0)e_i^0, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle b_j P^N, \tag{5.113a}$$

$$\begin{aligned}P^{N^\perp} P_{\mathscr{L}^\perp} I_2 P^N = P^{N^\perp} \Big\{ \Big( \Big\langle \frac{1}{3} R^{TX_G}(\mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_i^0})\mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle \\ + \Big\langle \frac{1}{4\pi} R^{TB}(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_i^0})e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle - \frac{3}{4\pi} \Big\langle A(\tfrac{\partial}{\partial \overline{z}_i^0})e_k^\perp, A(\tfrac{\partial}{\partial \overline{z}_j^0})e_k^\perp \Big\rangle \Big) b_i b_j \\ - \frac{4\pi}{3} \Big\langle R^{TX_G}(\mathcal{R}^0, \tfrac{\partial}{\partial z_j^0})\mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle \Big\} P^N.\end{aligned} \tag{5.113b}$$

By (3.6), (5.4), (5.93), (5.113a), (5.113b) and the fact that $R^{TX_G}(\, ,\, )$ is a $(1,1)$-form, we get

$$\begin{aligned}- \Big( (\mathscr{L}_2^0)^{-1} P^{N^\perp} P_{\mathscr{L}^\perp} (I_2 + \left\langle \Gamma_{ii}(\mathcal{R}), e_l \right\rangle \nabla_{0,e_l} P^N \Big) ((0, Z^\perp), (0, Z^\perp)) \\ = \frac{1}{6\pi} \Big\{ 3 \Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_j^0} + R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle \\ - 2 \Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, e_i^0)e_i^0 + R^{TX_G}(\tfrac{\partial}{\partial \overline{z}_i^0}, \tfrac{\partial}{\partial z_j^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle \Big\} P_{\mathscr{L}^\perp}(Z^\perp, Z^\perp) \\ = \frac{2}{3\pi} \Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle P_{\mathscr{L}^\perp}(Z^\perp, Z^\perp).\end{aligned} \tag{5.114}$$

Now by (5.25), (5.46a), (5.84), (5.108b) and (5.110),

$$- P^{N^\perp} P_{\mathscr{L}^\perp} \frac{1}{9} \sum_i \Big[ \sum_j (\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i) Z_j \Big]^2 P^N = \frac{\pi}{4} P^{N^\perp} \Big| T(\mathcal{R}^0, e_i^\perp) \Big|^2 P^N,$$

$$\begin{aligned}- \frac{1}{4} (\mathscr{L}_2^0)^{-1} P^{N^\perp} P_{\mathscr{L}^\perp} [K_2(\mathcal{R}), \mathscr{L}_2^0] P^N = \frac{1}{4} P^{N^\perp} P_{\mathscr{L}^\perp} K_2(\mathcal{R}) P^N \\ = \frac{1}{12} P^{N^\perp} \left\langle R^{TX_G}(\mathcal{R}^0, e_i^0)\mathcal{R}^0, e_i^0 \right\rangle P^N.\end{aligned} \tag{5.115}$$

By (5.13), (5.47), (5.49) and (5.50), as $T(\cdot,\cdot)\in TY$, we get

$$\frac{\sqrt{-1}}{2\pi}(\partial_j^0 R^{L_B})_{x_0}(\mathcal{R},e_i^0) = -\frac{1}{2}\left\langle J\mathcal{R}^\perp, T(e_j^0,e_i^0)\right\rangle + \left\langle JA(e_j^0)\mathcal{R}^\perp, e_i^0\right\rangle = 0. \tag{5.116}$$

Thus by (3.9), (5.27), (5.46a), (5.115) and (5.116), we get

$$\begin{aligned}(5.117)\quad -P^{N^\perp}P_{\mathscr{L}^\perp}\mathscr{O}_2'P^N = P^{N^\perp}P_{\mathscr{L}^\perp}\Big\{ &-I_1 - (I_2 + \langle\Gamma_{ii}(\mathcal{R}),e_l\rangle\nabla_{0,e_l}) \\ &-\frac{1}{4}[K_2(\mathcal{R}),\mathscr{L}_2^0] - R^{E_B}(\mathcal{R},\tfrac{\partial}{\partial\overline{z}_j^0})b_j - \frac{\pi}{4}\Big|T(\mathcal{R}^0,e_i^\perp)\Big|^2\Big\}P^N.\end{aligned}$$

Note that $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form, by (3.54), (5.4), (5.93), (5.103), (5.105), (5.109), (5.114) and (5.117),

$$\begin{aligned}(5.118)\quad &-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\mathscr{O}_2'P^N\Big)((0,Z^\perp),(0,Z^\perp)) \\ &= -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}(I_1+I_2+\langle\Gamma_{ii}(\mathcal{R}),e_l\rangle\nabla_{0,e_l})P^N\Big)((0,Z^\perp),(0,Z^\perp)) \\ &+\frac{1}{2\pi}\Big\{R^{E_B}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}) + \frac{1}{3}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},e_i^0)e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle + \frac{1}{4}\Big|T(\tfrac{\partial}{\partial\overline{z}_j^0},e_i^\perp)\Big|^2\Big\}P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp) \\ &= \Big\{\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle + \frac{3}{16\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle \\ &+\frac{3}{32\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 - \frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle + \frac{7}{64\pi}\Big|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 \\ &+\frac{5\sqrt{-1}}{32\pi}\Big\langle\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})) + \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^\perp\Big\rangle \\ &+\frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\}P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp).\end{aligned}$$

By (3.54), (5.63), (5.84), (5.108b), (5.110) and the arguments above (5.104),

$$\begin{aligned}(5.119)\quad 4\pi^2P^{N^\perp}P_{\mathscr{L}^\perp}\mathscr{O}_2''P^N &= 4\pi^2P^{N^\perp}P_{\mathscr{L}^\perp}\Big\{-\frac{1}{3}\left\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^0,\mathcal{R}^0)}J\mathcal{R}^\perp,J\mathcal{R}^\perp\right\rangle \\ &+\frac{1}{6}\left\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_j^\perp,J_{x_0}e_i^0))Z_j^\perp Z_i^0 + \nabla^{TY}_{\mathcal{R}^\perp}(T(e_j^0,J_{x_0}e_i^0))Z_j^0Z_i^0,J\mathcal{R}^\perp\right\rangle \\ &+\frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0,\mathcal{R}^\perp\right\rangle - \frac{1}{12}\sum_l\left\langle T(\mathcal{R}^0,e_l),J\mathcal{R}^\perp\right\rangle^2\Big\}P^N \\ &= \frac{\pi}{3}P^{N^\perp}\Big\{\frac{1}{2}\Big\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_k^\perp,J_{x_0}e_i^0))Z_i^0 + \nabla^{TY}_{e_k^\perp}(T(e_j^0,J_{x_0}e_i^0))Z_j^0Z_i^0,Je_k^\perp\Big\rangle \\ &-\left\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^0,\mathcal{R}^0)}Je_k^\perp,Je_k^\perp\right\rangle + \left\langle R^{TB}(e_k^\perp,\mathcal{R}^0)\mathcal{R}^0,e_k^\perp\right\rangle - \frac{1}{4}|T(\mathcal{R}^0,e_l)|^2\Big\}P^N.\end{aligned}$$

Let $\{f_l\}$ be an orthonormal frame of $TY$ on $X$.

As $\nabla^{TY}$ preserves the metric $g^{TY}$, by (1.4), (1.24),

$$\Big\langle(\nabla^{TY}_{e_i^0}\dot{g}^{TY}_{e_j^0})f_l,f_l\Big\rangle = \nabla_{e_i^0}\Big\langle\dot{g}^{TY}_{e_j^0}f_l,f_l\Big\rangle = 4\nabla_{e_i^0}\nabla_{e_j^0}\log h. \tag{5.120}$$

Now $\{Je_k^\perp\}$ is an orthonormal basis of $TY$ along the fiber $Y_{x_0}$ and $\{e_l\} = \{e_i^0\} \cup \{e_k^\perp\}$.

By (3.54), (5.93), (5.108a), (5.119) and (5.120),

$$
\begin{aligned}
(5.121)\quad & -4\pi^2 \Big( (\mathscr{L}_2^0)^{-1} P^{N^\perp} P_{\mathscr{L}^\perp} \mathscr{O}_2'' P^N \Big) ((0,Z^\perp),(0,Z^\perp)) \\
= \frac{1}{4\pi} \Big\{ & \frac{\sqrt{-1}}{6} \Big\langle -\nabla^{TY}_{\frac{\partial}{\partial z_j^0}} (T(e_k^\perp, \tfrac{\partial}{\partial \bar{z}_j^0})) + \nabla^{TY}_{\frac{\partial}{\partial \bar{z}_j^0}} (T(e_k^\perp, \tfrac{\partial}{\partial z_j^0})) - 2\nabla^{TY}_{e_k^\perp} (T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \bar{z}_j^0})), Je_k^\perp \Big\rangle \\
& - \frac{8}{3} \nabla_{\frac{\partial}{\partial z_j^0}} \nabla_{\frac{\partial}{\partial \bar{z}_j^0}} \log h - \frac{1}{3} \Big| T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \bar{z}_j^0}) \Big|^2 - \frac{1}{6} \Big| T(e_k^\perp, \tfrac{\partial}{\partial \bar{z}_j^0}) \Big|^2 \\
& - \frac{2}{3} \Big\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0}) e_k^\perp, \tfrac{\partial}{\partial \bar{z}_j^0} \Big\rangle \Big\} P_{\mathscr{L}^\perp}(Z^\perp, Z^\perp).
\end{aligned}
$$

By (5.74), (5.77), (5.118) and (5.121), we get (5.101). The proof of Lemma 5.11 is complete. □

### 5.4. Final computations: the proof of Theorem 0.6

By (3.40), (5.3), (5.5a), (5.6a) and (5.32), as $Je_k^\perp \in TY$ on $P$, we get at $x_0$,

$$
(5.122)\quad
\begin{aligned}
& \nabla^{TY}_{e_i^0} Je_k^\perp = P^{TY} \nabla^{TX}_{e_i^0} Je_k^\perp = P^{TY} J \nabla^{TX}_{e_i^0} e_k^\perp = 0, \\
& \nabla^{TB}_{e_i^0} Je_j^0 = \nabla^{TX_G}_{e_i^0} Je_j^0 + A(e_i^0) Je_j^0 = -\frac{1}{2} JT(e_i^0, e_j^0) = \nabla^{TB}_{e_i^0} (J_{x_0} e_j^0).
\end{aligned}
$$

By (1.6), (1.24), (5.5c) and (5.122), as in (5.120), at $x_0$,

$$
\begin{aligned}
(5.123)\quad \Big\langle \nabla^{TY}_{e_i^0} (T(e_k^\perp, e_j^0)), Je_k^\perp \Big\rangle_{x_0} &= -2 \Big\langle \nabla^{TY}_{e_i^0} (T(Je_j^0, Je_k^\perp)), Je_k^\perp \Big\rangle \\
&= - \Big\langle (\nabla^{TY}_{e_i^0} \dot{g}^{TY}_{Je_j^0}) Je_k^\perp, Je_k^\perp \Big\rangle = -4 \nabla_{e_i^0} \nabla_{J_{x_0} e_j^0} \log h.
\end{aligned}
$$

By (1.21) and (5.123), we get

$$
(5.124)\quad
\begin{aligned}
& \sqrt{-1} \Big\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}} (T(e_k^\perp, \tfrac{\partial}{\partial \bar{z}_j^0})), Je_k^\perp \Big\rangle = -4 \nabla_{\frac{\partial}{\partial z_j^0}} \nabla_{\frac{\partial}{\partial \bar{z}_j^0}} \log h = \Delta_{X_G} \log h, \\
& \sqrt{-1} \Big\langle \nabla^{TY}_{\frac{\partial}{\partial \bar{z}_j^0}} (T(e_k^\perp, \tfrac{\partial}{\partial z_j^0})), Je_k^\perp \Big\rangle = -\Delta_{X_G} \log h.
\end{aligned}
$$

Note that $T(e_i, e_j) = -[e_i^H, e_j^H]$, as $[e_i, e_j] = 0$. By (1.4) and (1.6) and the Jacobi identity,

$$
\begin{aligned}
(5.125)\quad \nabla^{TY}_{e_k^{\perp,H}} (T(e_i^{0,H}, e_j^{0,H})) &= -\Big[ e_k^{\perp,H}, [e_i^{0,H}, e_j^{0,H}] \Big] + T(e_k^{\perp,H}, T(e_i^{0,H}, e_j^{0,H})) \\
&= L_{e_i^{0,H}} (T(e_k^{\perp,H}, e_j^{0,H})) - L_{e_j^{0,H}} (T(e_k^{\perp,H}, e_i^{0,H})) + T(e_k^{\perp,H}, T(e_i^{0,H}, e_j^{0,H})) \\
&= \nabla^{TY}_{e_i^{0,H}} (T(e_k^{\perp,H}, e_j^{0,H})) - \nabla^{TY}_{e_j^{0,H}} (T(e_k^{\perp,H}, e_i^{0,H})) - T(e_i^{0,H}, T(e_k^{\perp,H}, e_j^{0,H})) \\
&\quad + T(e_j^{0,H}, T(e_k^{\perp,H}, e_i^{0,H})) + T(e_k^{\perp,H}, T(e_i^{0,H}, e_j^{0,H})).
\end{aligned}
$$

Thus by Theorem 5.1, (5.124) and (5.125),

$$\begin{aligned}(5.126)\quad & \sqrt{-1}\left\langle \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\right\rangle \\ &= \sqrt{-1}\Big\{2\Big\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\Big\rangle - \Big\langle T(\tfrac{\partial}{\partial z_j^0}, Je_k^\perp), T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle \\ &\quad + \Big\langle T(\tfrac{\partial}{\partial \overline{z}_j^0}, Je_k^\perp), T(e_k^\perp,\tfrac{\partial}{\partial z_j^0})\Big\rangle + \Big\langle T(e_k^\perp, Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\Big\} \\ &= 2\Delta_{X_G}\log h + |T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2 + \sqrt{-1}\Big\langle T(e_k^\perp, Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle.\end{aligned}$$

By $T(e_i,e_j) = -[e_i^H, e_j^H]$, (3.40), (5.6a) and (5.55), we have

$$\begin{aligned}(5.127)\quad & R^{TX}(e_k^H,e_j^H)e_i^H = \nabla^{TX}_{e_k^H}\nabla^{TX}_{e_j^H}e_i^H - \nabla^{TX}_{e_j^H}\nabla^{TX}_{e_k^H}e_i^H - \nabla^{TX}_{[e_k^H,e_j^H]}e_i^H \\ &= R^{TB}(e_k,e_j)e_i - \frac{1}{2}T(e_k,\nabla^{TB}_{e_j}e_i) + \frac{1}{2}T(e_j,\nabla^{TB}_{e_k}e_i) \\ &\qquad - \frac{1}{2}\nabla^{TX}_{e_k^H}(T(e_j,e_i)) + \frac{1}{2}\nabla^{TX}_{e_j^H}(T(e_k,e_i)) + \nabla^{TX}_{T(e_k^H,e_j^H)}e_i^H, \\ & \Big\langle R^{TX}(e_k^{\perp,H},e_j^{0,H})(J_{x_0}e_j^0)^H, J_{x_0}e_k^{\perp,H}\Big\rangle = \Big\langle R^{TX}(e_k^{\perp,H},e_j^{0,H})e_j^{0,H}, e_k^{\perp,H}\Big\rangle.\end{aligned}$$

By (5.5a), (5.6a), (5.13), (5.32), (5.122) and $T(e_k^\perp, e_i^0)\in TY$, at $x_0$, $(J_{x_0}e_i)^H = Je_i^H$ on $P$, we get

$$(5.128)\quad \begin{aligned} & \nabla^{TB}_{e_j^0}(J_{x_0}e_j^0) = 0, \quad \nabla^{TB}_{e_k^\perp}(J_{x_0}e_j^0) = \frac{1}{2}\left\langle T(e_j^0,e_l^0), Je_k^\perp\right\rangle e_l^0, \\ & \Big\langle \nabla^{TX}_{T(e_k^\perp,e_i^0)}(J_{x_0}e_i^0)^H, J_{x_0}e_k^\perp\Big\rangle = \Big\langle \nabla^{TX}_{T(e_k^\perp,e_i^0)}e_i^{0,H}, e_k^\perp\Big\rangle.\end{aligned}$$

We apply now the first equation of (5.127) into the second equation of (5.127), by using (1.8) and (5.128) and $T(\,,\,)$ is a $(1,1)$-form, we get at $x_0$,

$$\begin{aligned}(5.129)\quad & \frac{1}{4}|T(e_j^0,e_l^0)|^2 + \Big\langle -\frac{1}{2}\nabla^{TY}_{e_k^\perp}(T(e_j^0,J_{x_0}e_j^0)) + \frac{1}{2}\nabla^{TY}_{e_j^0}(T(e_k^\perp,J_{x_0}e_j^0)), Je_k^\perp\Big\rangle \\ &= \left\langle R^{TB}(e_k^\perp,e_j^0)e_j^0, e_k^\perp\right\rangle + \frac{1}{2}\left\langle \nabla^{TX}_{e_j^0}(T(e_k^\perp,e_j^0)), e_k^\perp\right\rangle \\ &= \left\langle R^{TB}(e_k^\perp,e_j^0)e_j^0, e_k^\perp\right\rangle - \frac{1}{4}|T(e_k^\perp,e_l^0)|^2.\end{aligned}$$

Finally, from (3.6), (5.124), (5.126) and (5.129) and $T(\,,\,)$ is a $(1,1)$-form, we get

$$\begin{aligned}(5.130)\quad & 4\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})\tfrac{\partial}{\partial \overline{z}_j^0}, e_k^\perp\Big\rangle = 2\sqrt{-1}\Big\langle \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\Big\rangle \\ &\quad - 2\sqrt{-1}\Big\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\Big\rangle + |T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2 + 2\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 \\ &= 2\Delta_{X_G}\log h + 3|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2 + 2\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 \\ &\qquad + 2\sqrt{-1}\Big\langle T(e_k^\perp, Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle.\end{aligned}$$

From (5.124)-(5.130),

(5.131)
$$\frac{\sqrt{-1}}{96\pi}\left\langle 11\,\nabla^{TY}_{\frac{\partial}{\partial \overline{z}^0_j}}(T(e^\perp_k,\tfrac{\partial}{\partial \overline{z}^0_j}))+4\,\nabla^{TY}_{\frac{\partial}{\partial \overline{z}^0_j}}(T(e^\perp_k,\tfrac{\partial}{\partial z^0_j}))+7\,\nabla^{TY}_{e^\perp_k}(T(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial \overline{z}^0_j})),Je^\perp_k\right\rangle$$
$$+\frac{1}{48\pi}\left\langle R^{TB}(e^\perp_k,\tfrac{\partial}{\partial z^0_j})e^\perp_k,\tfrac{\partial}{\partial \overline{z}^0_j}\right\rangle-\frac{\sqrt{-1}}{16\pi}\left\langle T(e^\perp_k,Je^\perp_k),T(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial \overline{z}^0_j})\right\rangle$$
$$=\frac{5}{24\pi}\Delta_{X_G}\log h+\frac{11}{192\pi}\left|T(e^\perp_k,\tfrac{\partial}{\partial \overline{z}^0_j})\right|^2-\frac{1}{96\pi}\left|T(\tfrac{\partial}{\partial z^0_i},\tfrac{\partial}{\partial \overline{z}^0_j})\right|^2.$$

By (3.19), (5.77), (5.82), (5.101) and (5.131),

$$\Phi_{1,1}+\Phi_{1,2}=\frac{1}{2\pi}\left\langle R^{TX_G}(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial \overline{z}^0_i})\tfrac{\partial}{\partial z^0_i},\tfrac{\partial}{\partial \overline{z}^0_j}\right\rangle+\frac{3}{8\pi}\Delta_{X_G}\log h+\frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial \overline{z}^0_j})$$
$$=\frac{1}{16\pi}r^{X_G}_{x_0}+\frac{3}{8\pi}\Delta_{X_G}\log h+\frac{1}{4\pi}R^{E_G}_{x_0}(w^0_j,\overline{w}^0_j). \tag{5.132}$$

From Lemma 5.10, (5.81) and (5.132), we get (0.25).

Recall that we compute everything on $\mathscr{C}^\infty(X,L^p\otimes E)$.

From (5.18), (5.19), (5.22), (5.23), comparing to (2.109), we know that in (0.20), $\Phi_r(x_0)\in \mathrm{End}(E_G)_{x_0}$, and the term $r^X$, $R^{\det}$ will not appear here, and $\tau=2\pi n$, thus we get the remainder part of Theorem 0.6 from Corollary 0.4.

The proof of Theorem 0.6 is complete.

## 5.5. Coefficient $\Phi_1$: general case

We use the general assumption at the beginning of this Chapter, but we do not suppose that $\mathbf{J}=J$ in (0.2).

Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X,L^p\otimes E)$ with the scalar product $\langle\ \ \rangle$ induced by $g^{TX}$, $h^L$, $h^E$ as in Section 2.2. Set

$$D_p=\sqrt{2}\left(\overline{\partial}^{L^p\otimes E}+\overline{\partial}^{L^p\otimes E,*}\right) \tag{5.133}$$

Then

$$D_p^2=2\left(\overline{\partial}^{L^p\otimes E}\overline{\partial}^{L^p\otimes E,*}+\overline{\partial}^{L^p\otimes E,*}\overline{\partial}^{L^p\otimes E}\right) \tag{5.134}$$

preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X,L^p\otimes E)$.

For $p$ large enough,

$$\operatorname{Ker} D_p=\operatorname{Ker} D_p^2=H^0(X,L^p\otimes E). \tag{5.135}$$

Here $D_p$ need not be a spin$^c$ Dirac operator on $\Omega^{0,\bullet}(X,L^p\otimes E)$.

Let $P_p^G(x,x')$ $(x,x'\in X)$ be the smooth kernel of the orthogonal projection $P_p^G$ from $(\mathscr{C}^\infty(X,L^p\otimes E),\langle\ \ \rangle)$ onto $(\operatorname{Ker} D_p^2)^G$ with respect to the Riemannian volume form $dv_X(x')$ for $p$ large enough.

We explain now how to reduce the study of the asymptotic expansion of $P_p^G(x,x')$ to the $\mathbf{J}=J$ case.

Let $g_\omega^{TX}(\cdot,\cdot):=\omega(\cdot,J\cdot)$ be the metric on $TX$ induced by $\omega,J$. We will use a subscript $\omega$ to indicate the objects corresponding to $g_\omega^{TX}$, especially $r_\omega^X$ is the scalar curvature of $(TX,g_\omega^{TX})$, and $\Delta_{X_G,\omega}$ is the Bochner-Laplace operator on $X_G$ as in (1.21) associated to $g_\omega^{TX_G}$.

Let $\det_{\mathbb{C}}$ denote the determinant function on the complex bundle $T^{(1,0)}X$, and $|\mathbf{J}|=(-\mathbf{J}^2)^{-1/2}$.

Let $h_\omega^E:=(\det_{\mathbb{C}}|\mathbf{J}|)^{-1}h^E$ define a metric on $E$. Let $R_\omega^E$ be the curvature associated to the holomorphic Hermitian connection on $(E,h_\omega^E)$.

Let $\langle\ \ \rangle_\omega$ be the Hermitian product on $\mathscr{C}^\infty(X,L^p\otimes E)$ induced by $g_\omega^{TX},h^L,h_\omega^E$ as in (1.19), then

$$(\mathscr{C}^\infty(X,L^p\otimes E),\langle\ \ \rangle_\omega)=(\mathscr{C}^\infty(X,L^p\otimes E),\langle\ \ \rangle),\quad dv_{X,\omega}=(\det{}_{\mathbb{C}}|\mathbf{J}|)dv_X. \tag{5.136}$$

Observe that $H^0(X,L^p\otimes E)$ does not depend on $g^{TX},h^L,h^E$.

Let $P_{\omega,p}^G(x,x')$ $(x,x'\in X)$ be the smooth kernel of the orthogonal projection $P_{\omega,p}^G$ from $(\mathscr{C}^\infty(X,L^p\otimes E),\langle\ \ \rangle_\omega)$ onto $H^0(X,L^p\otimes E)^G$ with respect to $dv_{X,\omega}(x)$.

By (5.136),

$$P_p^G(x,x')=(\det{}_{\mathbb{C}}|\mathbf{J}|)(x')P_{\omega,p}^G(x,x'). \tag{5.137}$$

We will use the trivialization in Introduction corresponding to $g_\omega^{TX}$.

Since $g_\omega^{TX}(\cdot,\cdot)=\omega(\cdot,J\cdot)$ is a Kähler metric on $TX$, $D_{\omega,p}$ is a Dirac operator (cf. Def. 2.1). Thus Theorems 0.1, 0.2 hold for $P_{\omega,p}^G(x,x')$.

Let $dv_B$ be the volume form on $B$ induced by $g^{TX}$ as in Introduction.

As in (0.11), let $\widetilde{\kappa}\in\mathscr{C}^\infty(TB|_{X_G},\mathbb{R})$ be defined by for $Z\in T_{x_0}B$, $x_0\in X_G$,

$$dv_B(x_0,Z)=\widetilde{\kappa}(x_0,Z)dv_{X_G,\omega}(x_0)dv_{N_{G,\omega,x_0}}. \tag{5.138}$$

As in (0.17), we introduce $\mathscr{I}_p(x_0)$ a section of $\operatorname{End}(E_G)$ on $X_G$,

$$\mathscr{I}_p(x_0)=\int_{Z\in N_{G,\omega},|Z|\leqslant\varepsilon_0}h^2(x_0,Z)P_p^G\circ\Psi_\omega((x_0,Z),(x_0,Z))\widetilde{\kappa}(x_0,Z)dv_{N_{G,\omega,x_0}}. \tag{5.139}$$

Then the analogue of (0.18) is

$$\dim(\operatorname{Ker} D_p)^G=\int_{X_G}\operatorname{Tr}[\mathscr{I}_p(x_0)]dv_{X_G,\omega}(x_0)+\mathscr{O}(p^{-\infty}).$$

Summarizes, we have the following result,

***Theorem 5.12***. — *The smooth kernel $P_p^G(x,x')$ has a full off–diagonal asymptotic expansion analogous to* (0.14) *with $\mathscr{Q}_0=(\det_{\mathbb{C}}|\mathbf{J}|)\operatorname{Id}_{E_G}$ as $p\to\infty$. There exist*

$\Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$ *polynomials in* $A_\omega$, $R^{TB}_\omega$, $R^{E_B}$, $\mu^E$, $R^E$ *(resp.* $h_\omega$, $R^{L_B}$*; resp.* $\mu$*) and their derivatives at* $x_0$ *to order* $2r-1$ *(resp.* $2r$*, resp.* $2r+1$*), and* $\Phi_0 = \mathrm{Id}_{E_G}$ *such that* (0.25) *holds for* $\mathscr{I}_p$*. Moreover*

(5.140)

$$\Phi_1(x_0) = \frac{1}{8\pi}\Big[r^{X_G}_\omega + 6\Delta_{X_G,\omega}\log\left(h_\omega|_{X_G}\right) - 2\Delta_{X_G,\omega}\Big(\log(\det_{\mathbb{C}}|\mathbf{J}|)\Big) + 4R^{E_G}(w^0_{\omega,j}, \overline{w}^0_{\omega,j})\Big].$$

*Here* $\{w_{\omega,j}\}$ *is an orthogonal basis of* $(T^{(1,0)}X_G, g^{TX_G}_\omega)$*.*

*Proof.* — By (5.136), $\det_{\mathbb{C}}|\mathbf{J}|\, h^2 dv_B = dv_{B,\omega} h^2_\omega$. Thus by (5.136)-(5.139),

(5.141)

$$\mathscr{I}_p(x_0) = \int_{Z\in N_{G,\omega}, |Z|\leqslant\varepsilon_0} h^2_\omega(x_0,Z) P^G_{\omega,p}\circ\Psi_\omega((x_0,Z),(x_0,Z))\kappa_\omega(x_0,Z) dv_{N_G,\omega}(Z).$$

From the above discussion, only (5.140) reminds to be proved. But

$$R^{E_G}_\omega = R^{E_G} - \overline{\partial}\partial\log(\det_{\mathbb{C}}|\mathbf{J}|), \tag{5.142}$$

Thus

$$2R^{E_G}_\omega(w^0_{\omega,j}, \overline{w}^0_{\omega,j}) = 2R^{E_G}(w^0_{\omega,j}, \overline{w}^0_{\omega,j}) - \Delta_{X_G,\omega}\log(\det_{\mathbb{C}}|\mathbf{J}|), \tag{5.143}$$

and (5.140) is from (0.7) and (5.141). ☐

# CHAPTER 6

# THE COEFFICIENT $P^{(2)}(0,0)$

The main purpose in this Chapter is to compute $P^{(2)}(0,0)$ in (0.16). The formula for $P^{(2)}(0,0)$ in Theorem 0.7 is quite complicate, it involves $h$, the volume function of the orbit and the curvature for the principal bundle $P \to X_G$.

This Chapter is organized as follows. In Section 6.1, we compute the contribution of $\Psi_{1,1}, \Psi_{1,3}, \Psi_{1,4}$ in (5.77) for $P^{(2)}(0,0)$. In Section 6.2, we compute the contribution of $\Psi_{1,2}$ in (5.77) for $P^{(2)}(0,0)$. In Section 6.3, we prove Theorem 0.7.

In this Chapter, we use the same notations and assumption as in Sections 5.1 and 5.2.

## 6.1. The terms $\Psi_{1,1}, \Psi_{1,3}, \Psi_{1,4}$

As in (5.81), we have

$$P^{(2)}(0,0) = (\Psi_{1,1} + \Psi_{1,2})(0) + (\Psi_{1,1} + \Psi_{1,2})^*(0) + (\Psi_{1,3} - \Psi_{1,4})(0). \tag{6.1}$$

For $k \in \mathbb{N}$, let $H_k(x)$ be the Hermite polynomial,

$$H_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{k!\,(2x)^{k-2j}}{j!\,(k-2j)!}. \tag{6.2}$$

Here $\lfloor k/2 \rfloor$ is the integer part of $k/2$.

By [**42**, §8.6] (cf. [**31**, Append. E]), (3.8) and $a_l^\perp = 2\pi$, we have

$$(b_l^\perp)^k e^{-\pi|Z_l^\perp|^2} = (2\pi)^{k/2} H_k(\sqrt{2\pi} Z_l^\perp) e^{-\pi|Z_l^\perp|^2}. \tag{6.3}$$

Especially, for $l$ fixed, $i \in \mathbb{N}$,

$$\begin{aligned}
&((b_l^\perp)^{2i+1} e^{-\pi|Z_l^\perp|^2})(0) = 0,\\
&((b_l^\perp)^{2} e^{-\pi|Z_l^\perp|^2})(0) = -4\pi, \quad ((b_l^\perp)^{4} e^{-\pi|Z_l^\perp|^2})(0) = 3\cdot(4\pi)^2,\\
&((b_l^\perp)^{6} e^{-\pi|Z_l^\perp|^2})(0) = 15\cdot(-4\pi)^3.
\end{aligned} \tag{6.4}$$

Recall that when we meet the operation $|\quad|^2$, we will first do this operation, then take the sum of the indices. Thus $|\mathcal{T}_{ijk}|^2$ means $\sum_{ijk}|\mathcal{T}_{ijk}|^2$, etc.

By (3.22), (5.95) and (6.4),

$$\mathcal{F}_2(\cdot,0) = -\frac{1}{8}\mathcal{T}_{kk};\quad P^N(0,0) = 2^{n_0/2}. \tag{6.5}$$

By (5.99), (6.4) and (6.5), we know

$$\Psi_{1,3}(0) = \frac{2^{n_0/2}}{\pi}\Big|\frac{1}{4}\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_i^0}) + \mathcal{F}_2(\tfrac{\partial}{\partial \overline{z}_i^0},0)\Big|^2 = \frac{2^{n_0/2}}{64\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2. \tag{6.6}$$

From (3.17), (3.18), (3.54), (5.100) and $a_i^\perp = 2\pi$,

$$\begin{aligned}\Psi_{1,4}(0) = G^\perp(0)^2\Big\{&\frac{1}{4\pi}\sum_k \mathcal{F}_1(e_k^\perp)^2 + \frac{6\cdot(4\pi)^3}{(192\pi^2)^2}|\mathcal{T}_{klm}|^2\\ &+\frac{1}{16\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2 + \frac{2\cdot(4\pi)^2}{\pi\cdot(32\pi)^2}\Big|\mathcal{T}_{kl}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2\Big\}\\ =\frac{2^{n_0/2}}{4\pi}\Big\{&\sum_k \mathcal{F}_1(e_k^\perp)^2 + \frac{1}{24}|\mathcal{T}_{klm}|^2 + \frac{1}{4}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2 + \frac{1}{8}\Big|\mathcal{T}_{kl}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2\Big\}.\end{aligned} \tag{6.7}$$

***Lemma 6.1***. — *The following identity holds,*

$$\begin{aligned}\Psi_{1,1}(0) = \Big\{&-\frac{19}{2^6\cdot 3\pi}|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})|^2 - \frac{11}{2^7\cdot 3\pi}\mathcal{T}_{klm}^2 + \frac{1}{2^8\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm}\\ &-\frac{5}{2^7\pi}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_l^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_l^0}) - \frac{1}{8\pi}\sum_k \mathcal{F}_1(e_k^\perp)^2 - \frac{1}{16\pi}\mathcal{F}_1(e_k^\perp)\mathcal{T}_{kll}\Big\}P^N(0,0).\end{aligned} \tag{6.8}$$

*Proof*. — Recall that $\mathcal{F}_1 \in N^*_{G,x_0}\otimes \mathrm{End}(E_{G,x_0})$ was defined in (5.95). Set

(6.9)
$$\begin{aligned}\mathcal{I}_1 &= -\sqrt{-1}\Big(\mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_i^0})\frac{b_i^+}{8\pi}(b_j^\perp b_{j'}^\perp + 4\pi\delta_{jj'}) + \frac{1}{4}\mathcal{T}_{jj'}(z^0)b_j^\perp b_{j'}^\perp\Big)\frac{\sqrt{-1}}{8\pi}b_l\mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_l^0})\\ \mathcal{I}_2 &= \sqrt{-1}\Big(\mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})b_i\frac{B_{jj'}^\perp}{8\pi} + \frac{1}{4}\mathcal{T}_{jj'}(\overline{z}^0)(b_j^{\perp+}b_{j'}^{\perp+} - b_j^\perp b_{j'}^\perp)\Big)\frac{-\sqrt{-1}}{32\pi}\mathcal{T}_{kl}(z^0)b_k^\perp b_l^\perp,\\ \mathcal{I}_3 &= -\frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijj'}(b_j^\perp b_{j'}^{\perp+} + b_j^\perp b_{j'}^\perp)b_i^{\perp+}\Big(\mathcal{F}_1(e_k^\perp)\frac{b_k^\perp}{4\pi} + \mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2}\Big).\end{aligned}$$

Observe that by (5.93), when we evaluate $\Psi_{1,1}$ in (5.77), in each monomial, if the total degree of $b_l$, $\overline{z}^0$ is not as same as the total degree of $b_l^+$, $z^0$, then the contribution of this term is 0. Thus by (3.9), (3.54), (5.77), (5.84), (5.87), (5.88), (5.95) and (6.9),

$$\begin{aligned}\Psi_{1,1}(Z^\perp) = \Big\{&(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3\\ &+\Big(\mathcal{F}_1(e_j^\perp)(b_j^{\perp+}+b_j^\perp) + \mathcal{T}_{ijj'}\frac{B_{ijj'}^\perp}{16\pi}\Big)\Big(\mathcal{F}_1(e_k^\perp)\frac{b_k^\perp}{4\pi} + \mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2}\Big)\Big]P^N\Big\}(Z^\perp,Z^\perp).\end{aligned} \tag{6.10}$$

By (3.8), (3.19) and (6.4),

$$(6.11)\quad (b_j z_i^0 P^N)(0,0) = -2\delta_{ij} P^N(0,0), \quad (b_k^\perp b_l^\perp b_j z_i^0 P^N)(0,0) = 8\pi\delta_{ij}\delta_{kl} P^N(0,0).$$

From Theorem 3.1, (3.9), (3.54), (6.4), (6.9) and (6.11),

$$\begin{aligned}(6.12)\quad &((\mathscr{L}_2^0)^{-1} P^{N^\perp} \mathcal{I}_1 P^N)(0,0)\\ &= \frac{1}{32\pi}\left\{(\mathscr{L}_2^0)^{-1} P^{N^\perp} \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_l^0})\Big(4\mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_l^0}) b_j^\perp b_{j'}^\perp + b_l b_j^\perp b_{j'}^\perp \mathcal{T}_{jj'}(z^0)\Big) P^N\right\}(0,0)\\ &= \frac{1}{32\pi}\mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_l^0})\Big\{\Big(\mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_l^0})\frac{b_j^\perp b_{j'}^\perp}{2\pi} + \frac{b_l b_j^\perp b_{j'}^\perp}{12\pi}\mathcal{T}_{jj'}(z^0)\Big)P^N\Big\}(0,0)\\ &= -\frac{1}{24\pi}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_l^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_l^0})P^N(0,0).\end{aligned}$$

By (3.9), (3.54), (5.5d), (5.14), (5.84) and (6.9),

$$\begin{aligned}(6.13)\quad &(P^{N^\perp}\mathcal{I}_2 P^N)(Z,(0,Z'^\perp)) = \frac{1}{2^8\pi^2}\Big\{P^{N^\perp}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})\\ &\Big[b_i \mathcal{T}_{kl}(z^0) B_{jj'}^\perp + (b_j^{\perp +} b_{j'}^{\perp +} - b_j^\perp b_{j'}^\perp)\mathcal{T}_{kl}(z^0) b_i\Big] b_k^\perp b_l^\perp P^N\Big\}(Z,(0,Z'^\perp))\\ &= \frac{1}{2^8\pi^2}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big\{P^{N^\perp}\Big[b_i\mathcal{T}_{kl}(z^0)(2b_j^{\perp +}b_{j'}^{\perp +} + 2b_j^\perp b_{j'}^{\perp +} + 4\pi\delta_{jj'})\\ &\quad + 2\mathcal{T}_{kl}(\tfrac{\partial}{\partial z_i^0})(b_j^{\perp +}b_{j'}^{\perp +} - b_j^\perp b_{j'}^\perp)\Big] b_k^\perp b_l^\perp P^N\Big\}(Z,(0,Z'^\perp))\\ &= \frac{1}{2^8\pi^2}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big\{b_i\Big(64\pi^2\mathcal{T}_{jj'}(z^0) + 16\pi\mathcal{T}_{kj'}(z^0)b_j^\perp b_k^\perp + 4\pi\delta_{jj'}\mathcal{T}_{kl}(z^0)b_k^\perp b_l^\perp\Big)\\ &\quad - 2\mathcal{T}_{kl}(\tfrac{\partial}{\partial z_i^0})b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp P^N\Big\}(Z,(0,Z'^\perp)).\end{aligned}$$

If $\alpha_{jj'}, \beta_{kl} \in \mathbb{C}$ for $j, j', k, l \in \{1, \cdots, n_0\}$ and $\beta_{kl}$ is symmetric on $k, l$, then by (3.22) and (6.4),

$$\begin{aligned}(6.14)\quad &\Big(\alpha_{jj'}\beta_{kl} b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp P^N\Big)(0,0)\\ &= \left\{\left[\sum_{k\neq l}\Big(2\alpha_{kl}\beta_{kl} + \alpha_{kk}\beta_{ll}\Big)(b_k^\perp)^2(b_l^\perp)^2 + \alpha_{ll}\beta_{ll}(b_l^\perp)^4\right]P^N\right\}(0,0)\\ &= (4\pi)^2\Big(\sum_{k\neq l}(2\alpha_{kl}\beta_{kl} + \alpha_{kk}\beta_{ll}) + 3\alpha_{ll}\beta_{ll}\Big)P^N(0,0)\\ &= (4\pi)^2(2\alpha_{kl}\beta_{kl} + \alpha_{kk}\beta_{ll})P^N(0,0).\end{aligned}$$

Thus by Theorem 3.1, (3.8), (6.4), (6.11), (6.13) and (6.14), we get

$$\begin{aligned}(6.15)\quad ((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{I}_2P^N)(0,0) &= \frac{1}{2^8\pi^2}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\bar{z}_i^0})\Big[\Big(16\pi b_i\mathcal{T}_{jj'}(z^0)\\ &+\frac{4}{3}b_i\mathcal{T}_{kj'}(z^0)b_j^\perp b_k^\perp+\frac{1}{3}\delta_{jj'}b_i\mathcal{T}_{kl}(z^0)b_k^\perp b_l^\perp-\frac{1}{8\pi}\mathcal{T}_{kl}(\tfrac{\partial}{\partial z_i^0})b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp\Big)P^N\Big](0,0)\\ &=\frac{1}{2^8\pi^2}\Big[-\frac{64\pi}{3}|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\bar{z}_i^0})|^2+\frac{8\pi}{3}\mathcal{T}_{jj}(\tfrac{\partial}{\partial\bar{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0})\\ &-2\pi\Big(2|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\bar{z}_i^0})|^2+\mathcal{T}_{jj}(\tfrac{\partial}{\partial\bar{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0})\Big)\Big]P^N(0,0)\\ &=\frac{1}{2^8\cdot 3\pi}\Big[-76|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\bar{z}_i^0})|^2+2\mathcal{T}_{jj}(\tfrac{\partial}{\partial\bar{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0})\Big]P^N(0,0).\end{aligned}$$

By (3.9), (3.54) and (6.9), we get

$$(6.16)\qquad \mathcal{I}_3P^N=-\frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijj'}\Big[b_j^\perp b_{j'}^\perp\mathcal{F}_1(e_i^\perp)+\mathcal{T}_{ilm}b_j^\perp b_{j'}^\perp\frac{b_l^\perp b_m^\perp}{16\pi}+\frac{1}{2}\mathcal{T}_{ilj'}b_j^\perp b_l^\perp\Big]P^N.$$

By (5.5e), (5.14), (6.4), (6.14) and (6.16), we get

$$(6.17)\qquad \Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{I}_3P^N\Big)(0,0)=\frac{\sqrt{-1}}{64\pi}\widetilde{\mathcal{T}}_{ijj'}\mathcal{T}_{ijj'}P^N(0,0)=0,$$

as $\widetilde{\mathcal{T}}_{ijj'}$ is anti-symmetric on $i,j$ and $\mathcal{T}_{ijj'}$ is symmetric on $i,j$.

By Theorem 3.1, (3.9), (3.54) and (6.4),

$$\begin{aligned}(6.18)\quad &\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{F}_1(e_j^\perp)(b_j^{\perp+}+b_j^\perp)\mathcal{F}_1(e_k^\perp)\frac{b_k^\perp}{4\pi}P^N\Big)(0,0)\\ &=\frac{1}{32\pi^2}\Big(\mathcal{F}_1(e_j^\perp)^2(b_j^\perp)^2P^N\Big)(0,0)=-\frac{1}{8\pi}\sum_j\mathcal{F}_1(e_j^\perp)^2P^N(0,0).\end{aligned}$$

Recall that $\mathcal{T}_{klm}$ is symmetric on $k,l,m$.

By Theorem 3.1, (3.9), (3.54), (5.84) and (6.4),

(6.19)

$$\begin{aligned}&\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big(\mathcal{F}_1(e_j^\perp)(b_j^{\perp+}+b_j^\perp)\mathcal{T}_{klm}\frac{b_m^\perp b_l^\perp b_k^\perp}{192\pi^2}+\mathcal{T}_{ijj'}\frac{B_{ijj'}^\perp}{64\pi^2}\mathcal{F}_1(e_k^\perp)b_k^\perp\Big)P^N\Big\}(0,0)\\ &=\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{F}_1(e_j^\perp)\Big(b_j^\perp\mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{48\pi^2}+\mathcal{T}_{jlm}\frac{b_l^\perp b_m^\perp}{4\pi}\Big)P^N\Big\}(0,0)\\ &=\frac{1}{32\pi^2}\Big\{\mathcal{F}_1(e_j^\perp)\Big(\mathcal{T}_{klm}\frac{b_j^\perp b_k^\perp b_l^\perp b_m^\perp}{24\pi}+\mathcal{T}_{jlm}b_l^\perp b_m^\perp\Big)P^N\Big\}(0,0)\\ &=\frac{1}{32\pi^2}\Big\{\mathcal{F}_1(e_j^\perp)\Big(\sum_{l\neq j}\mathcal{T}_{jll}\frac{(b_j^\perp)^2(b_l^\perp)^2}{8\pi}+\mathcal{T}_{jjj}\frac{(b_j^\perp)^4}{24\pi}+\mathcal{T}_{jll}(b_l^\perp)^2\Big)P^N\Big\}(0,0)\\ &=-\frac{1}{16\pi}\mathcal{F}_1(e_j^\perp)\mathcal{T}_{jll}P^N(0,0).\end{aligned}$$

As $\mathcal{T}_{klm}$ is symmetric on $k, l, m$, we know that

$$\mathcal{T}_{klm}^2 = 6 \sum_{k<l<m} \mathcal{T}_{klm}^2 + 3 \sum_{k\neq m} \mathcal{T}_{kkm}^2 + \mathcal{T}_{mmm}^2, \tag{6.20}$$
$$\mathcal{T}_{kkm}\mathcal{T}_{llm} = \sum_{k\neq l\neq m\neq k} \mathcal{T}_{kkm}\mathcal{T}_{llm} + \sum_{k\neq m} (2\mathcal{T}_{kkm}\mathcal{T}_{mmm} + \mathcal{T}_{kkm}^2) + \mathcal{T}_{mmm}^2.$$

From (6.4) and (6.20), we get

$$\begin{aligned} &\Big(\mathcal{T}_{ijj'}\mathcal{T}_{klm} b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp P^N\Big)(0,0) = \bigg\{\Big(36 \sum_{k<l<m} \mathcal{T}_{klm}^2 (b_k^\perp)^2 (b_l^\perp)^2 (b_m^\perp)^2 \\ &\quad + 9 \sum_{k\neq l\neq m\neq k} \mathcal{T}_{kkm}\mathcal{T}_{llm} (b_k^\perp)^2 (b_l^\perp)^2 (b_m^\perp)^2 + 6 \sum_{k\neq m} \mathcal{T}_{kkm}\mathcal{T}_{mmm} (b_k^\perp)^2 (b_m^\perp)^4 \\ &\quad + 9 \sum_{k\neq m} \mathcal{T}_{mmk}\mathcal{T}_{mmk} (b_k^\perp)^2 (b_m^\perp)^4 + \mathcal{T}_{mmm}^2 (b_m^\perp)^6\Big) P^N \bigg\}(0,0) \\ &= (-4\pi)^3 \Big(36 \sum_{k<l<m} \mathcal{T}_{klm}^2 + 9 \sum_{k\neq l\neq m\neq k} \mathcal{T}_{kkm}\mathcal{T}_{llm} \\ &\quad + 3 \sum_{k\neq m} (6\mathcal{T}_{kkm}\mathcal{T}_{mmm} + 9\mathcal{T}_{mmk}\mathcal{T}_{mmk}) + 15\mathcal{T}_{mmm}^2\Big) P^N(0,0) \\ &= (-4\pi)^3 \cdot 3\Big(2\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm}\mathcal{T}_{llm}\Big) P^N(0,0). \end{aligned} \tag{6.21}$$

By (3.9), (3.54) and (5.84), we have also

$$\begin{aligned} P^{N^\perp} \mathcal{T}_{ijj'} B_{ijj'}^\perp \mathcal{T}_{klm} b_k^\perp b_l^\perp b_m^\perp P^N = \Big(&\mathcal{T}_{ijj'}\mathcal{T}_{klm} b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp \\ &+ 36\pi \mathcal{T}_{ijm}\mathcal{T}_{klm} b_i^\perp b_j^\perp b_k^\perp b_l^\perp + 36\pi \cdot 8\pi \mathcal{T}_{ilm}\mathcal{T}_{klm} b_i^\perp b_k^\perp\Big) P^N. \end{aligned} \tag{6.22}$$

Thus from Theorem 3.1, (6.14), (6.21) and (6.22),

$$\begin{aligned} &\bigg\{\Big((\mathscr{L}_2^0)^{-1} P^{N^\perp} \frac{1}{16\pi} \mathcal{T}_{ijj'} B_{ijj'}^\perp \mathcal{T}_{klm} \frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2}\Big) P^N\bigg\}(0,0) \\ &\quad = \frac{1}{2^{10}\cdot 3\pi^3} \bigg\{\Big(\frac{1}{24\pi} \mathcal{T}_{ijj'}\mathcal{T}_{klm} b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp + \frac{9}{4} \mathcal{T}_{ijm}\mathcal{T}_{klm} b_i^\perp b_j^\perp b_k^\perp b_l^\perp \\ &\qquad + 36\pi \mathcal{T}_{ilm}\mathcal{T}_{klm} b_i^\perp b_k^\perp\Big) P^N\bigg\}(0,0) \\ &= \frac{1}{2^{10}\cdot 3\pi} \left\{-8(2\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm}\mathcal{T}_{llm}) + 36(2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm}\mathcal{T}_{llm}) - 144\mathcal{T}_{klm}^2\right\} P^N(0,0) \\ &= \frac{1}{2^8\cdot 3\pi} \left(-22\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm}\mathcal{T}_{llm}\right) P^N(0,0). \end{aligned} \tag{6.23}$$

From (6.10), (6.12), (6.15), (6.17), (6.18), (6.19) and (6.23), we get

(6.24)
$$\begin{aligned}\Psi_{1,1}(0) = \Big\{ \frac{1}{2^8\cdot 3\pi}\Big[ -76|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial \overline{z}_i^0})|^2 + 2\mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0}) - 22\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm}\mathcal{T}_{llm}\Big] \\ - \frac{1}{24\pi}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_l^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}_l^0}) - \frac{1}{8\pi}\sum_j \mathcal{F}_1(e_j^\perp)^2 - \frac{1}{16\pi}\mathcal{F}_1(e_j^\perp)\mathcal{T}_{jll}\Big\} P^N(0,0).\end{aligned}$$

From (6.24) we get (6.8). □

## 6.2. The term $\Psi_{1,2}$

Recall that $B(Z, e_l^\perp)$ was defined in (5.24).

***Lemma 6.2.*** — *The following identity holds,*

(6.25)
$$\begin{aligned}\frac{\sqrt{-1}}{\pi}B(Z,e_l^\perp) = \frac{1}{2}\left\langle R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e_l^\perp, J\mathcal{R}^0\right\rangle \\ - \frac{5}{4}\left\langle \nabla^{TY}_{\mathcal{R}}(T(e_k,e_l^\perp)), J\mathcal{R}^\perp\right\rangle Z_k \\ + \frac{1}{2}\left\langle \frac{1}{3}R^{TB}(\mathcal{R}^\perp,e_l^\perp)\mathcal{R}^\perp + \nabla^{TX_G}_{\mathcal{R}^0}(A(e_k^0)e_l^\perp)Z_k^0, J\mathcal{R}^0\right\rangle \\ + \frac{1}{8}\left\langle T(\mathcal{R}^0,e_j^0), Je_l^\perp\right\rangle\left\langle T(\mathcal{R}^\perp-\mathcal{R}^0, Je_j^0), J\mathcal{R}^\perp\right\rangle \\ + \frac{1}{4}\left\langle T(\mathcal{R}^\perp,e_j^0), Je_l^\perp\right\rangle\left\langle T(\mathcal{R}^0, Je_j^0), J\mathcal{R}^\perp\right\rangle \\ + \frac{1}{8}\left\langle T(\mathcal{R}^0,J\mathcal{R}^0), T(\mathcal{R}^\perp,e_l^\perp)\right\rangle - \frac{1}{8}\left\langle T(\mathcal{R},e_l^\perp), T(\mathcal{R}^\perp,J\mathcal{R}^0)\right\rangle \\ + \frac{1}{8}\left\langle T(\mathcal{R}^\perp, JT(\mathcal{R}^0,J\mathcal{R}^0)), Je_l^\perp\right\rangle + \frac{1}{2}\left\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp), T(\mathcal{R},e_l^\perp)\right\rangle.\end{aligned}$$

*Proof.* — By (5.34), (5.55) and $A(\mathcal{R}^0)A(\mathcal{R}^0)e_l^\perp \in N_G$, as $A$ exchanges $TX_G$ and $N_G$, we get

(6.26)
$$\begin{aligned}\langle J\mathcal{R}, (\nabla^{TX}\nabla^{TX}e_l^{\perp,H})_{(\mathcal{R},\mathcal{R})}\rangle = -\frac{1}{2}\left\langle J\mathcal{R}, T(\mathcal{R},\nabla^{TB}_{\mathcal{R}}e_l^\perp) + \nabla^{TX}_{\mathcal{R}}(T(e_i^H,e_l^\perp))Z_i\right\rangle \\ + \left\langle J\mathcal{R}^0, \frac{1}{3}R^{TB}(\mathcal{R}^\perp,e_l^\perp)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e_l^\perp + \nabla^{TX_G}_{\mathcal{R}^0}(A(e_k^0)e_l^\perp)Z_k^0\right\rangle.\end{aligned}$$

By (1.8), (5.13), (5.54), we have at $x_0$,

(6.27)
$$\begin{aligned}-\frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R},\nabla^{TB}_{\mathcal{R}}e_l^\perp)\right\rangle = \frac{1}{4}\left\langle Je_l^\perp, T(\mathcal{R}^0,e_j^0)\right\rangle\left\langle J\mathcal{R}^\perp, T(\mathcal{R},Je_j^0)\right\rangle, \\ -\frac{1}{2}\left\langle J\mathcal{R}^0, \nabla^{TX}_{\mathcal{R}}(T(e_i^H,e_l^{\perp,H}))Z_i\right\rangle = -\frac{1}{4}\left\langle T(\mathcal{R},e_l^\perp), T(\mathcal{R},J\mathcal{R}^0)\right\rangle.\end{aligned}$$

By (5.5a), (5.5d), (5.13), (5.54), (5.55) and $\nabla^{TX}_{\mathcal{R}}(T(e^H_i, e^H_k))Z_iZ_k = 0$, we have

$$
\begin{aligned}
\langle J(\nabla^{TX}\nabla^{TX}e^H_k)_{(\mathcal{R},\mathcal{R})}, e^\perp_l\rangle Z_k &= \frac{1}{2}\left\langle T(\mathcal{R}, \nabla^{TB}_{\mathcal{R}} e_k), Je^\perp_l\right\rangle Z_k \\
&= \frac{1}{2}\left\langle T(\mathcal{R}, 2A(\mathcal{R}^0)\mathcal{R}^\perp + A(\mathcal{R}^0)\mathcal{R}^0), Je^\perp_l\right\rangle \\
&= \frac{1}{2}\left\langle T(\mathcal{R}, e^0_j), Je^\perp_l\right\rangle \left\langle T(\mathcal{R}^0, Je^0_j), J\mathcal{R}^\perp\right\rangle \\
&\quad - \frac{1}{4}\left\langle T(\mathcal{R}^0, e^\perp_l), T(\mathcal{R}^0, J\mathcal{R}^0)\right\rangle + \frac{1}{4}\left\langle T(\mathcal{R}^\perp, JT(\mathcal{R}^0, J\mathcal{R}^0)), Je^\perp_l\right\rangle .
\end{aligned}
\tag{6.28}
$$

From (3.40), (5.5a), (5.13), (5.54) and the fact that $A$ exchanges $TX_G$ and $N_G$, we get

$$
\begin{aligned}
\left\langle J\nabla^{TX}_{\mathcal{R}} e^H_k, \nabla^{TX}_{\mathcal{R}} e^{\perp,H}_l\right\rangle Z_k &= \left\langle J\nabla^{TB}_{\mathcal{R}} e_k, A(\mathcal{R}^0)e^\perp_l - \frac{1}{2}T(\mathcal{R}, e^\perp_l)\right\rangle Z_k \\
&= \left\langle JA(\mathcal{R}^0)\mathcal{R}^0, -\frac{1}{2}T(\mathcal{R}, e^\perp_l)\right\rangle + 2\left\langle JA(\mathcal{R}^0)\mathcal{R}^\perp, A(\mathcal{R}^0)e^\perp_l\right\rangle , \\
&= \frac{1}{4}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e^\perp_l)\right\rangle - \frac{1}{2}\left\langle Je^\perp_l, T(\mathcal{R}^0, e^0_j)\right\rangle \left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0, Je^0_j)\right\rangle .
\end{aligned}
\tag{6.29}
$$

From (5.52), (5.53), (5.62), (6.26)-(6.29), we get

$$
\begin{aligned}
\frac{\sqrt{-1}}{\pi}B(Z, e^\perp_l) &= \frac{1}{8}\left\langle Je^\perp_l, T(\mathcal{R}^0, e^0_j)\right\rangle \left\langle J\mathcal{R}^\perp, T(\mathcal{R}, Je^0_j)\right\rangle \\
&\quad - \frac{1}{4}\left\langle J\mathcal{R}^\perp, \nabla^{TY}_{\mathcal{R}}(T(e_i, e^\perp_l))Z_i\right\rangle - \frac{1}{8}\left\langle T(\mathcal{R}, e^\perp_l), T(\mathcal{R}, J\mathcal{R}^0)\right\rangle \\
&\quad + \frac{1}{2}\left\langle J\mathcal{R}^0, \frac{1}{3}R^{TB}(\mathcal{R}^\perp, e^\perp_l)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)e^\perp_l + \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_k)e^\perp_l)Z^0_k\right\rangle \\
&\quad + \frac{1}{4}\left\langle T(\mathcal{R}, e^0_j), Je^\perp_l\right\rangle \left\langle T(\mathcal{R}^0, Je^0_j), J\mathcal{R}^\perp\right\rangle - \frac{1}{8}\left\langle T(\mathcal{R}^0, e^\perp_l), T(\mathcal{R}^0, J\mathcal{R}^0)\right\rangle \\
&\quad + \frac{1}{8}\left\langle T(\mathcal{R}^\perp, JT(\mathcal{R}^0, J\mathcal{R}^0)), Je^\perp_l\right\rangle + \frac{1}{4}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e^\perp_l)\right\rangle \\
&\quad - \frac{1}{2}\left\langle Je^\perp_l, T(\mathcal{R}^0, e^0_j)\right\rangle \left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0, Je^0_j)\right\rangle \\
&\quad + \frac{1}{2}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}, e^\perp_l)\right\rangle - \left\langle \nabla^{TY}_{\mathcal{R}}(T(e_k, e^\perp_l)), J\mathcal{R}^\perp\right\rangle Z_k.
\end{aligned}
\tag{6.30}
$$

From (6.30) we get (6.25). □

Now we need to compute the contribution from $-(\mathscr{L}^0_2)^{-1}P^{N^\perp}\mathscr{O}_2P^N$. Recall that $I_1$ was defined in (5.24).

***Lemma 6.3***. — *We have the following identity,*

$$
\begin{aligned}
(6.31)\quad -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}I_1P^N\Big)(0,0) = \Big\{-\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle \\
+\frac{7}{6}\Big[\frac{5\sqrt{-1}}{2^5\pi}\Big\langle Je_k^\perp, \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0}))+\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}))\Big\rangle \\
+\frac{3}{16\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle+\frac{3}{32\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})|^2 \\
-\frac{1}{2^6\pi}|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2-\frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\Big]\Big\}P^N(0,0).
\end{aligned}
$$

*Proof*. — From Theorem 3.1, (5.15), (5.84) and (6.4),

$$
(6.32)\qquad \Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}Z_k^\perp Z_l^\perp P^N\Big)(0,0)=\Big(\frac{b_k^\perp b_l^\perp}{2^7\pi^3}P^N\Big)(0,0)=-\frac{\delta_{kl}}{32\pi^2}P^N(0,0).
$$

Set

$$
(6.33)\qquad \mathcal{I}_4=-\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big(\tfrac{\partial}{\partial z_j^0}\Big(B(Z,\tfrac{\partial}{\partial \overline{z}_j^0})\Big)-\tfrac{\partial}{\partial \overline{z}_j^0}\Big(B(Z,\tfrac{\partial}{\partial z_j^0})\Big)\Big)P^N\Big\}(0,0).
$$

At first, if $Q$ is a monomial on $b_i, b_i^+$, $b_j^\perp, b_j^{\perp+}, Z_i$ and the total degree of $b_i, b_i^+, Z_i^0$ or $b_j^\perp, b_j^{\perp+}, Z_j^\perp$ is odd, then by Theorem 3.1,

$$
(6.34)\qquad \Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}QP^N\Big)(0,0)=0.
$$

By (6.34), only the monomials of $B(Z,e_l^0)$ with odd degree on $Z^0$ have contributions for $\mathcal{I}_4$.

If we denote by $\widetilde{B}_Z(e_l^0)$ the odd degree component on $Z^0$ of the difference of $B(Z,e_l^0)$ and of the sum of the first two and the last terms of $B(Z,e_l^0)$ in (5.46b), then by (5.46b) we know that $\widetilde{B}_Z(e_l^0)$ is a linear function on $Z^0$ and $\frac{\partial}{\partial z_j^0}\Big(\widetilde{B}_Z(\frac{\partial}{\partial \overline{z}_j^0})\Big)$ and $-\frac{\partial}{\partial \overline{z}_j^0}\Big(\widetilde{B}_Z(\frac{\partial}{\partial z_j^0})\Big)$ are equal.

Moreover, by $T(\frac{\partial}{\partial \overline{z}_j^0}, J\frac{\partial}{\partial z_j^0}) = T(\frac{\partial}{\partial z_j^0}, J\frac{\partial}{\partial \overline{z}_j^0})$ (or by (5.5e), (6.32)), we know the contribution of the last term of $B(Z,e_l^0)$ in (5.46b) is zero in $\mathcal{I}_4$.

Thus by Remark 5.2, (5.4), (5.46b) and (6.33),

$$
\begin{aligned}
(6.35)\quad \mathcal{I}_4 = \pi\sqrt{-1}\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[&\frac{1}{6}\tfrac{\partial}{\partial z_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0)\mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle \\
&-\frac{1}{6}\tfrac{\partial}{\partial \overline{z}_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0)\mathcal{R}^0, \tfrac{\partial}{\partial z_j^0}\Big\rangle \\
-\frac{5}{4}\Big\langle J\mathcal{R}^\perp, 2\nabla^{TY}_{\mathcal{R}^\perp}(T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})) &+ \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_i^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}))Z_i^\perp - \nabla^{TY}_{\frac{\partial}{\partial \overline{z}_j^0}}(T(e_i^\perp, \tfrac{\partial}{\partial z_j^0}))Z_i^\perp\Big\rangle \\
+3\sqrt{-1}\Big\langle R^{TB}(\mathcal{R}^\perp, \tfrac{\partial}{\partial z_j^0})\mathcal{R}^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle &- \frac{3\sqrt{-1}}{4}\Big\langle J\mathcal{R}^\perp, T(\tfrac{\partial}{\partial z_j^0}, e_i^0)\Big\rangle\Big\langle J\mathcal{R}^\perp, T(e_i^0, \tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle \\
-\frac{\sqrt{-1}}{4}\Big\langle T(\mathcal{R}^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}), T(\mathcal{R}^\perp, \tfrac{\partial}{\partial z_j^0})\Big\rangle &+ \Big\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\Big]P^N\Big\}(0,0).
\end{aligned}
$$

By (5.93), (5.108a), (6.32) and (6.35), comparing with (5.104) and (5.105), we get

$$
\begin{aligned}
(6.36)\quad \mathcal{I}_4 = \Big\{-\frac{1}{6\pi}\Big\langle &R^{TX_G}(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_j^0} + R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle \\
+\frac{5\sqrt{-1}}{2^7\pi}\Big\langle Je_k^\perp, 2\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})) &+ \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})) - \nabla^{TY}_{\frac{\partial}{\partial \overline{z}_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial z_j^0}))\Big\rangle \\
&+\frac{3}{32\pi}\Big\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0})e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{3}{64\pi}|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 \\
&-\frac{1}{2^7\pi}|T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 - \frac{\sqrt{-1}}{32\pi}\Big\langle T(e_k^\perp, Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\Big\}P^N(0,0).
\end{aligned}
$$

By (3.9), (3.54) and (5.84),

$$
(6.37)\quad
\begin{aligned}
&(z_i^0\overline{z}_j^0P^N)(Z,0) = (z_i^0\frac{b_j}{2\pi}P^N)(Z,0) = \frac{1}{2\pi}((b_jz_i^0 + 2\delta_{ij})P^N)(Z,0),\\
&Z_k^\perp Z_l^\perp P^N = \frac{1}{16\pi^2}(b_k^\perp b_l^\perp + 4\pi\delta_{kl})P^N,\\
&(4\pi)^4(Z_k^\perp)^4P^N = \big((b_k^\perp)^4 + 24\pi(b_k^\perp)^2 + 3\cdot(4\pi)^2\big)P^N.
\end{aligned}
$$

From Theorem 3.1, (3.9), (3.54), (5.93), (6.4), (6.11) and (6.37),

$$
\begin{aligned}
&(P^{N^\perp} Z_k^\perp Z_l^\perp P^N)(0,0) = \frac{1}{16\pi^2}(b_k^\perp b_l^\perp P^N)(0,0) = -\frac{\delta_{kl}}{4\pi} P^N(0,0),\\
&\Big((\mathscr{L}_2^0)^{-1} P^{N^\perp} b_j z_i^0 Z_k^\perp Z_l^\perp P^N\Big)(0,0)\\
&\quad = \frac{1}{16\pi^2}\Big\{\Big(\frac{1}{12\pi} b_k^\perp b_l^\perp b_j z_i^0 + \delta_{kl} b_j z_i^0\Big)P^N\Big\}(0,0) = -\frac{1}{12\pi^2}\delta_{ij}\delta_{kl}P^N(0,0),\\
&\big((\mathscr{L}_2^0)^{-1} b_l^\perp Z_k^\perp z_i^0 \overline{z}_j^0 P^N\big)(0,0) = \frac{1}{8\pi^2}\Big\{ b_l^\perp b_k^\perp \Big(\frac{b_j}{12\pi} z_i^0 + \frac{2}{8\pi}\delta_{ij}\Big)P^N\Big\}(0,0)\\
&\quad = -\frac{1}{24\pi^2}\delta_{ij}\delta_{kl}P^N(0,0),\\
&\Big((\mathscr{L}_2^0)^{-1} P^{N^\perp} Z_l^\perp Z_k^\perp z_i^0 \overline{z}_j^0 P^N\Big)(0,0)\\
&\quad = \frac{1}{4\pi}\Big\{(\mathscr{L}_2^0)^{-1} P^{N^\perp}(b_l^\perp Z_k^\perp + \delta_{kl}) z_i^0 \overline{z}_j^0 P^N\Big\}(0,0) = \frac{-7}{96\pi^3}\delta_{ij}\delta_{kl}P^N(0,0).
\end{aligned}
\tag{6.38}
$$

By (5.5e), (5.107), (5.108a), (6.38) and comparing with (5.109), we get

$$
\begin{aligned}
&-\Big((\mathscr{L}_2^0)^{-1} P^{N^\perp} b_j B(Z, \tfrac{\partial}{\partial \overline{z}_j^0}) P^N\Big)(0,0)\\
&\quad = \Big\{ -\frac{1}{12\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0} + R^{TX_G}(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\\
&\qquad + \frac{5\sqrt{-1}}{48\pi}\Big\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})) + \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})), J e_k^\perp\Big\rangle\\
&\qquad + \frac{1}{8\pi}\Big\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0}) e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{1}{16\pi}\Big|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2\\
&\qquad - \frac{1}{96\pi}\Big|T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 - \frac{\sqrt{-1}}{24\pi}\Big\langle T(e_k^\perp, J e_k^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\Big\} P(0,0).
\end{aligned}
\tag{6.39}
$$

From (6.25) and (6.34),

$$
\begin{aligned}
&\big((\mathscr{L}_2^0)^{-1} b_l^\perp B(Z, e_l^\perp) P^N\big)(0,0) = -\pi\sqrt{-1}\,\big\{(\mathscr{L}_2^0)^{-1} b_l^\perp\\
&\quad \Big[\frac{1}{2}\big\langle R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) e_l^\perp, J\mathcal{R}^0\big\rangle - \frac{5}{4}\big\langle \nabla^{TY}_{\mathcal{R}^\perp}(T(e_k^\perp, e_l^\perp)), J\mathcal{R}^\perp\big\rangle Z_k^\perp\\
&\quad - \frac{5}{4}\big\langle \nabla^{TY}_{\mathcal{R}^0}(T(e_k^0, e_l^\perp)), J\mathcal{R}^\perp\big\rangle Z_k^0 - \frac{1}{8}\big\langle T(\mathcal{R}^0, e_j^0), J e_l^\perp\big\rangle \big\langle T(\mathcal{R}^0, J e_j^0), J\mathcal{R}^\perp\big\rangle\\
&\quad + \frac{1}{8}\big\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^\perp, e_l^\perp)\big\rangle - \frac{1}{8}\big\langle T(\mathcal{R}^0, e_l^\perp), T(\mathcal{R}^\perp, J\mathcal{R}^0)\big\rangle\\
&\quad + \frac{1}{8}\big\langle T(\mathcal{R}^\perp, J T(\mathcal{R}^0, J\mathcal{R}^0)), J e_l^\perp\big\rangle + \frac{1}{2}\big\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}^\perp, e_l^\perp)\big\rangle\Big] P^N\Big\}(0,0).
\end{aligned}
\tag{6.40}
$$

As $T$ is anti-symmetric, from (3.9), (3.54), we get

$$
\begin{aligned}
& b_l^{\perp} \left\langle \nabla^{TY}_{\mathcal{R}^{\perp}} (T(e_k^{\perp}, e_l^{\perp})), J\mathcal{R}^{\perp} \right\rangle Z_k^{\perp} P^N \\
& \qquad = -\Big( \tfrac{\partial}{\partial Z_l^{\perp}} \left\langle \nabla^{TY}_{\mathcal{R}^{\perp}} (T(e_k^{\perp}, e_l^{\perp})), J\mathcal{R}^{\perp} \right\rangle \Big) Z_k^{\perp} P^N, \\
& b_l^{\perp} \left\langle T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}), T(\mathcal{R}^{\perp}, e_l^{\perp}) \right\rangle P^N \\
& \qquad = -\left\langle T(\mathcal{R}^{\perp}, Je_l^{\perp}) + T(e_l^{\perp}, J\mathcal{R}^{\perp}), T(\mathcal{R}^{\perp}, e_l^{\perp}) \right\rangle P^N.
\end{aligned} \tag{6.41}
$$

From (5.5e), (5.124), (6.32), (6.38), (6.40), (6.41) and the anti-symmetric property of $T$, we get

$$
\begin{aligned}
& -\frac{1}{2} \left( (\mathscr{L}_2^0)^{-1} b_l^{\perp} B(Z, e_l^{\perp}) P^N \right)(0,0) \\
& = \frac{\sqrt{-1}}{2\pi} \Big\{ -\frac{5}{2^7} \Big( \Big\langle \nabla^{TY}_{e_k^{\perp}} (T(e_k^{\perp}, e_l^{\perp})), Je_l^{\perp} \Big\rangle + \Big\langle \nabla^{TY}_{e_l^{\perp}} (T(e_k^{\perp}, e_l^{\perp})), Je_k^{\perp} \Big\rangle \Big) \\
& \qquad + \frac{5}{96} \Big\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}} (T(\tfrac{\partial}{\partial \overline{z}_j^0}, e_l^{\perp})) + \nabla^{TY}_{\frac{\partial}{\partial \overline{z}_j^0}} (T(\tfrac{\partial}{\partial z_j^0}, e_l^{\perp})), Je_l^{\perp} \Big\rangle \\
& \qquad\qquad + \frac{1}{2^6} \left\langle T(e_k^{\perp}, Je_l^{\perp}) + T(e_l^{\perp}, Je_k^{\perp}), T(e_k^{\perp}, e_l^{\perp}) \right\rangle \Big\} P^N(0,0) = 0.
\end{aligned} \tag{6.42}
$$

By (5.102), (5.124), (6.33), (6.36), (6.39), (6.42) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form, comparing with (5.105) and (5.109), we get (6.31). □

We compute $\Psi_{1,2}(0)$ now.

***Lemma 6.4***. — *The following identity holds,*

$$
\begin{aligned}
\Psi_{1,2}(0) = & \Big\{ \frac{1}{16\pi} r^{X_G}_{x_0} + \frac{1}{2\pi} R^{E_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0}) + \frac{1}{2\pi} \Delta_{X_G} \log h \\
& + \frac{29}{2^5 \cdot 3\pi} |T(e_k^{\perp}, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 + \frac{\sqrt{-1}}{16\pi} \Big\langle T(e_k^{\perp}, Je_k^{\perp}), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0}) \Big\rangle \\
& + \frac{1}{4\pi} \Big| T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}) \Big|^2 + \frac{1}{32\pi} \Big| \sum_j \mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0}) \Big|^2 \\
& + \frac{1}{2^7\pi} \widetilde{\mathcal{T}}_{ijk} (\widetilde{\mathcal{T}}_{kji} + \widetilde{\mathcal{T}}_{ijk}) + \frac{7}{2^8\pi} \Big( 2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jjm} \mathcal{T}_{kkm} \Big) \\
& - \frac{1}{2^6\pi} \Big\langle (\nabla^{TY}_{\cdot} \dot{g}^{TY}_{\cdot})_{(e_j^{\perp}, e_j^{\perp})} Je_k^{\perp} + 2 (\nabla^{TY}_{\cdot} \dot{g}^{TY}_{\cdot})_{(e_j^{\perp}, e_k^{\perp})} Je_j^{\perp}, Je_k^{\perp} \Big\rangle \\
& - \frac{\sqrt{-1}}{16\pi} \Big( \left\langle T(e_j^{\perp}, Je_j^{\perp}), \widetilde{\mu}^E \right\rangle - 2 \Big\langle Je_j^{\perp}, \nabla^{TY}_{e_j^{\perp}} \widetilde{\mu}^E \Big\rangle \Big) \Big\} P^N(0,0).
\end{aligned} \tag{6.43}
$$

*Proof.* — Recall that from (3.6), (5.5a), (5.5b) and (5.13),

$$
\begin{aligned}
&|A(e_i^0)e_k^\perp|^2 = 4|A(\tfrac{\partial}{\partial z_i^0})e_k^\perp|^2 = |T(\tfrac{\partial}{\partial z_i^0}, Je_j^0)|^2 = 2|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})|^2,\\
&\langle A(e_i^0)e_i^0, A(e_j^0)e_j^0\rangle = 4\Big|\sum_i T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2,\\
&|A(e_i^0)e_j^0|^2 = \frac{1}{4}|T(e_i^0, Je_j^0)|^2 = |T(\tfrac{\partial}{\partial z_i^0}, Je_j^0)|^2 = 2|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})|^2.
\end{aligned}
\tag{6.44}
$$

From (5.93), (5.111), (6.32), (6.44) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form (comparing with (5.113b), (5.114)), (note that in each monomial, if the total degree of $b_l$, $\overline{z}^0$ is not as same as the total degree of $b_l^+$, $z^0$, then the contribution of this term is 0 at $(0,0)$), we get

$$
\begin{aligned}
-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}I_2P^N\Big)(0,0) = \Big\{&\frac{4}{3\pi}\left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}\right\rangle\\
&-\frac{1}{8\pi}\left\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0})e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\right\rangle - \frac{1}{48\pi}\left\langle R^{TB}(e_k^\perp, e_j^\perp)e_k^\perp, e_j^\perp\right\rangle\\
&+\frac{3}{16\pi}|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})|^2\Big\}P^N(0,0).
\end{aligned}
\tag{6.45}
$$

By (3.6), (3.54), (5.25), (5.83), (5.93), (5.112), (6.32), (6.44) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form (comparing with (5.113a)), we get

$$
\begin{aligned}
-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}&\langle\Gamma_{ii}(\mathcal{R}), e_l\rangle\nabla_{0,e_l}P^N\Big)(0,0)\\
&= \Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big(\frac{2}{3}\left\langle R^{TX_G}(\mathcal{R}^0, e_i^0)e_i^0, \tfrac{\partial}{\partial \overline{z}_j^0}\right\rangle b_j\\
&\quad+\frac{1}{2}\left\langle R^{TB}(\mathcal{R}^\perp, e_i^0)e_i^0 + A(e_i^0)A(e_i^0)\mathcal{R}^\perp, e_k^\perp\right\rangle b_k^\perp\Big)P^N\Big\}(0,0)\\
= \Big\{-\frac{1}{3\pi}&\left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, e_i^0)e_i^0, \tfrac{\partial}{\partial \overline{z}_j^0}\right\rangle - \frac{1}{16\pi}\left\langle R^{TB}(e_k^\perp, e_i^0)e_i^0, e_k^\perp\right\rangle + \frac{1}{16\pi}|A(e_i^0)e_k^\perp|^2\Big\}P^N(0,0)\\
= \Big\{\Big\langle-\frac{2}{3\pi}&R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0} + \frac{1}{4\pi}R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0})e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{1}{8\pi}|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})|^2\Big\}P^N(0,0).
\end{aligned}
\tag{6.46}
$$

By $\mathscr{L}_2^0 P^N = 0$, (5.25), (5.93), (6.38), (6.44) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form (comparing with (5.115)), we get

$$
\begin{aligned}
(6.47)\quad & -\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\frac{1}{4}K_2(\mathcal{R}) - \frac{3}{8}\Big(\sum_l \big\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\big\rangle\Big)^2, \mathscr{L}_2^0\Big]P^N\Big\}(0,0)\\
&= \Big\{P^{N^\perp}\Big[\frac{1}{4}K_2(\mathcal{R}) - \frac{3}{8}\Big(\sum_l \big\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\big\rangle\Big)^2\Big]P^N\Big\}(0,0)\\
&= \frac{1}{4}\Big\{P^{N^\perp}\Big[\Big\langle \frac{1}{3}R^{TX_G}(\mathcal{R}^0, e_i^0)\mathcal{R}^0 + R^{TB}(\mathcal{R}^\perp, e_i^0)\mathcal{R}^\perp, e_i^0\Big\rangle\\
&\quad + \frac{1}{3}\big\langle R^{TB}(\mathcal{R}^\perp, e_i^\perp)\mathcal{R}^\perp, e_i^\perp\big\rangle + \frac{1}{2}\Big(\sum_i \big\langle A(e_i^0)e_i^0, \mathcal{R}^\perp\big\rangle\Big)^2 - |A(e_i^0)\mathcal{R}^\perp|^2\Big]P^N\Big\}(0,0)\\
&= \Big(\frac{1}{6\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, e_i^0)e_i^0, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle - \frac{1}{16\pi}\big\langle R^{TB}(e_k^\perp, e_i^0)e_k^\perp, e_i^0\big\rangle\\
&\quad - \frac{1}{48\pi}\big\langle R^{TB}(e_k^\perp, e_i^\perp)e_k^\perp, e_i^\perp\big\rangle - \frac{1}{32\pi}\Big|\sum_i A(e_i^0)e_i^0\Big|^2 + \frac{1}{16\pi}|A(e_i^0)e_k^\perp|^2\Big)P^N(0,0),\\
&= \Big(\frac{1}{3\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle - \frac{1}{4\pi}\Big\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0})e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\\
&\quad - \frac{1}{8\pi}\Big|\sum_i T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2 + \frac{1}{8\pi}|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 - \frac{1}{48\pi}\big\langle R^{TB}(e_k^\perp, e_j^\perp)e_k^\perp, e_j^\perp\big\rangle\Big)P^N(0,0).
\end{aligned}
$$

By (3.12), (3.54), (5.83), (5.93), (6.32) and (6.44),

$$
\begin{aligned}
(6.48)\quad & \\
& -\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big(-\frac{1}{2}\big\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\big\rangle\nabla_{A(e_k^0)e_k^0} + 2\big\langle A(e_i^0)e_j^0, \mathcal{R}^\perp\big\rangle\nabla_{A(e_i^0)e_j^0}\\
&\qquad + \frac{2}{3}\big\langle R^{TB}(\mathcal{R}^\perp, e_i^\perp)e_i^\perp, e_j\big\rangle\nabla_{0,e_j}\Big)P^N\Big\}(0,0)\\
&= -\frac{1}{16\pi}\Big(-\frac{1}{2}\Big|\sum_l A(e_l^0)e_l^0\Big|^2 + 2|A(e_i^0)e_j^0|^2 + \frac{2}{3}\big\langle R^{TB}(e_j^\perp, e_i^\perp)e_i^\perp, e_j^\perp\big\rangle\Big)P^N(0,0)\\
&= \Big(\frac{1}{8\pi}\Big|\sum_i T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2 - \frac{1}{4\pi}\Big|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2\\
&\qquad + \frac{1}{24\pi}\big\langle R^{TB}(e_k^\perp, e_j^\perp)e_k^\perp, e_j^\perp\big\rangle\Big)P^N(0,0),\\
& -\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}(-R^{E_B}(\mathcal{R}, e_i))\nabla_{0,e_i}P^N\Big\}(0,0) = \frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})P^N(0,0).
\end{aligned}
$$

For $F_{ij;kl} \in \mathbb{C}$, from Theorem 3.1, (5.15), (6.4), (6.37) and comparing with (6.14), we get

$$
\begin{aligned}
(6.49)\quad & \Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp} F_{ij;kl} Z_i^\perp Z_j^\perp Z_k^\perp Z_l^\perp P^N\Big\}(0,0)\\
&= \Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\sum_{j\neq k}(F_{jj;kk}+F_{kj;kj}+F_{kj;jk})(Z_j^\perp)^2(Z_k^\perp)^2 + F_{kk;kk}(Z_k^\perp)^4\Big]P^N\Big\}(0,0)\\
&= \frac{1}{2^8\pi^4}\Bigg\{P^{N^\perp}\Big[\sum_{j\neq k}(F_{jj;kk}+F_{kj;kj}+F_{kj;jk})\Big(\frac{(b_j^\perp)^2(b_k^\perp)^2}{16\pi}+\frac{1}{2}((b_j^\perp)^2+(b_k^\perp)^2)\Big)\\
&\qquad + F_{kk;kk}\Big(\frac{(b_k^\perp)^4}{16\pi}+3(b_k^\perp)^2\Big)\Big]P^N\Bigg\}(0,0)\\
&= \frac{-3}{2^8\pi^3}(F_{jj;kk}+F_{kj;kj}+F_{kj;jk})P^N(0,0).
\end{aligned}
$$

By (5.46a),

$$
\begin{aligned}
(6.50)\quad \frac{1}{9}\sum_i\Big[(\partial_{\mathcal{R}}R^{L_B})_{x_0}(\mathcal{R},e_i)\Big]^2 &= -\pi^2\sum_i\big\langle JT(\mathcal{R}^\perp,e_i^0),\mathcal{R}^\perp\big\rangle^2\\
&\quad -\pi^2\sum_j\big\langle JT(\mathcal{R},e_j^\perp),\mathcal{R}^\perp\big\rangle^2.
\end{aligned}
$$

By (3.6), (5.14), (6.49) and $\mathcal{T}_{kl}(e_i^0)$ is symmetric on $k,l$, we get

$$
\begin{aligned}
(6.51)\quad & -\pi^2\sum_i\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\big\langle JT(\mathcal{R}^\perp,e_i^0),\mathcal{R}^\perp\big\rangle^2 P^N\Big)(0,0)\\
&= -\pi^2\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{T}_{jj'}(e_i^0)\mathcal{T}_{kl}(e_i^0)Z_j^\perp Z_{j'}^\perp Z_k^\perp Z_l^\perp P^N\Big)(0,0)\\
&= \frac{3}{2^8\pi}\Big(2\mathcal{T}_{jk}(e_i^0)^2+\mathcal{T}_{jj}(e_i^0)\mathcal{T}_{kk}(e_i^0)\Big)P^N(0,0)\\
&= \frac{3}{2^6\pi}\Big(2|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2+\Big|\sum_j\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2\Big)P^N(0,0).
\end{aligned}
$$

In the same way, by (5.5e), (5.14), (6.49), we get

$$
\begin{aligned}
(6.52)\quad & -\pi^2\sum_j\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\big\langle JT(\mathcal{R}^\perp,e_j^\perp),\mathcal{R}^\perp\big\rangle^2 P^N\Big)(0,0)\\
&= \frac{3}{2^8\pi}\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji})P^N(0,0).
\end{aligned}
$$

By (5.14) and (6.38),

$$
\begin{aligned}
(6.53)\quad & -\pi^2\sum_j\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\big\langle JT(\mathcal{R}^0,e_j^\perp),\mathcal{R}^\perp\big\rangle^2 P^N\Big)(0,0)\\
&= \frac{7}{48\pi}|\mathcal{T}_{jk}(\tfrac{\partial}{\partial\overline{z}_i^0})|^2P^N(0,0) = \frac{7}{48\pi}|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2P^N(0,0).
\end{aligned}
$$

By (5.46a) and (5.116), the total degree of $Z^0$, $\nabla_{0,e_i^0}$ in the fourth term of $\mathcal{O}_2'$ in (5.27) is 1, thus the contribution of the fourth term of $\mathcal{O}_2'$ in (5.27) for $-((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2'P^N)(0,0)$ is zero. By (5.27), (6.31), (6.45)-(6.48) and (6.50)-(6.53), comparing with (5.118), we get

$$
\begin{aligned}
(6.54)\quad & -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2'P^N\Big)(0,0) = \bigg\{\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle \\
& + \frac{7}{6}\bigg[\frac{5\sqrt{-1}}{2^5\pi}\bigg\langle Je_k^\perp, \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})) + \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}))\bigg\rangle \\
& + \frac{3}{16\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{3}{32\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})|^2 \\
& - \frac{1}{2^6\pi}|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2 - \frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp,Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\bigg] \\
& + (\frac{3}{32\pi}+\frac{7}{48\pi})|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2 - \frac{1}{8\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle \\
& + \frac{3}{16\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})|^2 + \frac{3}{64\pi}\Big|\sum_j \mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2 \\
& + \frac{3}{2^8\pi}\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji}) + \frac{1}{2\pi}R^{E_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\bigg\}P^N(0,0).
\end{aligned}
$$

By (5.63) and (6.34),

$$
\begin{aligned}
(6.55)\quad & -4\pi^2\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2''P^N\Big)(0,0) = -4\pi^2\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp} \\
& \Big[-\frac{1}{3}\big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^0,\mathcal{R}^0)}J\mathcal{R}^\perp + (\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^\perp,\mathcal{R}^\perp)}J\mathcal{R}^\perp, J\mathcal{R}^\perp\big\rangle \\
& + \frac{1}{6}\big\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_j^\perp,J_{x_0}e_i^0))Z_j^\perp Z_i^0 + \nabla^{TY}_{\mathcal{R}^\perp}(T(e_j^0,J_{x_0}e_i^0))Z_j^0 Z_i^0, J\mathcal{R}^\perp\big\rangle \\
& + \frac{1}{3}\big\langle R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0,\mathcal{R}^\perp\big\rangle - \frac{1}{12}\sum_l\big\langle T(\mathcal{R}^0,e_l),J\mathcal{R}^\perp\big\rangle^2 \\
& - \frac{1}{12}\sum_l\big\langle T(\mathcal{R}^\perp,e_l),J\mathcal{R}^\perp\big\rangle^2 + \frac{7}{12}|T(\mathcal{R}^\perp,J\mathcal{R}^\perp)|^2\Big]P^N\Big\}(0,0).
\end{aligned}
$$

Now $\{e_l\} = \{e_i^0\}\cup\{e_k^\perp\}$.

By Theorem 5.1, (5.108a), (5.120), (5.124), (6.38), (6.49), (6.51), (6.52), (6.55) and comparing with (5.121),

$$\begin{aligned}(6.56)\quad & -4\pi^2\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2''P^N\Big)(0,0) = \Big\{\frac{7}{24\pi}\Big[-\frac{8}{3}\nabla_{\frac{\partial}{\partial z_j^0}}\nabla_{\frac{\partial}{\partial \overline{z}_j^0}}\log h \\ & + \frac{\sqrt{-1}}{3}\Big\langle -\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})) - \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\Big\rangle \\ & -\frac{1}{3}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 - \frac{1}{6}\Big|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 - \frac{2}{3}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\Big] \\ & - \frac{1}{2^6\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_j^\perp)}Je_k^\perp + 2(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_k^\perp)}Je_j^\perp, Je_k^\perp\Big\rangle \\ & - \frac{1}{2^8\pi}\Big(8|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2 + 4\Big|\sum_j \mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2 + \widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji})\Big) \\ & + \frac{7}{2^8\pi}\Big(2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jjm}\mathcal{T}_{kkm}\Big)\Big\}P^N(0,0).\end{aligned}$$

By (5.74), (5.77), (6.32), (6.54) and (6.56), comparing with (5.101), we have

$$\begin{aligned}(6.57)\quad \Psi_{1,2}(0) = & -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}(\mathcal{O}_2' + 4\pi^2\mathcal{O}_2'')P^N\Big)(0,0) \\ & - \frac{\sqrt{-1}}{16\pi}\Big(\big\langle T(e_j^\perp,Je_j^\perp),\widetilde{\mu}^E\big\rangle - 2\Big\langle Je_j^\perp,\nabla^{TY}_{e_j^\perp}\widetilde{\mu}^E\Big\rangle\Big)P^N(0,0) \\ = & \Big\{\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{1}{2\pi}R^{E_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0}) \\ & + \frac{7}{6}\Big[\frac{1}{6\pi}\Delta_{X_G}\log h + \frac{1}{48\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{1}{96\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 \\ & - \frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle + \frac{13}{192\pi}\Big|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2 \\ & + \frac{7\sqrt{-1}}{96\pi}\Big\langle\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})) + \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\Big\rangle\Big] \\ & + \frac{1}{16\pi}|T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})|^2 - \frac{1}{8\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle + \frac{3}{16\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})|^2 \\ & + \frac{1}{32\pi}\Big|\sum_j \mathcal{T}_{jj}(\tfrac{\partial}{\partial \overline{z}_i^0})\Big|^2 + \frac{1}{2^7\pi}\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji}) + \frac{7}{2^8\pi}\Big(2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jjm}\mathcal{T}_{kkm}\Big) \\ & - \frac{1}{2^6\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_j^\perp)}Je_k^\perp + 2(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_k^\perp)}Je_j^\perp, Je_k^\perp\Big\rangle \\ & - \frac{\sqrt{-1}}{16\pi}\Big(\big\langle T(e_j^\perp,Je_j^\perp),\widetilde{\mu}^E\big\rangle - 2\Big\langle Je_j^\perp,\nabla^{TY}_{e_j^\perp}\widetilde{\mu}^E\Big\rangle\Big)\Big\}P^N(0,0).\end{aligned}$$

By (5.124), (5.131), the term $\frac{7}{6}[\cdots]$ in (6.57) is $\frac{7}{6}\Big(\frac{3}{8\pi}\Delta_{X_G}\log h + \frac{1}{8\pi}|T(e_k^\perp,\frac{\partial}{\partial \overline{z}_j^0})|^2\Big)$. By (5.130) and (6.57), we get (6.43).

The proof of Lemma 6.4 is complete. □

***Lemma 6.5***. — *The following identity holds,*

$$\begin{aligned}
&\left\langle(\nabla^{TY}_{e_k^\perp}\dot g^{TY}_{e_k^\perp})Je_l^\perp, Je_l^\perp\right\rangle = 4\nabla_{e_k^\perp}\nabla_{e_k^\perp}\log h,\\
&\left\langle(\nabla^{TY}_{e_k^\perp}\dot g^{TY}_{e_l^\perp})Je_l^\perp, Je_k^\perp\right\rangle = 4\nabla_{e_k^\perp}\nabla_{e_k^\perp}\log h + 2\Big|\sum_l \mathcal{T}_{ll}(\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2\\
&\qquad\qquad - 2|T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 - \frac{1}{2}(\widetilde{\mathcal{T}}_{jki}+\widetilde{\mathcal{T}}_{ijk})\widetilde{\mathcal{T}}_{ijk}.
\end{aligned}\tag{6.58}$$

*Proof*. — By using the same argument as in (5.120), we get the first equation of (6.58).

Recall that $P^{T^HX}, P^{TY}$ are the projections from $TX = T^HX \oplus TY$ onto $T^HX, TY$.

By (1.3), (1.7), (3.1), (3.40) and (3.41) (cf. also (5.32)),

$$(P^{T^HX}Je_l^{\perp,H})|_{\mu^{-1}(0)} = 0, \quad (Je_l^{\perp,H})_{x_0} \in TY,\tag{6.59a}$$

$$\begin{aligned}
&(\nabla^{TX}_{e_k^{\perp,H}}e_l^{\perp,H})_{x_0} = -\frac{1}{2}T(e_k^\perp, e_l^\perp),\\
&(\nabla^{TX}_{e_j^0}e_l^{\perp,H})_{x_0} = (A(e_j^0)e_l^\perp)^H - \frac{1}{2}T(e_j^0, e_l^\perp),
\end{aligned}\tag{6.59b}$$

$$(\nabla^{TX}_{Je_l^{\perp,H}}e_k^H)_{x_0} = \frac{1}{2}\left\langle T(e_k,e_j), Je_l^\perp\right\rangle e_j^H + \left\langle T(e_k, Je_l^\perp), Je_j^\perp\right\rangle Je_j^\perp.\tag{6.59c}$$

From (6.59a), we get

$$\nabla^{TX}_{e_i^0}P^{T^HX}Je_l^{\perp,H} = \nabla^{TX}_{Je_k^{\perp,H}}P^{T^HX}Je_l^{\perp,H} = 0.\tag{6.60}$$

By (3.40), (5.14), (5.72) and (6.59b), we get at $x_0$,

$$\begin{aligned}
\nabla^{TX}_{e_k^{\perp,H}}P^{T^HX}Je_l^{\perp,H} &= \nabla^{T^HX}_{e_k^{\perp,H}}P^{T^HX}Je_l^{\perp,H}\\
&= -\frac{1}{2}JT(e_k^\perp, e_l^\perp) + \frac{1}{2}\left\langle JT(e_k^\perp, e_j), e_l^\perp\right\rangle e_j\\
&= -\frac{1}{2}(\widetilde{\mathcal{T}}_{klj} - \widetilde{\mathcal{T}}_{kjl})e_j^\perp + \frac{1}{2}\left\langle JT(e_k^\perp, e_j^0), e_l^\perp\right\rangle e_j^0.
\end{aligned}\tag{6.61}$$

By (5.6a), (6.59b), (6.60) and (6.61), at $x_0$,

$$\begin{aligned}
&\nabla^{TX}_{e_j^{0,H}}P^{TY}Je_l^{\perp,H} = J\nabla^{TX}_{e_j^{0,H}}e_l^{\perp,H} = JA(e_j^0)e_l^\perp - \frac{1}{2}JT(e_j^0, e_l^\perp),\\
&\nabla^{TX}_{e_k^{\perp,H}}P^{TY}Je_l^{\perp,H} = J\nabla^{TX}_{e_k^{\perp,H}}e_l^{\perp,H} - \nabla^{TX}_{e_k^{\perp,H}}P^{T^HX}Je_l^{\perp,H}\\
&\qquad = -\frac{1}{2}\left\langle JT(e_k^\perp, e_j), e_l^\perp\right\rangle e_j = -\frac{1}{2}\widetilde{\mathcal{T}}_{kjl}e_j^\perp - \frac{1}{2}\left\langle JT(e_k^\perp, e_j^0), e_l^\perp\right\rangle e_j^0.
\end{aligned}\tag{6.62}$$

Thus by (6.62), at $x_0$,

$$\nabla^{TY}_{e_k^{\perp,H}}P^{TY}Je_l^{\perp,H} = P^{TY}\nabla^{TX}_{e_k^{\perp,H}}P^{TY}Je_l^{\perp,H} = 0.\tag{6.63}$$

By (1.3), (1.6), (1.7) and (6.63), at $x_0$,

$$\begin{aligned}(6.64)\quad \Big\langle (\nabla^{TY}_{e^\perp_k}\dot g^{TY}_{e^\perp_l})Je^\perp_l, Je^\perp_k\Big\rangle &= e^\perp_k\Big\langle \dot g^{TY}_{e^\perp_l}P^{TY}Je^\perp_l, P^{TY}Je^\perp_k\Big\rangle\\ &= 2e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}e^\perp_l, P^{TY}Je^\perp_k\Big\rangle\\ &= 2e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}e^\perp_l, Je^\perp_k\Big\rangle - 2e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}e^\perp_l, P^{T^HX}Je^\perp_k\Big\rangle.\end{aligned}$$

By (5.5e), (5.14), (6.59a), (6.59c) and (6.61), at $x_0$, we have

$$\begin{aligned}(6.65)\quad -2e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}e^\perp_l, P^{T^HX}Je^\perp_k\Big\rangle &= -2\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}e^\perp_l, \nabla^{TX}_{e^\perp_k}P^{T^HX}Je^\perp_k\Big\rangle\\ &= -\frac{1}{2}\left\langle T(e^\perp_l,e_j), Je^\perp_l\right\rangle\left\langle JT(e^\perp_k,e_j), e^\perp_k\right\rangle = \frac{1}{2}\mathcal{T}_{ll}(e^0_j)\mathcal{T}_{kk}(e^0_j).\end{aligned}$$

Now by (5.6a),

$$\begin{aligned}(6.66)\quad e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}e^\perp_l, Je^\perp_k\Big\rangle &= -e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}Je^\perp_l, e^\perp_k\Big\rangle\\ &= -e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}P^{TY}Je^\perp_l + \nabla^{TX}_{P^{TY}Je^\perp_l}P^{T^HX}Je^\perp_l, e^\perp_k\Big\rangle.\end{aligned}$$

Observe that for any $Y\in\mathscr{C}^\infty(X,TY)$, $[e^{\perp,H}_k, Y]\in TY$. Thus

$$(6.67)\qquad [e^{\perp,H}_k, P^{TY}Je^{\perp,H}_l]\in TY.$$

From (6.59a) and (6.67), at $x_0$,

$$(6.68)\qquad \nabla^{TX}_{[e^{\perp,H}_k, P^{TY}Je^{\perp,H}_l]}P^{T^HX}Je^\perp_l = 0.$$

And by (5.5d), (6.59a)-(6.59c), (6.60) and (6.61), as $\widetilde{\mathcal{T}}_{klj}$, $\mathcal{T}_{kl}(e^0_j)$ are constant functions along the fiber $Gx_0$, at $x_0$,

$$\begin{aligned}(6.69)\quad & -2e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}P^{T^HX}Je^\perp_l, e^\perp_k\Big\rangle\\ &= -2\Big\langle\Big(\nabla^{TX}_{P^{TY}Je^\perp_l}\nabla^{TX}_{e^\perp_k} + \nabla^{TX}_{[e^{\perp,H}_k, P^{TY}Je^{\perp,H}_l]}\Big)P^{T^HX}Je^{\perp,H}_l, e^\perp_k\Big\rangle\\ &= -\Big\langle T\Big(-\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})e^\perp_j + \frac{1}{2}\left\langle JT(e^\perp_k,e^0_j), e^\perp_l\right\rangle e^0_j, e^\perp_k\Big), Je^\perp_l\Big\rangle\\ &= -\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})\widetilde{\mathcal{T}}_{jkl} - \frac{1}{2}|T(e^\perp_k,e^0_j)|^2.\end{aligned}$$

Finally, by (1.4), (1.7), (1.24) and (6.63), as in (5.120),

$$\begin{aligned}(6.70)\quad -2e^\perp_k\Big\langle \nabla^{TX}_{P^{TY}Je^\perp_l}P^{TY}Je^\perp_l, e^\perp_k\Big\rangle &= 2e^\perp_k\left\langle T(e^\perp_k, P^{TY}Je^\perp_l), P^{TY}Je^\perp_l\right\rangle\\ &= \Big\langle (\nabla^{TY}_{e^\perp_k}\dot g^{TY}_{e^\perp_k})Je^\perp_l, Je^\perp_l\Big\rangle = 4\nabla_{e^\perp_k}\nabla_{e^\perp_k}\log h.\end{aligned}$$

Thus by (6.64)-(6.70),

$$\Big\langle (\nabla^{TY}_{e^\perp_k}\dot g^{TY}_{e^\perp_l})Je^\perp_l, Je^\perp_k\Big\rangle = 4\nabla_{e^\perp_k}\nabla_{e^\perp_k}\log h + \frac{1}{2}\mathcal{T}_{ll}(e^0_j)\mathcal{T}_{kk}(e^0_j) - \frac{1}{2}|T(e^\perp_k, e^0_j)|^2 - \frac{1}{2}(\widetilde{\mathcal{T}}_{klj} - \widetilde{\mathcal{T}}_{kjl})\widetilde{\mathcal{T}}_{jkl}. \tag{6.71}$$

From (3.6), (5.14) and (6.71), we get (6.58). □

## 6.3. Proof of Theorem 0.7

By (5.14), (5.95),

$$\begin{aligned}
\sum_k \mathcal{F}_1(e^\perp_k)^2 = &- \langle \widetilde{\mu}^E_{x_0}, \widetilde{\mu}^E_{x_0}\rangle_{g^{TY}} - \Big\langle \widetilde{\mu}^E, \frac{3}{2}\sqrt{-1}T(e^\perp_l, Je^\perp_l) + 2T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \overline{z}^0_j})\Big\rangle\\
&+ \Big|\sum_j T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \overline{z}^0_j})\Big|^2 + \frac{9}{16}\mathcal{T}_{llm}\mathcal{T}_{kkm} - \frac{3\sqrt{-1}}{2}\Big\langle T(e^\perp_l, Je^\perp_l), T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \overline{z}^0_j})\Big\rangle,\\
\mathcal{F}_1(e^\perp_k)\mathcal{T}_{kll} = &-\sqrt{-1}\Big\langle T(e^\perp_l, Je^\perp_l), \widetilde{\mu}^E + T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \overline{z}^0_j})\Big\rangle + \frac{3}{4}\mathcal{T}_{llm}\mathcal{T}_{kkm}.
\end{aligned} \tag{6.72}$$

By (5.14), (6.6), (6.7), (6.8) and (6.72), we have

$$\begin{aligned}
(\Psi_{1,1} + \Psi^*_{1,1} + \Psi_{1,3} - \Psi_{1,4})(0) = &\Big\{-\frac{1}{2\pi}\sum_k \mathcal{F}_1(e^\perp_k)^2 - \frac{1}{8\pi}\mathcal{F}_1(e^\perp_k)\mathcal{T}_{kll}\\
&- \frac{11}{48\pi}\Big|\mathcal{T}_{kl}(\tfrac{\partial}{\partial \overline{z}^0_i})\Big|^2 - \frac{13}{2^6\cdot 3\pi}\mathcal{T}^2_{klm} + \frac{1}{2^7\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm} - \frac{1}{8\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}^0_i})\Big|^2\Big\} P^N(0,0)\\
= &\Big\{\frac{1}{2\pi}\langle \widetilde{\mu}^E_{x_0}, \widetilde{\mu}^E_{x_0}\rangle_{g^{TY}} + \frac{1}{\pi}\Big\langle \widetilde{\mu}^E, \frac{7}{8}\sqrt{-1}T(e^\perp_l, Je^\perp_l) + T(\tfrac{\partial}{\partial z^0_l}, \tfrac{\partial}{\partial \overline{z}^0_l})\Big\rangle\\
&- \frac{1}{2\pi}\Big|\sum_j T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \overline{z}^0_j})\Big|^2 + \frac{7\sqrt{-1}}{8\pi}\Big\langle T(e^\perp_l, Je^\perp_l), T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \overline{z}^0_j})\Big\rangle - \frac{47}{2^7\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm}\\
&- \frac{11}{48\pi}\Big|T(e^\perp_k, \tfrac{\partial}{\partial \overline{z}^0_j})\Big|^2 - \frac{13}{2^6\cdot 3\pi}\mathcal{T}^2_{klm} - \frac{1}{8\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \overline{z}^0_i})\Big|^2\Big\} P^N(0,0).
\end{aligned} \tag{6.73}$$

By (6.43) and (6.58), we get

$$
\begin{aligned}
(6.74)\quad \Psi_{1,2}(0)+\Psi_{1,2}(0)^* = \Bigg\{ & \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{1}{\pi} R^{E_G}(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \bar z^0_j}) + \frac{1}{\pi}\Delta_{X_G}\log h \\
& - \frac{3}{8\pi}\nabla_{e^\perp_k}\nabla_{e^\perp_k}\log h + \frac{35}{48\pi}|T(e^\perp_k, \tfrac{\partial}{\partial \bar z^0_j})|^2 + \frac{\sqrt{-1}}{8\pi}\left\langle T(e^\perp_l, Je^\perp_l), T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \bar z^0_j})\right\rangle \\
& + \frac{1}{2\pi}\Big|T(\tfrac{\partial}{\partial z^0_i}, \tfrac{\partial}{\partial \bar z^0_j})\Big|^2 - \frac{1}{16\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \bar z^0_i})\Big|^2 \\
& + \frac{1}{2^6\pi}\Big[\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{kji}+\widetilde{\mathcal{T}}_{ijk}) + 2(\widetilde{\mathcal{T}}_{jki}+\widetilde{\mathcal{T}}_{ijk})\widetilde{\mathcal{T}}_{ijk}\Big] + \frac{7}{2^7\pi}\Big(2\mathcal{T}^2_{jkm} + \mathcal{T}_{jjm}\mathcal{T}_{kkm}\Big) \\
& - \frac{\sqrt{-1}}{8\pi}\Big(\left\langle T(e^\perp_l, Je^\perp_l), \widetilde{\mu}^E\right\rangle - 2\left\langle Je^\perp_k, \nabla^{TY}_{e^\perp_k}\widetilde{\mu}^E\right\rangle\Big)\Bigg\} P^N(0,0).
\end{aligned}
$$

Thus by (6.1), (6.73) and (6.74), as $\widetilde{\mathcal{T}}_{ijk}$ is anti-symmetric on $i, j$, we get

$$
\begin{aligned}
(6.75)\quad P^{(2)}(0,0) = \Bigg\{ & \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{1}{\pi} R^{E_G}(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \bar z^0_j}) + \frac{1}{\pi}\Delta_{X_G}\log h - \frac{3}{8\pi}\nabla_{e^\perp_k}\nabla_{e^\perp_k}\log h \\
& + \frac{1}{2\pi}|T(e^\perp_k, \tfrac{\partial}{\partial \bar z^0_j})|^2 + \frac{1}{2\pi}\Big|T(\tfrac{\partial}{\partial z^0_i}, \tfrac{\partial}{\partial \bar z^0_j})\Big|^2 - \frac{1}{2\pi}\Big|\sum_j T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \bar z^0_j})\Big|^2 \\
& + \frac{\sqrt{-1}}{\pi}\left\langle T(e^\perp_l, Je^\perp_l), T(\tfrac{\partial}{\partial z^0_j}, \tfrac{\partial}{\partial \bar z^0_j})\right\rangle - \frac{3}{16\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial \bar z^0_i})\Big|^2 + \frac{1}{24\pi}\mathcal{T}^2_{klm} \\
& - \frac{5}{16\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm} + \frac{1}{2^6\pi}\widetilde{\mathcal{T}}_{ijk}(-\widetilde{\mathcal{T}}_{kji}+3\widetilde{\mathcal{T}}_{ijk}) + \frac{1}{2\pi}\left\langle \widetilde{\mu}^E_{x_0}, \widetilde{\mu}^E_{x_0}\right\rangle_{g^{TY}} \\
& + \frac{1}{\pi}\left\langle \widetilde{\mu}^E, \frac{3}{4}\sqrt{-1}T(e^\perp_l, Je^\perp_l) + T(\tfrac{\partial}{\partial z^0_l}, \tfrac{\partial}{\partial \bar z^0_l})\right\rangle + \frac{\sqrt{-1}}{4\pi}\left\langle Je^\perp_k, \nabla^{TY}_{e^\perp_k}\widetilde{\mu}^E\right\rangle\Bigg\} P^N(0,0).
\end{aligned}
$$

By Theorem 5.1, (1.4), (1.24), (5.5c) and (5.14), as same as in (5.120), we get for $U \in T_{x_0}X_G$,

$$
\begin{aligned}
& \mathcal{T}_{llm} = -\left\langle T(e^\perp_m, Je^\perp_l), Je^\perp_l\right\rangle = -2\nabla_{e^\perp_m}\log h, \\
(6.76)\qquad & T(e^\perp_l, Je^\perp_l) = 2(\nabla_{e^\perp_k}\log h)Je^\perp_k, \\
& \mathcal{T}_{kk}(U) = -2\left\langle T(JU, Je^\perp_k), Je^\perp_k\right\rangle = -\left\langle \dot g^{TY}_{JU} Je^\perp_k, Je^\perp_k\right\rangle = -4\nabla_{JU}\log h.
\end{aligned}
$$

By (6.5), (6.75) and (6.76), we get Theorem 0.7.

# CHAPTER 7

# BERGMAN KERNEL AND GEOMETRIC QUANTIZATION

In this Chapter, we prove Theorems 0.10, 0.12.

*Proof of Theorem 0.10.* — We use the notations in Section 4.5.

By Lemma 4.6 and Theorem 4.8, we know that $p^{-\frac{n_0}{4}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}$ is a Toeplitz operator with principal symbol $(2^{\frac{n_0}{4}}/\widetilde{h}(x_0))\,\mathrm{Id}_{E_G}$ in the sense of Definition 4.3, and its kernel has an expansion analogous to (4.79) and $Q_{0,0}$ therein is $2^{\frac{n_0}{4}}/\widetilde{h}(x_0)$.

We claim that

$$\mathbb{T}_p = p^{-\frac{n_0}{2}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}\widetilde{h}^2(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \tag{7.1}$$

is a Toeplitz operator with principal symbol $2^{\frac{n_0}{2}}\,\mathrm{Id}_{E_G}$.

Indeed, when $E = \mathbb{C}$, this is a consequence of [**9**] on the composition of the Toeplitz operators.

To get the above claim for general $E$, we need just keep in mind that the kernel $\mathbb{T}_p(x_0, x_0')$ of $\mathbb{T}_p$ with respect $dv_{X_G}(x_0')$ has the expansion analogous to (4.79) and $Q_{0,0}$ therein is $2^{\frac{n_0}{2}}\,\mathrm{Id}_{E_G}$.

By Theorem 4.4, our claim then follows from the composition of the expansion of the kernel of $p^{-\frac{n_0}{4}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}$, as well as the Taylor expansion of $\widetilde{h}^2$ (cf. also [**31**, Chap. 7]).

Now we still denote by $\langle\, ,\, \rangle$ the $L^2$-scalar product on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ induced by $h^{L_G^p}$, $h^{E_G}$, $g^{TX_G}$ as in (1.19).

Let $\{s_i^p\}$ be an orthonormal basis of $(H^0(X, L^p \otimes E)^G, \langle\, ,\, \rangle)$, then

$$\varphi_i^p = (\sigma_p \circ \sigma_p^*)^{-\frac{1}{2}}\sigma_p s_i^p$$

is an orthonormal basis of $(H^0(X_G, L_G^p \otimes E_G), \langle\, ,\, \rangle)$.

From Definition 4.3, (0.28), (1.19) and (7.1), we get

$$(2p)^{-\frac{n_0}{2}} \left\langle \sigma_p s_i^p, \sigma_p s_j^p \right\rangle_{\widetilde{h}} = (2p)^{-\frac{n_0}{2}} \left\langle (\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \varphi_i^p, (\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \varphi_j^p \right\rangle_{\widetilde{h}} = 2^{-\frac{n_0}{2}} \left\langle \mathbb{T}_p \varphi_i^p, \varphi_j^p \right\rangle = \delta_{ij} + \mathscr{O}\left(\frac{1}{p}\right). \tag{7.2}$$

The proof of Theorem 0.10 is complete. □

In the symplectic case, we use (4.88) to define $\sigma_p : (\ker D_p)^G \to \ker D_{G,p}$ which is an isomorphism for $p$ large enough. Now by Theorems 4.4, 4.12, Corollary 4.13 as the above argument, we know $(2p)^{-n_0/4}\sigma_p$ is an asymptotic isometry is the sense of (0.29).

*Proof of Theorem 0.12.* — Set

$$\widetilde{h}^{E_G} = \widetilde{h}^2 h^{E_G}.$$

Then $\widetilde{P}_p^{X_G}$ is the orthogonal projection from $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ onto $H^0(X, L_G^p \otimes E_G)$, associated to the Hermitian product on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ induced by the metrics $h^{L_G}$, $\widetilde{h}^{E_G}$, $g^{TX_G}$ as in (1.19).

Let $\widetilde{P}_{p,\omega}^{X_G}(x_0, x_0')$ be the smooth kernel of $\widetilde{P}_p^{X_G}$ with respect to $dv_{X_G}(x_0')$. Then

$$\widetilde{P}_{p,\omega}^{X_G}(x_0, x_0') = \widetilde{h}^2(x_0')\widetilde{P}_p^{X_G}(x_0, x_0'). \tag{7.3}$$

Let $\widetilde{\nabla}^{E_G}$ be the Hermitian holomorphic connection on $(E_G, \widetilde{h}^{E_G})$ with curvature $\widetilde{R}^{E_G}$. Then

$$\widetilde{\nabla}^{E_G} = \nabla^{E_G} + \partial \log(\widetilde{h}^2), \quad \widetilde{R}^{E_G} = R^{E_G} + 2\overline{\partial}\partial \log \widetilde{h}. \tag{7.4}$$

Thus from (7.4),

$$\widetilde{R}^{E_G}(w_j^0, \overline{w}_j^0) = 2\widetilde{R}^{E_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0}) = R^{E_G}(w_j^0, \overline{w}_j^0) + \Delta_{X_G} \log \widetilde{h}. \tag{7.5}$$

By (5.19), (7.3) and (7.5), Theorem 0.12 is a direct consequence of [**17**, Theorem 1.3] (or Theorem 0.6 with $G = \{1\}$) for $\widetilde{P}_{p,\omega}^{X_G}(x_0, x_0)$. □

# BIBLIOGRAPHY


[1] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundl. Math. Wiss. Band 298, Springer-Verlag, Berlin, 1992.

[2] J.-M. Bismut, *The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs*, Invent. Math. **83** (1986), no. 1, 91–151.

[3] ______, *Koszul complexes, harmonic oscillators, and the Todd class*, J. Amer. Math. Soc. **3** (1990), no. 1, 159–256, With an appendix by the author and C. Soulé.

[4] ______, *Equivariant immersions and Quillen metrics*, J. Differential Geom. **41** (1995), no. 1, 53–157.

[5] ______, *Holomorphic families of immersions and higher analytic torsion forms*, Astérisque (1997), no. 244, viii+275.

[6] J.-M. Bismut, H. Gillet, and C. Soulé, *Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott-Chern forms*, Comm. Math. Phys. **115** (1988), no. 1, 79–126.

[7] J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 74, ii+298 pp. (1992).

[8] J.-M. Bismut and É. Vasserot, *The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle*, Comm. Math. Phys. **125** (1989), no. 2, 355–367.

[9] M. Bordemann and E. Meinrenken and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* $\mathrm{gl}(N)$*,* $N \to \infty$ *limits*, Comm. Math. Phys. **165** (1994), no. 2, 281–296.

[10] D. Borthwick and A. Uribe, *Almost complex structures and geometric quantization*, Math. Res. Lett. **3** (1996), no. 6, 845–861. Erratum: **5** (1998),211–212.

[11] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ, 1981.

[12] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, Journées: Équations aux Dérivées Partielles de Rennes (1975), Soc. Math. France, Paris, 1976, pp. 123–164. Astérisque, No. 34–35.

[13] M. Braverman, *Vanishing theorems on covering manifolds*, Contemp. Math., vol. 231, Amer. Math. Soc., Providence, RI, 1999, pp. 1–23.

[14] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1–23.

[15] L. Charles, *Toeplitz operators and hamiltonian torus action*, J. Funct. Anal. **236** (2006), 299-350.

[16] J. Chazarain and A. Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*, Gauthier-Villars, Paris, 1981.

[17] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), 1-41; announced in C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 193–198.

[18] S. K. Donaldson, *Scalar curvature and projective embeddings. I*, J. Differential Geom. **59** (2001), no. 3, 479–522.

[19] H. Fegan, *Introduction to compact Lie groups*, Series in Pure Mathematics, vol. 13, World Scientific Publishing Co. Inc., River Edge, NJ, 1991.

[20] V. Guillemin, *Star products on compact pre-quantizable symplectic manifolds*, Lett. Math. Phys. **35** (1995), no. 1, 85–89.

[21] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538.

[22] ______, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.

[23] A. V. Karabegov and M. Schlichenmaier, *Identification of Berezin-Toeplitz deformation quantization*, J. Reine Angew. Math. **540** (2001), 49–76.

[24] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.

[25] Z. Lu, *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. **122** (2000), no. 2, 235–273.

[26] X. Ma, *Orbifolds and analytic torsions*, Trans. Amer. Math. Soc. **357** (2005), 2205–2233.

[27] X. Ma and G. Marinescu, *The* Spin$^c$ *Dirac operator on high tensor powers of a line bundle*, Math. Z. **240** (2002), no. 3, 651–664.

[28] ______, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 493–498. The full version: math.DG/0411559, Adv. Math. **217** (2008), 1756-1815.

[29] ______, *The first coefficients of the asymptotic expansion of the Bergman kernel of the spin$^c$ Dirac operator*, Internat. J. Math. **17** (2006), 737–759.

[30] ______, *Toeplitz operators on symplectic manifolds*, Preprint (2005), J. Geom. Anal., **18** (2008), 565-611.

[31] ______, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics **254**, Birkhäuser Boston, Inc., Boston, MA 2007, 422 pp.

[32] X. Ma and W. Zhang, *Bergman kernels and symplectic reduction*, C. R. Math. Acad. Sci. Paris **341** (2005), 297-302.

[33] ______, *Toeplitz quantization and symplectic reduction*, Differential Geometry and Physics. Eds. M.-L. Ge and Weiping Zhang, Nankai Tracts in Mathematics Vol. **10**, World Scientific, (2006), 343-349.

[34] E. Meinrenken, *On Riemann-Roch formulas for multiplicities*, J. Amer. Math. Soc. **9** (1996), no. 2, 373–389.

[35] ______, *Symplectic surgery and the* Spin$^c$*-Dirac operator*, Adv. Math. **134** (1998), no. 2, 240–277.

[36] R. Paoletti, *Moment maps and equivariant Szegö kernels*, J. Symplectic Geom. **2** (2003), 133–175.

[37] ______, *The Szegö kernel of a symplectic quotient*, Adv. Math. **197** (2005), 523-553.

[38] W. Ruan, *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom. **6** (1998), 589–631.

[39] M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 289–306.

[40] B. Shiffman and S. Zelditch, *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds*, J. Reine Angew. Math. **544** (2002), 181–222.

[41] M. E. Taylor, *Pseudodifferential operators*, Princeton Mathematical Series, vol. 34, Princeton University Press, Princeton, N.J., 1981.

[42] ______, *Partial differential equations. I, II*, Applied Mathematical Sciences, vol. 115, 116, Springer-Verlag, New York, 1996.

[43] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130.

[44] Y. Tian and W. Zhang, *An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg*, Invent. Math. **132** (1998), no. 2, 229–259.

[45] M. Vergne, *Multiplicities formula for geometric quantization. I, II*, Duke Math. J. **82** (1996), no. 1, 143–179, 181–194.

[46] S. -T.Yau, *Nonlinear analysis in geometry*. Enseign. Math. (2) **33**(1987), no. 1-2, 109–158.

[47] S. Zelditch, *Szegö kernels and a theorem of Tian*, Intern. Math. Res. Notices (1998), no. 6, 317–331.

[48] W. Zhang, *Holomorphic quantization formula in singular reduction*, Commun. Contemp. Math. **1** (1999), no. 3, 281–293.