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Holomorphic immersions and equivariant torsion forms

By *Jean Michel Bismut* at Orsay and *Xiaonan Ma* at Palaiseau

Abstract. We compute the behaviour of the equivariant torsion forms of a Kähler fibration under composition of an immersion and a submersion. This extends previous results by the first author.

0. Introduction

The Quillen metric [Q2], [BGS3] is a natural metric on the determinant of the cohomology of a holomorphic Hermitian vector bundle, which one constructs using the Ray-Singer analytic torsion [RaS]. This metric has a number of remarkable properties [BGS3]. In particular the curvature of the corresponding holomorphic Hermitian connection on the determinant of a direct image is given by an explicit local formula, which is compatible with the theorem of Riemann-Roch-Grothendieck at the level of differential forms.

In [BL], Bismut and Lebeau have studied the behaviour of Quillen metrics under embeddings. In their formula, the additive R genus of Gillet-Soulé [GS1] appears. Using this result, Gillet and Soulé have established in [GS2] an arithmetic Riemann-Roch theorem for the determinant. In [B5], the result of Bismut-Lebeau was extended to the analytic torsion forms constructed in [BGS2], [BK]. This result was used by Roessler [R] to establish an arithmetic Riemann-Roch theorem for all the Chern classes.

In [B3], Bismut has obtained an equivariant version of the genus R, the genus $R(\theta, x)$. It was conjectured in [B3] that this genus should appear in an arithmetic Lefschetz formula. In [B4], an equivariant version of the Quillen metric was defined, and an equivariant analogue of the embedding formula of [BL] was obtained.

In [KR1], Köhler and Roessler proved a Lefschetz formula in Arakelov Geometry for the determinant. Various applications of this formula have been given in [KaK], [KR2], [KR3]. In [KR2], they also conjectured a higher degree version of their formula.

The purpose of this paper is to extend the main result of [B5] to the equivariant setting. We could also say that our paper extends the formula of [B4] for the determinant to higher Chern classes. The present paper provides the analytic arguments which should lead to the proof of the formula conjectured in [KR2]. Applications of this formula have been given in [K], [MR].

Let us now describe the geometric setting in more detail. Let $i : W \to V$ be an embedding of smooth complex manifolds. Let S be a complex manifold. Let $\pi_V : V \to S$ be a holomorphic submersion with compact fibre X, whose restriction $\pi_W : W \to S$ is a holomorphic submersion with compact fibre Y. Then we have the diagram of holomorphic maps:

$$\begin{array}{ccccc} Y & \longrightarrow & W & & \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i} & \overset{\pi_W}{\searrow} & \\ X & \longrightarrow & V & \underset{\pi_V}{\longrightarrow} & S \end{array}$$

Let η be a holomorphic vector bundle on W. Let (ξ, v) be a holomorphic complex of vector bundles on V, which together with a holomorphic restriction map $r : \xi_0|_W \to \eta$, provides a resolution of $i_*\eta$.

Let G be a compact Lie group acting holomorphically, fibrewise on W, V, whose action lifts holomorphically to $\big((\xi, v), \eta\big)$.

Let $R\pi_{V*}\xi, R\pi_{W*}\eta$ be the direct images of ξ, η. We make the assumption that the $R^i\pi_{W*}\eta$ are locally free. Then $R\pi_{V*}\xi$ is also locally free, and moreover we have a canonical isomorphism of $\mathbb{Z}$-graded G-holomorphic vector bundles on S,

$$R\pi_{V*}\xi \simeq R\pi_{W*}\eta. \tag{0.1}$$

Let $H(X, \xi|_X), H(Y, \eta|_Y)$ be the hypercohomology of $\xi|_X$, and the cohomology of $\eta|_Y$. Then we have the canonical identification of G-bundles on S,

$$R\pi_{V*}\xi \simeq H(X, \xi|_X), \quad R\pi_{W*}\eta \simeq H(Y, \eta|_Y). \tag{0.2}$$

Let ω^V, ω^W be real, closed, G-invariant $(1,1)$ forms on V, W which, when restricted to the relative tangent bundles TX, TY, are the Kähler forms of Hermitian metrics h^{TX}, h^{TY} on TX, TY. By identifying the normal bundles $N_{W/V} \simeq N_{Y/X}$ to the orthogonal bundle to TY in $TX|_Y, N_{Y/X}$ inherits a G-invariant metric $h^{N_{Y/X}}$. Let $h^{\xi_0}, \dots, h^{\xi_m}, h^\eta$ be G-invariant Hermitian metrics on $\xi_0, \dots, \xi_m, \eta$.

By identifying $H(Y, \eta|_Y)$ to the corresponding fibrewise harmonic forms in $\Omega(Y, \eta|_Y)$, the $\mathbb{Z}$-graded vector bundle $H(Y, \eta|_Y)$ is naturally equipped with a suitably normalized L_2 metric $h^{H(Y, \eta|_Y)}$.

For $g \in G$, set $V_g = \{x \in V, gx = x\}$, $W_g = \{x \in W, gx = x\}$. Then $\pi_{V_g} : V_g \to S$, $\pi_{W_g} : W_g \to S$ are holomorphic submersions with compact fibre X_g, Y_g. Let $\mathrm{Td}_g(TX, h^{TX})$ be the Chern-Weil g-Todd form on V_g associated to the holomorphic Hermitian connection on (TX, h^{TX}) [B4], §2a). Other Chern-Weil forms will be denoted in a similar way. In particular, $\mathrm{ch}_g(\eta, h^\eta)$ denote the g-Chern character form of (η, h^η).

Let P^S be the vector space of smooth forms on S, which are sums of forms of type (p,p). Let $P^{S,0}$ be the vector space of the forms $\alpha \in P^S$ such that there exist smooth forms β, γ on S, for which $\alpha = \partial\beta + \bar{\partial}\gamma$. We define $P^{W_g}, P^{W_g,0}$ in the same way.

Let $T_g(\omega^V, h^{\xi_i}), T_g(\omega^W, h^\eta) \in P^S$ be equivariant analytic torsion forms constructed in [Ma], Definition 2.11, which generalize the construction of Bismut-Gillet-Soulé [BGS2] and Bismut-Köhler [BK], which are such that

$$\frac{\bar{\partial}\partial}{2i\pi} T_g(\omega^W, h^\eta) = \mathrm{ch}_g\big(H(Y,\eta|_Y), h^{H(Y,\eta|_Y)}\big) - \int_{Y_g} \mathrm{Td}_g(TY, h^{TY})\,\mathrm{ch}_g(\eta, h^\eta). \tag{0.3}$$

Let $\big(\Omega(X,\xi|_X), \bar{\partial}^X + v\big)$ be the family of relative Dolbeault double complexes, whose cohomology coincides with the hypercohomology $H(X,\xi|_X)$. Let $h^{H(X,\xi|_X)}$ be the corresponding L_2 metric on $H(X,\xi|_X)$. Put $\mathrm{ch}_g(\xi, h^\xi) = \sum_{i=0}^{m} (-1)^i \mathrm{ch}_g(\xi_i, h^{\xi_i})$. Then we can construct the equivariant analytic torsion forms $T_g(\omega^V, h^\xi) \in P^S$ as in [BGS2], §2, [BK], §2, [Ma], §2, such that

$$\frac{\bar{\partial}\partial}{2i\pi} T_g(\omega^V, h^\xi) = \mathrm{ch}_g\big(H(X,\xi|_X), h^{H(X,\xi|_X)}\big) - \int_{X_g} \mathrm{Td}_g(TX, h^{TX})\,\mathrm{ch}_g(\xi, h^\xi). \tag{0.4}$$

In the sequel, we assume that the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ verify assumption (A) of [B2], §1b), with respect to $h^{N_{Y/X}}, h^\eta$. By [B4], Proposition 3.5, such metrics do exist.

Let $T_g(\xi, h^\xi)$ be the Bott-Chern current of [B4], §6, on V_g such that

$$\frac{\bar{\partial}\partial}{2i\pi} T_g(\xi, h^\xi) = (\mathrm{Td}_g)^{-1}(N_{Y/X}, h^{N_{Y/X}})\,\mathrm{ch}_g(\eta, h^\eta)\delta_{\{W_g\}} - \mathrm{ch}_g(\xi, h^\xi). \tag{0.5}$$

Let $\widetilde{\mathrm{Td}}_g(TY, TX|_{W_g}, h^{TX}) \in P^{W_g}/P^{W_g,0}$ be the Bott-Chern class of [BGS1] associated to the exact sequence of holomorphic Hermitian vector bundles on W_g, $0 \to TY \to TX|_{W_g} \to N_{Y/X} \to 0$, such that

$$\frac{\bar{\partial}\partial}{2i\pi} \widetilde{\mathrm{Td}}_g(TY, TX|_{W_g}, h^{TX}) = \mathrm{Td}_g(TX|_{W_g}, h^{TX}) - \mathrm{Td}_g(TY, h^{TY})\,\mathrm{Td}_g(N_{Y/X}, h^{N_{Y/X}}). \tag{0.6}$$

Recall that $H(X,\xi|_X) \simeq H(Y,\eta|_Y)$. Let $\widetilde{\mathrm{ch}}_g\big(H(Y,\eta|_Y), h^{H(X,\xi|_X)}, h^{H(Y,\eta|_Y)}\big) \in P^S/P^{S,0}$ be the Bott-Chern class of [BGS1], Theorem 1.29, such that

$$\frac{\bar{\partial}\partial}{2i\pi} \widetilde{\mathrm{ch}}_g\big(H(Y,\eta|_Y), h^{H(X,\xi|_X)}, h^{H(Y,\eta|_Y)}\big) = \mathrm{ch}_g\big(H(Y,\eta|_Y), h^{H(Y,\eta|_Y)}\big) - \mathrm{ch}_g\big(H(X,\xi|_X), h^{H(X,\xi|_X)}\big). \tag{0.7}$$

Let $\zeta(\theta,s),\eta(\theta,s)$ be the real and imaginary parts of the Lerch series, i.e.

$$\zeta(\theta,s)=\sum_{n=1}^{+\infty}\frac{\cos(n\theta)}{n^s},\quad \eta(\theta,s)=\sum_{n=1}^{+\infty}\frac{\sin(n\theta)}{n^s}. \tag{0.8}$$

Recall that $R(\theta,x)$, defined in [B3], is given by the formula

$$\begin{aligned}R(\theta,x)=&\sum_{\substack{n\geqq 0\\ n\text{ even}}} i\left(\sum_{j=1}^{n}\frac{1}{j}\eta(\theta,-n)+2\frac{\partial\eta}{\partial s}(\theta,-n)\right)\frac{x^n}{n!}\\ &+\sum_{\substack{n\geqq 1\\ n\text{ odd}}}\left(\sum_{j=1}^{n}\frac{1}{j}\zeta(\theta,-n)+2\frac{\partial\zeta}{\partial s}(\theta,-n)\right)\frac{x^n}{n!}.\end{aligned} \tag{0.9}$$

Then $R(0,x)$ is just the Gillet-Soulé's series $R(x)$ [GS1]. We identify $R(\theta,\cdot)$ to the corresponding additive genus.

Over V_g, TX splits as direct sum $TX=\bigoplus TX^{\theta}$, where the $\theta\in[0,2\pi[$ are distinct and locally constant, and g acts on TX^{θ} by multiplication by $e^{i\theta}$. Set $R_g(TX)=\sum_{\theta}R(\theta,TX^{\theta}|_{V_g})$. We use a similar notation for $R_g(TY)$.

The purpose of this paper is to prove an extension of [B4], Theorem 0.1, [B5], Theorem 0.1.

Theorem 0.1. *The following identity holds*:

$$\begin{aligned}&\widetilde{\mathrm{ch}}_g\big(H(Y,\eta|_Y),h^{H(X,\xi|_X)},h^{H(Y,\eta|_Y)}\big)-T_g(\omega^W,h^{\eta})+T_g(\omega^V,h^{\xi})\\ &\quad=\int_{X_g}\mathrm{Td}_g(TX,h^{TX})T_g(\xi,h^{\xi})-\int_{Y_g}\frac{\widetilde{\mathrm{Td}}_g(TY,TX|_{W_g},h^{TX})}{\mathrm{Td}_g(N_{Y/X},h^{N_{Y/X}})}\mathrm{ch}_g(\eta,h^{\eta})\\ &\qquad+\int_{X_g}\mathrm{Td}_g(TX)R_g(TX)\,\mathrm{ch}_g(\xi)-\int_{Y_g}\mathrm{Td}_g(TY)R_g(TY)\,\mathrm{ch}_g(\eta)\quad\text{in } P^S/P^{S,0}.\end{aligned} \tag{0.10}$$

Assume now that for $j>0$, $R^j\pi_{V*}\xi_k=0$ $(0\leqq k\leqq m)$, $R^j\pi_{W*}\eta=0$. Then we have an acyclic complex of holomorphic G-vector bundles $\mathscr{K}$ on S,

$$\mathscr{K}:0\to H^0(X,\xi_m)\xrightarrow{v}H^0(X,\xi_{m-1})\cdots\xrightarrow{v}H^0(X,\xi_0)\xrightarrow{v}H^0(X,\xi|_X)\to 0. \tag{0.11}$$

Let $h^{\mathscr{K}}$ be the obvious L_2 metric on $\mathscr{K}$. Let $\widetilde{\mathrm{ch}}_g(\mathscr{K},h^{\mathscr{K}})\in P^S/P^{S,0}$ be the Bott-Chern class of [BGS1] such that

$$\begin{aligned}\frac{\bar\partial\partial}{2i\pi}\widetilde{\mathrm{ch}}_g(\mathscr{K},h^{\mathscr{K}})=&\,\mathrm{ch}_g\big(H^0(X,\xi|_X),h^{H^0(X,\xi|_X)}\big)\\ &-\sum_{i=0}^{m}(-1)^i\,\mathrm{ch}_g\big(H^0(X,\xi_i|_X),h^{H^0(X,\xi_i|_X)}\big).\end{aligned} \tag{0.12}$$

The following theorem is an extension of [B5], Theorem 0.2.

Theorem 0.2. *The following identity holds*:

$$T_g(\omega^V, h^{\xi}) - \sum_{i=0}^{m}(-1)^i T_g(\omega^V, h^{\xi_i}) - \widetilde{\mathrm{ch}}_g(\mathscr{K}, h^{\mathscr{K}}) = 0 \quad \text{in } P^S/P^{S,0}. \tag{0.13}$$

The references [B4], [B5] provide all the techniques what we need in this paper. The main point of this paper is to explain how to put these papers together to get our results. While the general organization of our paper follows [B5], we will use at certain key point the arguments of [B4].

Our paper is organized as follows. In Section 1, we recall the construction of equivariant torsion forms. In Section 2, we describe the basic geometric setting, and the objects which appear in Theorem 0.1. In Section 3, which corresponds to [B4], §8, [B5], §6, we prove Theorem 0.1. The proof is based on several intermediate results whose proof occupies Sections 4–7. Finally in Section 8, we prove Theorem 0.2.

The results contained in this paper were announced in [BMa].

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1. Equivariant analytic torsion forms

In this section, we briefly describe the construction of the equivariant analytic torsion forms. This section is organized as follows. In Section 1.1, we recall elementary results on Clifford algebras and complex vector spaces. In Section 1.2, we construct the Levi-Civita superconnection in the sense of [B1]. In Section 1.3, we construct the equivariant analytic torsion forms.

1.1. Clifford algebras and complex vector spaces. Let V be a complex Hermitian vector space of complex dimension k, let $\overline{V}$ be the conjugate vector space. If $z \in V$, z represents $Z = z + \bar{z} \in V_{\mathbb{R}}$, so that $|Z|^2 = 2|z|^2$. Let $J \in \mathrm{End}(V_{\mathbb{R}})$ be the complex structure of $V_{\mathbb{R}}$.

Let $c(V_{\mathbb{R}})$ be the Clifford algebra of $V_{\mathbb{R}}$. Then $\Lambda(\overline{V}^*)$ and $\Lambda(V^*)$ are Clifford modules. Namely if $X \in V$, $X' \in \overline{V}$, let $X^* \in \overline{V}^*$, $X'^* \in V^*$ correspond to X, X' by the Hermitian product of V. Set

$$\begin{aligned} c(X) &= \sqrt{2}X^*\wedge, \quad & c(X') &= -\sqrt{2}i_{X'}, \\ \hat{c}(X) &= \sqrt{2}i_X, \quad & \hat{c}(X') &= -\sqrt{2}X'^*\wedge. \end{aligned} \tag{1.1}$$

Note that our conventions in (1.1) for $\hat{c}$ differ from the conventions in [BL], §5a), and are the same as in [B5], §2.2.

Then if $U, U' \in V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$,

$$c(U)c(U') + c(U')c(U) = -2\langle U, U' \rangle, \tag{1.2}$$
$$\hat{c}(U)\hat{c}(U') + \hat{c}(U')\hat{c}(U) = -2\langle U, U' \rangle.$$

Also $c(U), \hat{c}(U)$ act as odd operators on $\Lambda(\overline{V}^*) \hat{\otimes} \Lambda(V^*)$. If $U, U' \in V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, then

$$c(U)\hat{c}(U') + \hat{c}(U')c(U) = 0. \tag{1.3}$$

1.2. The Levi-Civita superconnection of a Kähler fibration. Let $\pi : V \to S$ be a holomorphic submersion with compact fibre X. Let TV, TS be the holomorphic tangent bundles to V, S. Let TX be the holomorphic relative tangent bundle TV/S. Let J^{TX} be the complex structure on the real tangent bundle $T_{\mathbb{R}}X$.

Let ω^V be a real, closed smooth $(1,1)$-form on V such that

$$h^{TX}(X, Y) = \omega^V(J^{TX}X, Y) \quad (X, Y \in T_{\mathbb{R}}X)$$

defines a Hermitian metric h^{TX} on TX. For $x \in M$, set

$$T_x^H V = \{Y \in T_x V;\ \text{for any } X \in T_x X, \omega^V(X, \overline{Y}) = 0\}. \tag{1.4}$$

Then $T^H V$ is a sub-bundle of TV such that we have the C^∞ splitting $TV = T^H V \oplus TX$. Also $(\pi, h^{TX}, T^H V)$ is a Kähler fibration in the sense of [BGS2], Definition 1.4, and ω^V is an associated $(1,1)$-form.

If $U \in T_{\mathbb{R}}S$, let U^H be the lift of U in $T_{\mathbb{R}}^H V$, so that $\pi_* U^H = U$.

Let ξ be a complex vector bundle on V. Let h^ξ be a Hermitian metric on ξ. Let ∇^{TX}, ∇^ξ be the holomorphic Hermitian connections on $(TX, h^{TX}), (\xi, h^\xi)$. Let R^{TX}, R^ξ be the curvatures of ∇^{TX}, ∇^ξ. Let $\nabla^{\Lambda(T^{*(0,1)}X)}$ be the connection induced by ∇^{TX} on $\Lambda(T^{*(0,1)}X)$. Let $\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}$ be the connection on $\Lambda(T^{*(0,1)}X) \otimes \xi$,

$$\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi} = \nabla^{\Lambda(T^{*(0,1)}X)} \otimes 1 + 1 \otimes \nabla^\xi. \tag{1.5}$$

Definition 1.1. For $0 \leqq p \leqq \dim X$, $s \in S$, let E_s^p be the vector space of smooth sections of $\left(\Lambda^p(T^{*(0,1)}X) \otimes \xi\right)|_{X_s}$ over X_s. Set

$$E_s = \bigoplus_{p=0}^{\dim X} E_s^p, \quad E_s^+ = \bigoplus_{p \text{ even}} E_s^p, \quad E_s^- = \bigoplus_{p \text{ odd}} E_s^p. \tag{1.6}$$

As in [B1], §1f), [BGS2], §1d), we can regard the E_s's as the fibres of a smooth $\mathbb{Z}$-graded infinite dimensional vector bundle E over the base S. Smooth sections of E over S will be identified with smooth sections of $\Lambda(T^{*(0,1)}X) \otimes \xi$ over V.

Let dv_X be the Riemannian volume form on X associated to h^{TX}. Let $*$ be the Hodge operator attached to the metric h^{TX}. Let $\langle\ \rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}$ be the Hermitian product induced by h^{TX}, h^ξ on $\Lambda(T^{*(0,1)}X) \otimes \xi$. If $s, s' \in E$, set

$$\langle s, s' \rangle = \left(\frac{1}{2\pi}\right)^{\dim X} \int_X \langle s, s' \rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}\, dv_X = \left(\frac{1}{2\pi}\right)^{\dim X} \int_X \langle s \wedge *s' \rangle_{h^\xi}. \tag{1.7}$$

Definition 1.2. If $U \in T_{\mathbb{R}}S$, if s is a smooth section of E over S, set

$$\nabla^E_U s = \nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}_{U^H} s. \tag{1.8}$$

By [B1], §1f), ∇^E is a connection on the infinite dimensional vector bundle E. Let $\nabla^{E'}$ and $\nabla^{E''}$ be the holomorphic and anti-holomorphic parts of ∇^E.

For $s \in S$, let $\bar{\partial}^{X_s}$ be the Dolbeault operator acting on E_s, and let $\bar{\partial}^{X_s*}$ be its formal adjoint with respect to the Hermitian product (1.7). Set

$$D^{X_s} = \bar{\partial}^{X_s} + \bar{\partial}^{X_s*}. \tag{1.9}$$

Let $h^{T_{\mathbb{R}}S}$ be a Euclidean metric on $T_{\mathbb{R}}S$. Let $\nabla^{T_{\mathbb{R}}S}$ be the Levi-Civita connection on $(T_{\mathbb{R}}S, h^{T_{\mathbb{R}}S})$. Let $\nabla^{T_{\mathbb{R}}V} = \pi^*\nabla^{T_{\mathbb{R}}S} \oplus \nabla^{T_{\mathbb{R}}X}$ be the connection on $T_{\mathbb{R}}V = T^H_{\mathbb{R}}V \oplus T_{\mathbb{R}}X$. Let T be the torsion of $\nabla^{T_{\mathbb{R}}V}$.

Let P^{TX} be the projection $TV \simeq T^H V \oplus TX \to TX$. If U, V are smooth vector fields on S, then

$$T(U^H, V^H) = -P^{TX}[U^H, V^H]. \tag{1.10}$$

By [BGS2], Theorem 1.7, we know that as a 2-form, T is of complex type $(1,1)$.

Let $f_1, \dots, f_{2m}$ be a base of $T_{\mathbb{R}}S$, and let $f^1, \dots, f^{2m}$ be the dual base of $T^*_{\mathbb{R}}S$.

Definition 1.3. Set

$$c(T) = \frac{1}{2}\sum_{1\leqq\alpha,\beta\leqq 2m} f^\alpha f^\beta c\big(T(f^H_\alpha, f^H_\beta)\big). \tag{1.11}$$

Then $c(T)$ is a section of $\big(\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\mathrm{End}\big(\Lambda(T^{*(0,1)}X)\otimes\xi\big)\big)^{\mathrm{odd}}$. Similarly, if $T^{(1,0)}, T^{(0,1)}$ denote the components of T in $T^{(1,0)}X, T^{(0,1)}X$, we also define $c(T^{(1,0)}), c(T^{(0,1)})$ as in (1.11), so that

$$c(T) = c(T^{(1,0)}) + c(T^{(0,1)}). \tag{1.12}$$

Definition 1.4. For $u > 0$, set

$$\begin{aligned} B''_u &= \sqrt{u}\bar{\partial}^X + \nabla^{E''} - \frac{c(T^{(1,0)})}{2\sqrt{2u}}, \\ B'_u &= \sqrt{u}\bar{\partial}^{X*} + \nabla^{E'} - \frac{c(T^{(0,1)})}{2\sqrt{2u}}, \\ B_u &= B'_u + B''_u. \end{aligned} \tag{1.13}$$

Then B_u is the Levi-Civita superconnection constructed in [B1], §3, [BGS2], §2a).

Let N_V be the number operator defining the $\mathbb{Z}$-grading on $\Lambda(T^{*(0,1)}X) \otimes \xi$ and on E. N_V acts by multiplication by p on $\Lambda^p(T^{*(0,1)}X) \otimes \xi$. If $U, V \in T_{\mathbb{R}}S$, set

$$\omega^H(U, V) = \omega^V(U^H, V^H). \tag{1.14}$$

Definition 1.5. For $u > 0$, set

$$N_u = N_V + \frac{i\omega^H}{u}. \tag{1.15}$$

1.3. Higher analytic torsion forms. First, we assume that the direct image $R^\bullet\pi_*\xi$ of ξ by π is locally free. For $s \in S$, let $H(X_s, \xi|_{X_s})$ be the cohomology of the sheaf of holomorphic sections of $\xi|_{X_s}$. Then the $H(X_s, \xi|_{X_s})$'s are the fibres of a $\mathbb{Z}$-graded holomorphic vector bundle $H(X, \xi|_X)$ on S, and $R^\bullet\pi_*\xi = H(X, \xi|_X)$. So we will write indifferently $R^\bullet\pi_*\xi$ or $H(X, \xi|_X)$.

For $s \in S$, set

$$K(X_s, \xi|_{X_s}) = \operatorname{Ker} D^{X_s}. \tag{1.16}$$

By Hodge theory, $K(X_s, \xi|_{X_s}) \simeq H(X_s, \xi|_{X_s})$. So the $K(X_s, \xi|_{X_s})$ are the fibres of a smooth vector bundle $K(X, \xi|_X)$ over S. By [BGS3], Theorem 3.5, this isomorphism induces a smooth isomorphism of $\mathbb{Z}$-graded vector bundles on S

$$H(X, \xi|_X) \simeq K(X, \xi|_X). \tag{1.17}$$

Then $K(X, \xi|_X)$ inherits a Hermitian product from $(E, \langle\ \rangle)$. Let $h^{H(X,\xi|_X)}$ be the corresponding smooth metric on $H(X, \xi|_X)$. Let $\nabla^{H(X,\xi|_X)}$ be the holomorphic Hermitian connection on $\left(H(X, \xi|_X), h^{H(X,\xi|_X)}\right)$.

Let G be a compact Lie group. We assume that G acts holomorphically on V, and preserves the fibres X. Also we assume that the action of G lifts to a holomorphic action on ξ. Suppose that ω^V, h^ξ are G-invariant. Then $R^\bullet\pi_*\xi$ is also a G-equivariant vector bundle over S, and the metric $h^{H(X,\xi|_X)}$ is also G-invariant.

For $g \in G$, set

$$V_g = \{x \in V, gx = x\}. \tag{1.18}$$

Then we have a holomorphic submersion $\pi_g : V_g \to S$ with compact fibre X_g.

Let Φ be the homomorphism of $\Lambda^{\mathrm{even}}(T_{\mathbb{R}}^*S)$ into itself: $\alpha \to (2i\pi)^{-\deg\alpha/2}\alpha$.

Let $1, e^{i\theta_1}, \dots, e^{i\theta_q}$ $(0 < \theta_j < 2\pi)$ be the locally constant distinct eigenvalues of g acting on TX on V_g. Let $TX^{\theta_0}, TX^{\theta_1}, \dots, TX^{\theta_q}$ $(\theta_0 = 0)$ be the corresponding eigenbundles. Then TX splits holomorphically as an orthogonal sum

$$TX = TX^{\theta_0} \oplus \cdots \oplus TX^{\theta_q}. \tag{1.19}$$

Let $h^{TX^{\theta_0}}, \dots, h^{TX^{\theta_q}}$ be the Hermitian metrics on $TX^{\theta_0}, \dots, TX^{\theta_q}$ induced by h^{TX}. Then ∇^{TX} induces the holomorphic Hermitian connections $\nabla^{TX^{\theta_0}}, \dots, \nabla^{TX^{\theta_q}}$ on $(TX^{\theta_0}, h^{TX^{\theta_0}}), \dots, (TX^{\theta_q}, h^{TX^{\theta_q}})$. Let $R^{TX^{\theta_0}}, \dots, R^{TX^{\theta_q}}$ be their curvatures.

If A is (q,q) matrix, set

$$\mathrm{Td}(A) = \det\left(\frac{A}{1 - e^{-A}}\right), \tag{1.20}$$

$$\mathrm{e}(A) = \det(A), \quad \mathrm{ch}(A) = \mathrm{Tr}[\exp(A)].$$

The genera associated to Td and e are called the Todd genus and the Euler genus.

Definition 1.6. Set

$$\mathrm{Td}_g(TX, h^{TX}) = \mathrm{Td}\left(\frac{-R^{TX^{\theta_0}}}{2i\pi}\right) \prod_{j=1}^{q} \frac{\mathrm{Td}}{e}\left(\frac{-R^{TX^{\theta_j}}}{2i\pi} + i\theta_j\right), \tag{1.21}$$

$$\mathrm{Td}_g'(TX, h^{TX}) = \frac{\partial}{\partial b}\left[\mathrm{Td}\left(\frac{-R^{TX^{\theta_0}}}{2i\pi} + b\right) \times \prod_{j=1}^{q} \frac{\mathrm{Td}}{e}\left(\frac{-R^{TX^{\theta_j}}}{2i\pi} + i\theta_j + b\right)\right]_{b=0},$$

$$(\mathrm{Td}_g^{-1})'(TX, h^{TX}) = \frac{\partial}{\partial b}\left[\mathrm{Td}^{-1}\left(\frac{-R^{TX^{\theta_0}}}{2i\pi} + b\right) \times \prod_{j=1}^{q} \left(\frac{\mathrm{Td}}{e}\right)^{-1}\left(\frac{-R^{TX^{\theta_j}}}{2i\pi} + i\theta_j + b\right)\right]_{b=0},$$

$$\mathrm{ch}_g(\xi, h^{\xi}) = \mathrm{Tr}\left[g \exp\left(\frac{-R^{\xi}}{2i\pi}\right)\right].$$

Then the forms in (1.21) are closed forms on V_g, and their cohomology class does not depend on the g-invariant metric h^{TX}. We denote these cohomology classes by $\mathrm{Td}_g(TX), \mathrm{Td}_g'(TX), \dots, \mathrm{ch}_g(\xi)$. In the same way, set

$$\mathrm{ch}_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) = \sum_{k=0}^{\dim X} (-1)^k \mathrm{ch}_g\big(H^k(X, \xi|_X), h^{H(X,\xi|_X)}\big), \tag{1.22}$$

$$\mathrm{ch}_g'\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) = \sum_{k=0}^{\dim X} (-1)^k k \,\mathrm{ch}_g\big(H^k(X, \xi|_X), h^{H(X,\xi|_X)}\big).$$

In [Ma], §2d), we constructed an equivariant analytic torsion form $T_g(\omega^V, h^{\xi}) \in P^S$ which generalized the construction of [BK], §2 to the equivariant case. Moreover,

$$\frac{\bar{\partial}\partial}{2i\pi} T_g(\omega^V, h^{\xi}) = \mathrm{ch}_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) - \int_{X_g} \mathrm{Td}_g(TX, h^{TX})\, \mathrm{ch}_g(\xi, h^{\xi}). \tag{1.23}$$

More precisely, put

$$(1.24)\qquad C_{-1,g} = \int_{X_g} \frac{\omega^V}{2\pi} \operatorname{Td}_g(TX, h^{TX}) \operatorname{ch}_g(\xi, h^{\xi}),$$

$$C_{0,g} = \int_{X_g} \big(-\operatorname{Td}'_g(TX, h^{TX}) + \dim X \operatorname{Td}_g(TX, h^{TX})\big) \operatorname{ch}_g(\xi, h^{\xi}).$$

Then, by [Ma], (2.27),

$$(1.25)\qquad T_g(\omega^V, h^{\xi}) = -\int_0^1 \left(\Phi \operatorname{Tr}_s[gN_u \exp(-B_u^2)] - \frac{C_{-1,g}}{u} - C'_{0,g}\right)\frac{du}{u}$$
$$- \int_1^{+\infty} \big(\Phi \operatorname{Tr}_s[gN_u \exp(-B_u^2)] - \operatorname{ch}'_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big)\big)\frac{du}{u}$$
$$+ C_{-1,g} + \Gamma'(1)\big(C'_{0,g} - \operatorname{ch}'_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big)\big).$$

Here $C'_{0,g} = C_{0,g}$ in $P^S/P^{S,0}$.

2. Resolutions, Bott-Chern currents, and equivariant analytic torsion forms

This section is the obvious extension of [B5], §3, to the equivariant case. When S is a point, the corresponding result was obtained in [B4], §3, §6.

This section is organized as follows. In Section 2.1, we describe the geometric setting. In Section 2.2, we construct the equivariant analytic torsion forms of the family of double complexes. In Section 2.3, we give various assumptions on the metrics on TX, TY, ξ, η. In Section 2.4, we describe the Bott-Chern currents of [B5].

2.1. A family of double complexes. Let $i : W \to V$ be an embedding of smooth complex manifolds. Let S be a complex manifold. Let $\pi_V : V \to S$ be a holomorphic submersion with compact fibre X of complex dimension l, whose restriction $\pi_W : W \to S$ is a holomorphic submersion with compact fibre Y.

Let η be a holomorphic vector bundle on W. Let

$$(2.1)\qquad (\xi, v) : 0 \to \xi_m \xrightarrow{v} \xi_{m-1} \xrightarrow{v} \cdots \xrightarrow{v} \xi_0 \to 0$$

be a holomorphic complex of vector bundles on V. We identify ξ with $\bigoplus_{i=0}^{m} \xi_i$. Let $r : \xi_0|_W \to \eta$ be a holomorphic restriction map. We make the assumption that (ξ, v) is a resolution of $i_*\eta$, or equivalently that we have the exact sequence of $\mathcal{O}_V$ sheaves

$$(2.2)\qquad 0 \to \mathcal{O}_V(\xi_m) \xrightarrow{v} \mathcal{O}_V(\xi_{m-1}) \xrightarrow{v} \cdots \xrightarrow{v} \mathcal{O}_V(\xi_0) \to i_*\mathcal{O}_W(\eta) \to 0.$$

Then for every $s \in S$, $(\xi, v)|_{X_s}$ provides a resolution of $i_*\eta|_{Y_s}$.

Let $N_{\mathbf{H}}$ be the number operator of ξ, i.e. $N_{\mathbf{H}}$ acts on ξ_k by multiplication by k. Let $N_{\mathbf{V}}^X, N_{\mathbf{V}}^Y$ be the operators defining the $\mathbb{Z}$-grading on $\Lambda(T^{*(0,1)}X), \Lambda(T^{*(0,1)}Y)$.

Definition 2.1. For $s \in S$, $0 \leqq p \leqq l$, $0 \leqq i \leqq m$, let E_i^p be the vector space of smooth sections of $\Lambda^p(T^{*(0,1)}X) \hat{\otimes} \xi_i$ on the fibre X_s. Set

$$E_+ = \bigoplus_{p-i \text{ even}} E_i^p, \quad E_- = \bigoplus_{p-i \text{ odd}} E_i^p, \quad E = E_+ \oplus E_-. \tag{2.3}$$

Then E is the set of smooth sections of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ on X. It is $\mathbb{Z}$-graded by the operator $N_{\mathbf{V}}^X - N_{\mathbf{H}}$.

For $s \in S$, $1 \leqq q \leqq \dim Y$, let F_s^q be the set of smooth sections of $\Lambda^q(T^{*(0,1)}Y) \hat{\otimes} \eta|_Y$ on the fibre Y. Set

$$F_{+,s} = \bigoplus_{q \text{ even}} F_s^q, \quad F_{-,s} = \bigoplus_{q \text{ odd}} F_s^q, \quad F_s = F_{+,s} \oplus F_{-,s}. \tag{2.4}$$

Let $H(X_s, \xi|_{X_s})$ be the hypercohomology of $(\mathcal{O}_{X_s}(\xi|_{X_s}), v)$, let $H(Y_s, \eta|_{Y_s})$ be the cohomology of $\mathcal{O}_{Y_s}(\eta|_{Y_s})$. For any $s \in S$, the map $r : \mathcal{O}_{X_s}(\xi|_{X_s}) \to \mathcal{O}_{Y_s}(\eta|_{Y_s})$ is a quasi-isomorphism, and so

$$H(X_s, \xi|_{X_s}) \simeq H(Y_s, \eta|_{Y_s}). \tag{2.5}$$

Let $\bar{\partial}^X, \bar{\partial}^Y$ be the Dolbeault operators acting on E, F. Then $\bar{\partial}^X + v$ is a chain map on E. By [BL], Proposition 1.5, for every $s \in S$,

$$H(E_s, \bar{\partial}^X + v) \simeq H(X_s, \xi|_{X_s}), \quad H(F_s, \bar{\partial}^Y) \simeq H(Y_s, \eta|_{Y_s}). \tag{2.6}$$

We extend r to a morphism $\xi|_W \to \eta$, with $r = 0$ on ξ_i, $i > 0$. For $s \in S$, let r_s be the restriction map $r_s : \alpha \in E_s \to (i^* \hat{\otimes} r)\alpha \in F_s$.

Now we recall a result in [BL], Theorem 1.7.

Theorem 2.2. *The map $r : (E, \bar{\partial}^X + v) \to (F, \bar{\partial}^Y)$ is a quasi-isomorphism of $\mathbb{Z}$-graded complexes. It induces the canonical identification $H(E, \bar{\partial}^X + v) \simeq H(Y, \eta|_Y)$.*

In the whole paper, we assume that $\dim H^i(X, \xi|_X)$ $(i \geqq 0)$ is locally constant. Then the $H(X_s, \xi|_{X_s})$'s are the fibres of a holomorphic vector bundle $H(X, \xi|_X)$ on S. By (2.5), the $H(Y_s, \eta|_{Y_s})$'s also are the fibres of a holomorphic vector bundle $H(Y, \eta|_Y)$ on S. By (2.5), (2.6), Theorem 2.2, we get the identification of holomorphic $\mathbb{Z}$-graded vector bundles on S,

$$H(X, \xi|_X) \simeq H(Y, \eta|_Y), \quad H(E, \bar{\partial}^X + v) \simeq H(Y, \eta|_Y). \tag{2.7}$$

2.2. The equivariant analytic torsion forms of the double complex. We now extend the equivariant setting of Section 1.3. Let G be a compact Lie group. We assume that G acts holomorphically fibrewise on V and preserves W. Also we assume that the action of G on V and W lifts to holomorphic actions on the chain complex (ξ, v) and on η, and the restriction map $r : \xi_0|_Y \to \eta$ is G-invariant.

Then G acts naturally by chain maps on $(E, \bar{\partial}^X + v) \to (F, \bar{\partial}^Y)$. Therefore G acts on $H^*(X, \xi_i|_X)$ $(0 \leqq i \leqq m)$, $H^*(E, \bar{\partial}^X + v)$ and $H^*(Y, \eta|_Y)$. Finally the canonical identification $H(E, \bar{\partial}^X + v) \simeq H(Y, \eta|_Y)$ is an identification of G-vector bundles on S.

Let ω^V, ω^W be real, closed, G-invariant smooth $(1,1)$ forms on V, W which, when restricted to each fibre X, Y, are Kähler forms of metrics h^{TX}, h^{TY} on TX, TY. To ω^V, ω^W, we associate the objects considered in Section 1, to distinguish them from one another, we will often denote them with a superscript V or W.

Let $h^\xi = \bigoplus h^{\xi_i}, h^\eta$ be smooth G-invariant Hermitian metrics on $\xi = \bigoplus_{i=0}^m \xi_i, \eta$. We equip the fibres of E (resp. F) with the Hermitian product (1.7) associated to h^{TX}, h^ξ (resp. h^{TY}, h^η). Let v^* be the adjoint of v with respect to h^ξ. Let $\bar{\partial}^{X*}$ (resp. $\bar{\partial}^{Y*}$) be the formal adjoint of $\bar{\partial}^X$ (resp. $\bar{\partial}^Y$) with respect to the Hermitian product $\langle\ \rangle$ on E (resp. F). Set

$$D^X = \bar{\partial}^X + \bar{\partial}^{X*}, \quad D^Y = \bar{\partial}^Y + \bar{\partial}^{Y*}, \quad V = v + v^*. \tag{2.8}$$

For $g \in G$, set

$$V_g = \{x \in V, gx = x\}, \quad W_g = \{x \in W, gx = x\}.$$

Then $\pi_{V_g} : V_g \to S$, $\pi_{W_g} : W_g \to S$ are holomorphic submersions with compact fibres X_g, Y_g.

For $u > 0$, let B_u^{V,ξ_i} $(0 \leqq i \leqq m)$, B_u^W be the superconnections on E_i, F associated to (ω^V, h^{ξ_i}) and to (ω^W, h^η), whose construction was given in Definition 1.4. Then we can construct the equivariant analytic torsion forms $T_g(\omega^V, h^{\xi_i})$ and $T_g(\omega^W, h^\eta)$ as in (1.23). By (1.23),

$$\frac{\bar{\partial}\partial}{2i\pi} T_g(\omega^W, h^\eta) = \mathrm{ch}_g\big(H(Y, \eta|_Y), h^{H(Y,\eta|_Y)}\big) - \int_{Y_g} \mathrm{Td}_g(TY, h^{TY})\, \mathrm{ch}_g(\eta, h^\eta). \tag{2.9}$$

To describe the analytic torsion forms associated to (ω^V, h^ξ), we modify the constructions of Section 1.3. For $s \in S$, by Hodge theory, we have a canonical identification of $\mathbb{Z}$-graded vector spaces $H(X_s, \xi|_{X_s}) \simeq K_s^V = \{f \in E_s, (D^X + V)f = 0\}$. Let $h^{H(X_s, \xi|_{X_s})}$ be the corresponding metric on $H(X_s, \xi|_{X_s})$ as in Section 1.3. Set

$$\begin{aligned} \bar{B}_u^{\prime\prime V} &= \sqrt{u}(\bar{\partial}^X + v) + \nabla^{E''} - \frac{c(T^{(1,0)})}{2\sqrt{2u}}, \\ \bar{B}_u^{\prime V} &= \sqrt{u}(\bar{\partial}^{X*} + v^*) + \nabla^{E'} - \frac{c(T^{(0,1)})}{2\sqrt{2u}}, \\ \bar{B}_u^V &= \bar{B}_u^{\prime V} + \bar{B}_u^{\prime\prime V}. \end{aligned} \tag{2.10}$$

For $u > 0$, set

$$\bar{N}_u^V = N_{\mathbf{V}}^X - N_{\mathbf{H}} + \frac{i\omega^{V,H}}{u}. \tag{2.11}$$

Put

$$\operatorname{ch}_g(\xi, h^\xi) = \sum_{i=0}^{m} (-1)^i \operatorname{ch}_g(\xi_i, h^{\xi_i}), \tag{2.12}$$

$$\operatorname{ch}'_g(\xi, h^\xi) = \sum_{i=0}^{m} (-1)^i i \operatorname{ch}_g(\xi_i, h^{\xi_i}),$$

$$\operatorname{ch}_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) = \sum_{k=0}^{\dim X} (-1)^k \operatorname{ch}_g\big(H^k(X, \xi|_X), h^{H(X,\xi|_X)}\big),$$

$$\operatorname{ch}'_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) = \sum_{k=0}^{\dim X} (-1)^k k \operatorname{ch}_g\big(H^k(X, \xi|_X), h^{H(X,\xi|_X)}\big).$$

If $(\alpha_u)_{u>0}$ is a family of smooth forms on S, we will write that as $u \to 0$, $\alpha_u = \mathcal{O}(u^{k+1})$ if for any compact set $K \subset S$, and any $p \in \mathbb{N}$, there is $C > 0$ such that the sup of α_u and its derivatives of order $\leqq p$ on K are dominated by Cu^{k+1}.

Then by combining the techniques of [BGS2], Theorems 2.2 and 2.16, and [B6], Theorems 4.9–4.11, the following analogue of [Ma], Theorem 2.10, holds.

Theorem 2.3. *As $u \to 0$*

$$\Phi \operatorname{Tr}_s[g \exp(-\bar{B}_u^{V,2})] = \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) \operatorname{ch}_g(\xi, h^\xi) + \mathcal{O}(u). \tag{2.13}$$

There are forms $D^V_{j,g} \in P^S$ $(j \geqq -1)$ such that for $k \in \mathbb{N}$, as $u \to 0$

$$\Phi \operatorname{Tr}_s[g \bar{N}_u^V \exp(-\bar{B}_u^{V,2})] = \sum_{j=-1}^{k} D^V_{j,g} u^j + \mathcal{O}(u^{k+1}). \tag{2.14}$$

Also

$$D^V_{-1,g} = \int_{X_g} \frac{\omega^V}{2\pi} \operatorname{Td}_g(TX, h^{TX}) \operatorname{ch}_g(\xi, h^\xi), \tag{2.15}$$

$$D^V_{0,g} = \int_{X_g} \big(\dim X \operatorname{Td}_g(TX) - \operatorname{Td}'_g(TX)\big) \operatorname{ch}_g(\xi)$$

$$- \int_{X_g} \operatorname{Td}_g(TX) \operatorname{ch}'_g(\xi) \quad \text{in } P^S/P^{S,0}.$$

As $u \to +\infty$

$$\Phi \operatorname{Tr}_s[g \exp(-\bar{B}_u^{V,2})] = \operatorname{ch}_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) + \mathcal{O}\Big(\frac{1}{\sqrt{u}}\Big), \tag{2.16}$$

$$\Phi \operatorname{Tr}_s[g \bar{N}_u^V \exp(-\bar{B}_u^{V,2})] = \operatorname{ch}'_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) + \mathcal{O}\Big(\frac{1}{\sqrt{u}}\Big).$$

By replacing in (1.25), B_u by $\bar{B}_u^V$, N_u by $\overline{N}_u^V$, as in (1.25), we construct a form $T_g(\omega^V, h^\xi) \in P^S$ such that the analogue of (1.23) holds, i.e.

$$\frac{\bar{\partial}\partial}{2i\pi} T_g(\omega^V, h^\xi) = \mathrm{ch}_g\big(H(X, \xi|_X), h^{H(X,\xi|_X)}\big) - \int_{X_g} \mathrm{Td}_g(TX, h^{TX})\,\mathrm{ch}_g(\xi, h^\xi). \tag{2.17}$$

A simple modification of the argument of [Ma], §2e), shows that the analogue of the anomaly formulas [Ma], Theorem 2.13, still holds.

2.3. Assumption on the metrics on TX, TY, ξ, η. Let $N_{Y/X}$ be the fibrewise normal bundle to Y in X. Let $N_{W/V}$ be the normal bundle to W in V. Clearly, $N_{W/V} \simeq N_{Y/X}$. We identify $N_{Y/X}$ as a smooth vector bundle to the orthogonal bundle to TY in $TX|_W$ with respect to $h^{TX|_W}$. Let $h^{N_{Y/X}}$ be the metric induced by $h^{TX|_W}$ on $N_{Y/X}$.

In the sequel, we assume that the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ verify assumption (A) of [B2], §1b), with respect to $h^{N_{Y/X}}, h^\eta$. We describe this assumption in more detail.

On W, we have the exact sequence of G-equivariant holomorphic vector bundles

$$0 \to TW \to TV|_W \to N_{Y/X} \to 0. \tag{2.18}$$

For $y \in W$, let $H_y(\xi, v)$ be the homology of the complex $(\xi, v)_y$. If $y \in W$, $U \in TV_y$, let $\partial_U v(y)$ be the derivative of v at y in the direction U in any given holomorphic trivialization of (ξ, v) near y. By the arguments of [B2], §1b), [B4], §3d), [B5], §3d), we know that:

a) The $H_y(\xi, v)$ are the fibres of a holomorphic $\mathbb{Z}$-graded vector bundle $H(\xi, v)$ on W. The map $\partial_U v(y)$ acts on $H_y(\xi, v)$ as a chain map, this action does not depend on the trivialization of (ξ, v), and only depend on the image z of U in $N_{W/V,y} = N_{Y/X,y}$. From now on, we will write $\partial_z v(y)$ instead of $\partial_U v(y)$.

b) Let π be the projection $N_{Y/X} \to W$. Then over $N_{Y/X}$, we have a canonical identification of $\mathbb{Z}$-graded chain complexes of bundles

$$\big(\pi^* H(\xi, v), \partial_z v(y)\big) \simeq \big(\pi^*\big(\Lambda(N_{Y/X}^*) \otimes \eta\big), \sqrt{-1} i_z\big). \tag{2.19}$$

The group G acts on both complexes in (2.19) by holomorphic chain maps, and (2.19) is an identification of G-bundles.

By finite dimensional Hodge theory, we know that there is a canonical isomorphism of $\mathbb{Z}$-graded vector bundles over W

$$H(\xi, v) \simeq \{f \in \xi, vf = 0, v^* f = 0\}. \tag{2.20}$$

Let $h^{H(\xi,v)}$ be the metric on $H(\xi, v)$ induced by h^ξ by identification (2.20). Let $h^{\Lambda(N_{Y/X}^*)\otimes\eta}$ be the metric on $\Lambda(N_{Y/X}^*) \otimes \eta$ induced by $h^{N_{Y/X}}$ and h^η. Then the metrics $h^{H(\xi,v)}, h^{\Lambda(N_{Y/X}^*)\otimes\eta}$ are G-invariant.

We say that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ verify assumption (A) with respect to $h^{N_{Y/X}}, h^{\eta}$, if the identification (2.19) also identifies the metrics.

By [B4], Proposition 3.5, given G-invariant metrics $h^{N_{Y/X}}, h^{\eta}$, there exist G-invariant metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ which verify assumption (A) with respect to $h^{N_{Y/X}}, h^{\eta}$.

2.4. A singular Bott-Chern current. In this section, we make the same assumptions and we use the same notation as in Section 2.3.

Let $\nabla^{\xi} = \bigoplus_{i=0}^{m} \nabla^{\xi_i}$ be the holomorphic Hermitian connection on $(\xi, h^{\xi}) = \bigoplus_{i=0}^{m} (\xi_i, h^{\xi_i})$. For $u > 0$, set

$$C_u = \nabla^{\xi} + \sqrt{u} V. \tag{2.21}$$

Then C_u is a G-invariant superconnection [Q1] on the $\mathbb{Z}_2$-graded vector bundle ξ on V.

Take $g \in G$. Then we construct forms $\Phi \operatorname{Tr}_s[g \exp(-C_u^2)]$, $\Phi \operatorname{Tr}_s[g N_{\mathbf{H}} \exp(-C_u^2)]$ and $T_g(\xi, h^{\xi})$ as in [B4], §6, on V_g. Let $\delta_{\{W_g\}}$ be the current of integration on W_g. Then by [B4], Theorem 6.7, $T_g(\xi, h^{\xi})$ is a sum of currents of type (p, p) over V_g, such that

$$\frac{\bar{\partial}\partial}{2i\pi} T_g(\xi, h^{\xi}) = (\mathrm{Td}_g)^{-1}(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^{\eta}) \delta_{\{W_g\}} - \operatorname{ch}_g(\xi, h^{\xi}). \tag{2.22}$$

More precisely, we have the formula,

(2.23)

$$\begin{aligned} T_g(\xi, h^{\xi}) = {} & \int_0^1 \Phi \operatorname{Tr}_s\big[g N_{\mathbf{H}} \big(\exp(-C_u^2) - \exp(-C_0^2)\big)\big] \frac{du}{u} \\ & + \int_1^{+\infty} \big(\Phi \operatorname{Tr}_s[g N_{\mathbf{H}} \exp(-C_u^2)] + (\mathrm{Td}_g^{-1})'(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^{\eta}) \delta_{\{W_g\}}\big) \frac{du}{u} \\ & - \Gamma'(1)\big(\operatorname{ch}_g'(\xi, h^{\xi}) + (\mathrm{Td}_g^{-1})'(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^{\eta}) \delta_{\{W_g\}}\big). \end{aligned}$$

3. A proof of Theorem 0.1

This section is an extension of [B5], §4, §6, to the equivariant setting, and of [B4], §3, §8, to the case of general S.

This section is organized as follows. In Section 3.1, we state our main result. In Section 3.2, we introduce a contour integral. In Section 3.3, we state five intermediate results, the proofs of which are delayed to the next sections. In Section 3.4, we establish our main result.

In this section, we use the assumptions and notation of Section 2. We fix $g \in G$ in the rest of our paper.

3.1. The main theorem. Recall that by (2.7), we have the canonical isomorphism of G-equivariant holomorphic $\mathbb{Z}$-graded vector bundles on S

$$H(X,\xi|_X) \simeq H(Y,\eta|_Y). \tag{3.1}$$

From now on, we identify $H(X,\xi|_X)$ and $H(Y,\eta|_Y)$ by (3.1). Also, in Section 2.1, smooth G-invariant Hermitian metrics $h^{H(X,\xi|_X)}$ and $h^{H(Y,\eta|_Y)}$ were constructed on $H(X,\xi|_X)$ and $H(Y,\eta|_Y)$. Then $h^{H(X,\xi|_X)}$ can be considered as a metric on $H(Y,\eta|_Y)$. Let $\widetilde{\mathrm{ch}}_g\big(H(Y,\eta|_Y),h^{H(X,\xi|_X)},h^{H(Y,\eta|_Y)}\big) \in P^S/P^{S,0}$ be the Bott-Chern class of [BGS1], Theorem 1.29, such that

$$\begin{aligned}\frac{\bar\partial\partial}{2i\pi}\,\widetilde{\mathrm{ch}}_g\big(H(Y,\eta|_Y),h^{H(X,\xi|_X)},h^{H(Y,\eta|_Y)}\big)\\ = \mathrm{ch}_g\big(H(Y,\eta|_Y),h^{H(Y,\eta|_Y)}\big) - \mathrm{ch}_g\big(H(X,\xi|_X),h^{H(X,\xi|_X)}\big).\end{aligned} \tag{3.2}$$

Consider the exact sequence of holomorphic G-equivariant Hermitian vector bundles on W_g,

$$0 \to TY \to TX|_{W_g} \to N_{Y/X} \to 0. \tag{3.3}$$

Let $\widetilde{\mathrm{Td}}_g(TY,TX|_{W_g},h^{TX}) \in P^{W_g}/P^{W_g,0}$ be the Bott-Chern class constructed as in [BGS1], Theorem 1.29, such that

$$\begin{aligned}\frac{\bar\partial\partial}{2i\pi}\,\widetilde{\mathrm{Td}}_g(TY,TX|_{W_g},h^{TX}) = \mathrm{Td}_g(TX|_{W_g},h^{TX})\\ - \mathrm{Td}_g(TY,h^{TY})\,\mathrm{Td}_g(N_{Y/X},h^{N_{Y/X}}).\end{aligned} \tag{3.4}$$

For $y \in \mathbb{R}$, $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, set

$$\zeta(y,s) = \sum_{n=1}^{+\infty}\frac{\cos(ny)}{n^s}, \quad \eta(y,s) = \sum_{n=1}^{+\infty}\frac{\sin(ny)}{n^s}. \tag{3.5}$$

Then for a fixed $y \in \mathbb{R}\backslash 2\pi\mathbb{Z}$, both functions in (3.5) extend to a holomorphic function of s for $\mathrm{Re}(s) < 1$. If $y \in 2\pi\mathbb{Z}$, $\zeta(y,s)$ has a simple pole at $s = 1$.

Following [B3], Theorem 7.8, we introduce the following genus $R(\theta,x)$.

Definition 3.1. For $\theta \in \mathbb{R}$, and $x \in \mathbb{C}$ are such that $|x| < 2\pi$ if $\theta \in 2\pi\mathbb{Z}$; $|x| < \inf_{k\in\mathbb{Z}}|\theta + 2k\pi|$ if $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$, let $R(\theta,x)$ be the convergent power series

$$\begin{aligned}R(\theta,x) = \sum_{\substack{n\geqq 0\\ n \text{ even}}} i\left(\sum_{j=1}^{n}\frac{1}{j}\eta(\theta,-n) + 2\frac{\partial\eta}{\partial s}(\theta,-n)\right)\frac{x^n}{n!}\\ + \sum_{\substack{n\geqq 1\\ n \text{ odd}}}\left(\sum_{j=1}^{n}\frac{1}{j}\zeta(\theta,-n) + 2\frac{\partial\zeta}{\partial s}(\theta,-n)\right)\frac{x^n}{n!}.\end{aligned} \tag{3.6}$$

Then for $x \in \mathbb{C}$, $|x| < 2\pi$, $R(0,x)$ is the Gillet-Soulé's power series $R(x)$ [GS1]. For $\theta \in \mathbb{R}$, we identify $R(\theta, \cdot)$ with the corresponding additive genus.

On $W_g, N_{Y/X}$ splits holomorphically as an orthogonal sum of holomorphic vector bundles N^{θ_j} $(0 \leqq \theta_j < 2\pi, j = 1, \dots, q)$. Let $h^{N^{\theta_j}}$ be the metrics on N^{θ_j} induced by $h^{N_{Y/X}}$. Let $R^{N^{\theta_j}}$ be the curvature of the holomorphic Hermitian connection on $(N^{\theta_j}, h^{N^{\theta_j}})$. Set

$$R(\theta_j, N^{\theta_j}, h^{N^{\theta_j}}) = \operatorname{Tr}\left[R\left(\theta_j, \frac{-R^{N^{\theta_j}}}{2i\pi}\right)\right], \tag{3.7}$$

$$R_g(N_{Y/X}, h^{N_{Y/X}}) = \sum_{j=1}^{q} R(\theta_j, N^{\theta_j}, h^{N^{\theta_j}}).$$

Then the cohomology class of the closed form $R_g(N_{Y/X}, h^{N_{Y/X}})$ does not depend on the metric $h^{N_{Y/X}}$. We denote this class by $R_g(N_{Y/X})$.

Let $R_g(TX) \in H^*(V_g, \mathbb{C})$, $R_g(TY) \in H^*(W_g, \mathbb{C})$ be the cohomology classes of TX, TY defined as (3.7).

By [B4], Remark 6.8, the wave front set of $T_g(\xi, h^\xi)$ is included in $N^*_{W_g/V_g,\mathbb{R}} \simeq N^*_{Y_g/X_g,\mathbb{R}}$. It follows from [H], Theorem 8.2.12, that the integral along the fibre $\int_{X_g} \operatorname{Td}_g(TX, h^{TX}) T_g(\xi, h^\xi)$ lies in P^S.

Theorem 3.2. *The following identities hold*:

$$\begin{aligned} &\widetilde{\operatorname{ch}}_g\big(H(Y,\eta|_Y), h^{H(X,\xi|_X)}, h^{H(Y,\eta|_Y)}\big) - T_g(\omega^W, h^\eta) + T_g(\omega^V, h^\xi) \\ &\quad = \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) T_g(\xi, h^\xi) - \int_{Y_g} \frac{\widetilde{\operatorname{Td}}_g(TY, TX|_{W_g}, h^{TX})}{\operatorname{Td}_g(N_{Y/X}, h^{N_{Y/X}})} \operatorname{ch}_g(\eta, h^\eta) \\ &\quad\quad + \int_{Y_g} \operatorname{Td}_g(TY) R_g(N_{Y/X}) \operatorname{ch}_g(\eta) \quad \text{in } P^S/P^{S,0}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\widetilde{\operatorname{ch}}_g\big(H(Y,\eta|_Y), h^{H(X,\xi|_X)}, h^{H(Y,\eta|_Y)}\big) - T_g(\omega^W, h^\eta) + T_g(\omega^V, h^\xi) \\ &\quad = \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) T_g(\xi, h^\xi) - \int_{Y_g} \frac{\widetilde{\operatorname{Td}}_g(TY, TX|_{W_g}, h^{TX})}{\operatorname{Td}_g(N_{Y/X}, h^{N_{Y/X}})} \operatorname{ch}_g(\eta, h^\eta) \\ &\quad\quad + \int_{X_g} \operatorname{Td}_g(TX) R_g(TX) \operatorname{ch}_g(\xi) - \int_{Y_g} \operatorname{Td}_g(TY) R_g(TY) \operatorname{ch}_g(\eta) \quad \text{in } P^S/P^{S,0}. \end{aligned}$$

Proof. The remainder of this section is devoted to the proof of Theorem 3.2. □

By the anomaly formula of [Ma], Theorem 2.13, one verifies easily that we only need to establish Theorem 3.2 for one single choice of ω^W. In the sequel, we will assume that $\omega^W = i^*\omega^V$, and we will prove Theorem 3.2 in this case.

3.2. A contour integral. For $u > 0$, $T > 0$, set

(3.9)
$$A_{u,T} = B^V_{u^2} + TV; \quad B_{u,T} = A_{u,uT};$$
$$N^V_u = N^X_{\mathbf{V}} + \frac{i\omega^{V,H}}{u}.$$

Then $A_{u,T}, B_{u,T}$ are superconnections on E.

Let $d_{u,T}$ be the standard de Rham operator acting on smooth forms on $\mathbb{R}^*_+ \times \mathbb{R}^*_+$.

Theorem 3.3. *Let $\beta_{u,T}$ be the form on $\mathbb{R}^*_+ \times \mathbb{R}^*_+ \times S$,*

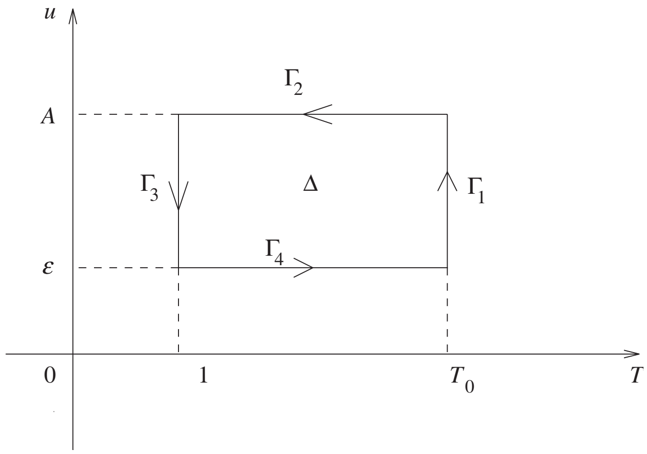
(3.10)
$$\beta_{u,T} = \frac{du}{u} \operatorname{Tr}_s[g(N^V_{u^2} - N_{\mathbf{H}}) \exp(-B^2_{u,T})] - \frac{dT}{T} \operatorname{Tr}_s[gN_{\mathbf{H}} \exp(-B^2_{u,T})].$$

The following identity holds:

(3.11)
$$\begin{aligned} d_{u,T}\beta_{u,T} = u\,dT\,du\Bigg[&\bar{\partial}\frac{\partial}{\partial b}\Bigg\{\operatorname{Tr}_s\left[\frac{1}{u} gN^V_{u^2} \exp(-B^2_{u,T} - bv^*)\right] \\ &+ \operatorname{Tr}_s\left[\frac{1}{uT} gN_{\mathbf{H}} \exp\left(-B^2_{u,T} - b\frac{\partial}{\partial u} B^{V'}_{u^2}\right)\right]\Bigg\}_{b=0} \\ &+ \partial\frac{\partial}{\partial b}\Bigg\{\operatorname{Tr}_s\left[\frac{1}{u} gN^V_{u^2} \exp(-B^2_{u,T} - bv)\right] \\ &+ \operatorname{Tr}_s\left[\frac{1}{uT} gN_{\mathbf{H}} \exp\left(-B^2_{u,T} - b\frac{\partial}{\partial u} B^{V''}_{u^2}\right)\right]\Bigg\}_{b=0}\Bigg]. \end{aligned}$$

Proof. The proof of Theorem 3.3 is identical to the proof of [B5], Theorem 4.3. □

Take ε, A, T_0, $0 < \varepsilon \leqq 1 \leqq A < +\infty$, $1 \leqq T_0 < +\infty$. Let $\Gamma = \Gamma_{\varepsilon,A,T_0}$ be the oriented contour in $\mathbb{R}^*_+ \times \mathbb{R}^*_+$,



The contour Γ is made of the four oriented pieces $\Gamma_1, \dots, \Gamma_4$ indicated above. Also Γ bounds an oriented rectangular domain Δ. For $1 \leqq k \leqq 4$, set

(3.12)
$$I^0_k = \int_{\Gamma_k} \Phi\beta_{u,T}.$$

Definition 3.4. Let γ, δ be the forms on S

$$\gamma = \int_\Delta \frac{\partial}{\partial b}\left\{\operatorname{Tr}_s[gN^V_{u^2}\exp(-B^2_{u,T} - bv^*)] + \operatorname{Tr}_s\left[\frac{1}{T}gN_{\mathbf{H}}\exp\left(-B^2_{u,T} - b\frac{\partial}{\partial u}B^{V'}_{u^2}\right)\right]\right\}_{b=0} dT\,du, \tag{3.13}$$

$$\delta = \int_\Delta \frac{\partial}{\partial b}\left\{\operatorname{Tr}_s[gN^V_{u^2}\exp(-B^2_{u,T} - bv)] + \operatorname{Tr}_s\left[\frac{1}{T}gN_{\mathbf{H}}\exp\left(-B^2_{u,T} - b\frac{\partial}{\partial u}B^{V''}_{u^2}\right)\right]\right\}_{b=0} dT\,du.$$

Theorem 3.5. *The following identity holds*:

$$\sum_{k=1}^{4} I^0_k = \Phi(\bar{\partial}\gamma + \partial\delta). \tag{3.14}$$

Proof. This follows from Theorem 3.3. □

As in [B5], the proof of Theorem 3.2 will consist in making $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ in this order in identity (3.14).

3.3. Five intermediate results.

Definition 3.6. For $T > 0$, we denote by $\langle\,,\rangle_T$ the Hermitian product on E associated with the metrics $h^{TX}, h^{\xi_0}, h^{\xi_1}/T^2, \dots, h^{\xi_m}/T^{2m}$ on TX, $\xi_0, \dots, \xi_m$ respectively. Set

$$K_T = \{s \in E; (\bar{\partial}^X + v)s = 0, (\bar{\partial}^{X*} + T^2v^*)s = 0\}. \tag{3.15}$$

Let P_T be the orthogonal projection operator from E on K_T with respect to the Hermitian product $\langle\,,\rangle_T$.

By Hodge theory, for any $T > 0$, there is a canonical isomorphism of G-equivariant $\mathbb{Z}$-graded vector bundles,

$$K_T \cong H^*(E, \bar{\partial}^X + v). \tag{3.16}$$

Let $h^{H(X,\xi|_X)}_T$ be the G-invariant metric on $H(X,\xi|_X)$ inherited from the metric $\langle\,,\rangle_T$ restricted to K_T. Let $\nabla^{H(X,\xi|_X)}_T$ be the holomorphic Hermitian connection on $\big(H(X,\xi|_X), h^{H(X,\xi|_X)}_T\big)$.

We now state five intermediate results contained in Theorems 3.7–3.11, which are the obvious extension of [B4], Theorems 8.4–8.8, [B5], Theorems 6.5–6.9. The proofs of Theorems 3.7–3.11 are deferred to Sections 4–7.

Theorem 3.7. *For any compact set $K \subset S$, for any $u_0 > 0$, there exist $C > 0$, $\delta \in \,]0,1]$ such that on K, for $u \geqq u_0$, $T \geqq 1$,*

$$(3.17)\qquad |\mathrm{Tr}_s[g(N_{u^2}^V - N_{\mathbf{H}})\exp(-B_{u,T}^2)] - \mathrm{Tr}_s[gN_{u^2}^W \exp(-B_{u^2}^{W,2})]| \leqq \frac{C}{T^\delta},$$

$$\left|\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-B_{u,T}^2)] - \frac{1}{2}\dim N_{Y/X}\,\mathrm{Tr}_s[g\exp(-B_{u^2}^{W,2})]\right| \leqq \frac{C}{T^\delta}.$$

Theorem 3.8. *For any compact set $K \subset S$, there exists $C > 0$ such that on K, for $u \geqq 1, T \geqq 1$,*

$$(3.18)\qquad |\mathrm{Tr}_s[gN_{u^2}^V \exp(-B_{u,T}^2)] - \mathrm{Tr}_s[gP_T N_{\mathbf{V}}^X P_T \exp(-\nabla_T^{H(X,\xi|_X),2})]| \leqq \frac{C}{u},$$

$$|\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-B_{u,T}^2)] - \mathrm{Tr}_s[gP_T N_{\mathbf{H}} P_T \exp(-\nabla_T^{H(X,\xi|_X),2})]| \leqq \frac{C}{u}.$$

Theorem 3.9. *For any compact set $K \subset S$, there exist $C > 0$, $\gamma \in]0,1]$, such that on K, for $u \in]0,1]$, $0 \leqq T \leqq 1/u$, then*

$$(3.19)\qquad \left|\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2)] - \int_{X_g}\mathrm{Td}_g(TX,h^{TX})\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-C_{T^2}^2)]\right| \leqq C\big(u(1+T)\big)^\gamma.$$

There exists $C' > 0$, such that on K, for $u \in]0,1]$, $0 \leqq T \leqq 1$,

$$(3.20)\qquad |\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2)] - \mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,0}^2)]| \leqq C'T.$$

In the sequel, we use the notation of [B4], §7, applied to the exact sequence (3.3). In particular, for $u > 0$, we consider the operator $\mathscr{B}_u^2$ of [B4], Definition 7.4, and $\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-\mathscr{B}_{T^2}^2)]$ is the generalized supertrace in the sense of [B3], Definition 2.1 (cf. [B4], Definition 7.5). We denote by $\mathbf{B}_g(TY, TX|_{W_g}, h^{TX}) \in P^{W_g}$ the generalized analytic torsion forms associated to (3.3) as in [B3], §6 (cf. [B4], Definition 7.9). Then

$$(3.21)\qquad \frac{\bar\partial\partial}{2i\pi}\mathbf{B}_g(TY, TX|_{W_g}, h^{TX}) = \mathrm{Td}_g(TY, h^{TY}) - \frac{\mathrm{Td}_g(TX|_{W_g}, h^{TX})}{\mathrm{Td}_g(N_{Y/X}, h^{N_{Y/X}})}.$$

Theorem 3.10. *For any $T > 0$, the following identity holds:*

$$(3.22)\qquad \lim_{u\to 0}\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T/u}^2)] = \int_{Y_g}\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-\mathscr{B}_{T^2}^2)]\,\mathrm{ch}_g(\eta, h^\eta).$$

Theorem 3.11. *For any compact set $K \subset S$, there exist $C > 0$, $\delta \in]0,1]$, such that on K, for $u \in]0,1]$, $T \geqq 1$,*

$$(3.23)\qquad \left|\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T/u}^2)] - \frac{1}{2}\dim N_{Y/X}\,\mathrm{Tr}_s[g\exp(-B_{u^2}^{W,2})]\right| \leqq \frac{C}{T^\delta}.$$

In Sections 4–7, we will assume for simplicity that S is compact. If S is not compact, then we consider instead the compact subsets $K \subset S$, and the various constants $C > 0$ depend explicitly on K.

3.4. The asymptotics of the I_k^0's. Because of the formal analogies with [B5], Theorems 6.5–6.9, which were indicated before, the discussion of the asymptotics of the I_k^0's for $k = 1, 3, 4$ as $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ can be formally transferred from [B5], §6.4, §6.5. Then we obtain I_k^3 for $k = 1, 3, 4$ formally as in [B5], §6.4. For $k = 2$, by [B5], Theorem 6.10, which gives us the asymptotics of $h_T^{H(X,\xi|_X)}$ as $T \to +\infty$. Then by using [B5], Theorem 6.10, as in [B5], pp. 77–78, we get

$$\begin{aligned}(3.24)\qquad I_3^2 = I_2^2 &= \int_1^{+\infty} \Phi\bigg(\mathrm{Tr}_s[gP_T N_{\mathbf{H}} P_T \exp(-\nabla_T^{H(X,\xi|_X),2})] \\ &\quad - \frac{1}{2}\dim N_{Y/X}\,\mathrm{Tr}_s[g\exp(-\nabla^{H(Y,\eta|_Y),2})]\bigg)\frac{dT}{T} \\ &= \frac{1}{2}\,\widetilde{\mathrm{ch}}_g\big(H(Y,\eta|_Y), h^{H(X,\xi|_X)}, h^{H(Y,\eta|_Y)}\big) \quad \text{in } P^S/P^{S,0}.\end{aligned}$$

If S is compact and Kähler, $P^{S,0}$ is closed under uniform convergence. In the case, since $\sum_{k=1}^{4} I_k^0 \in P^{S,0}$, then $\sum_{k=1}^{4} I_k^3 \in P^{S,0}$.

In the case of a general $S, P^{S,0}$ is not necessary closed. In [B5], (6.170), an explicit formula is given for $\sum_{k=1}^{4} I_k^3$ as

$$(3.25)\qquad \sum_{k=1}^{4} I_k^3 = \Phi(\bar\partial\mu^3 + \partial\nu^3) - \frac{\bar\partial\partial}{i\pi}\Phi\lambda^3,$$

and the same proofs for μ^3, ν^3, λ^3 as in [B5], §6.6–§6.8, work as well. To keep this paper short, we will not discuss μ^3, ν^3, λ^3 in more detail.

Ultimately, we obtain an extension of [B4], Theorem 8.12, [B5], Theorem 6.22:

Theorem 3.12. *The following identity holds in $P^S/P^{S,0}$:*

$$\begin{aligned}(3.26)\qquad &\widetilde{\mathrm{ch}}_g\big(H(Y,\eta|_Y), h^{H(X,\xi|_X)}, h^{H(Y,\eta|_Y)}\big) - T_g(\omega^W, h^\eta) + T_g(\omega^V, h^\xi) \\ &\quad = \int_{X_g} \mathrm{Td}_g(TX, h^{TX}) T_g(\xi, h^\xi) + \int_{Y_g} \mathbf{B}_g(TY, TX|_{W_g}, h^{TX})\,\mathrm{ch}_g(\eta, h^\eta) \\ &\qquad - \Gamma'(1)\int_{Y_g} \mathrm{Td}_g(TY)\left(\frac{\mathrm{Td}_g'}{\mathrm{Td}_g}(N_{Y/X}) - \frac{1}{2}\dim N_{Y/X}\right)\mathrm{ch}_g(\eta).\end{aligned}$$

By [B4], Theorem 7.14, (7.38), (8.26), we know that, in $P^{W_g}/P^{W_g,0}$

$$\begin{aligned}(3.27)\qquad \mathbf{B}_g(TY, TX|_{W_g}, h^{TX}) &= -\mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}})\,\widetilde{\mathrm{Td}}_g(TY, TX|_{W_g}, h^{TX}) \\ &\quad + \mathrm{Td}_g(TY)\left\{R_g(N_{Y/X}) + \Gamma'(1)\left(\frac{\mathrm{Td}_g'}{\mathrm{Td}_g}(N_{Y/X}) - \frac{1}{2}\dim N_{Y/X}\right)\right\}.\end{aligned}$$

Let $i'_g : W_g \to V_g$ be the obvious embedding. Then we have the identities in $H^*(W_g, \mathbb{C})$,

$$\mathrm{Td}_g(TY) = \frac{i'^*_g \,\mathrm{Td}_g(TX)}{\mathrm{Td}_g(N_{Y/X})}, \tag{3.28}$$

$$R_g(N_{Y/X}) = i'^*_g R_g(TX) - R_g(TY).$$

Now the identity (3.8) follows from Theorem 3.12, (3.27) and (3.28).

The proof of Theorem 3.2 is completed. □

4. A proof of Theorems 3.7 and 3.8

In this section, we give a proof of Theorems 3.7 and 3.8. This proof relies essentially on the results of [B5], §9, where the corresponding results were established when G is trivial. Theorems 3.7 and 3.8 are the obvious extension of [B4], Theorems 8.4, 8.5, [B5], Theorems 6.5, 6.6.

This section is organized as follows. In Section 4.1, we recall the construction of an extension of $T^H W$ to V [B5], §7.6. In Section 4.2, we give a proof of Theorems 3.7 and 3.8.

4.1. An extension of $T^H W$ to V. In the discussion which follows, we will assume for simplicity that S is compact. We proceed as in [B5], §7.6.

If $y \in W$, $Z \in N_{Y/X,\mathbb{R},y}$, let $t \in \mathbb{R} \to x_t = \exp^X_y(tZ) \in V$ be the geodesic in the fibre $X_{\pi_W(y)}$ with respect to h^{TX}, such that $x_0 = y$, $\left.\frac{dx_t}{dt}\right|_{t=0} = Z$.

For $\varepsilon > 0$, set $B_\varepsilon = \{Z \in N_{Y/X,\mathbb{R}}, |Z| < \varepsilon\}$. For $\varepsilon_0 > 0$ small enough, the map $(y, Z) \in N_{Y/X,\mathbb{R}} \to \exp^X_y(Z) \in V$ is a diffeomorphism from $B_{2\varepsilon_0}$ on a tubular neighbourhood $\mathscr{U}_{2\varepsilon_0}$ of W in V. From now on, we use the notation $x = (y, Z)$ instead of $x = \exp^X_y(Z)$. We identify $y \in W$ with $(y, 0) \in N_{Y/X,\mathbb{R}}$. Since $g \in G$ is an isometry which preserves W, g preserves the geodesics in the fibre $X_{\pi_W(y)}$ which are normal to Y. Of course, under this identification, g acts linearly in the fibre of $N_{Y/X,\mathbb{R}}$.

As explained in [B5], §7.5, in general, $T^H V|_W \neq T^H W$. This is a potential source of difficulties. Thus we are forced to modify the horizontal bundle $T^H V$ near W.

Recall that ∇^{TX} is the holomorphic Hermitian connection on (TX, h^{TX}). Let $\nabla^{\pi_V^* TS}$ be the trivial connection on $\pi_V^* TS$ along the fibres X. We equip $TV = T^H V \oplus TX$ with the connection along the fibres X, $\nabla^{TV} = \nabla^{\pi_V^* TS} \oplus \nabla^{TX}$. Now the tensor T^V is defined in Section 1.2.

Definition 4.1. If $(y, Z) \in N_{Y/X,\mathbb{R}}$, if $A \in T_\mathbb{R} S$, let $A' \in T_\mathbb{R} V$ be the solution of the differential equation along $t \in \mathbb{R} \to x_t = \exp^X_y(tZ)$,

$$\nabla^{TV}_{\frac{dx}{dt}} A' + T^V_{x_t}\left(A', \frac{dx}{dt}\right) = 0, \tag{4.1}$$

$$A'_0 = A^{H,W}.$$

Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\gamma(a) = 1 \quad \text{for } a \leqq 1/2, \qquad = 0 \quad \text{for } a \geqq 1. \tag{4.2}$$

Then $\gamma\left(\frac{|Z|}{\varepsilon_0}\right)$ can be considered as a smooth function on V with values in $[0,1]$, which vanishes on $V \setminus \mathscr{U}_{\varepsilon_0}$.

Definition 4.2. If $A \in TS$, set

$$A^{H,W} = \gamma\left(\frac{|Z|}{\varepsilon_0}\right) A' + \left(1 - \gamma\left(\frac{|Z|}{\varepsilon_0}\right)\right) A^{H,V}, \tag{4.3}$$

$$A^{H,N_{Y/X}} = A^{H,W} - A^{H,V}.$$

Let $T^H W$ be the smooth subbundle of TV which is the image of TS by the map $A \to A^{H,W}$.

By (4.1), it is clear that $T^H W$ extends G-equivariantly the given vector bundle $T^H W$ on W to the whole V.

Let $f_1, \dots, f_{2m_0}$ be a locally defined basis of $T_{\mathbb{R}}S$, and let $f^1, \dots, f^{2m_0}$ be the corresponding dual basis of $T^*_{\mathbb{R}}S$. Let $e_1, \dots, e_{2l}$ be an orthonormal basis of $T_{\mathbb{R}}X$.

Definition 4.3. For $u > 0$, $T > 0$, set

$$\tilde{A}_{u,T} = \exp\left\{-f^\alpha \frac{1}{\sqrt{2u}} c(f_\alpha^{H,N_{Y/X}})\right\} A_{u,T} \exp\left\{f^\alpha \frac{1}{\sqrt{2u}} c(f_\alpha^{H,N_{Y/X}})\right\}, \tag{4.4}$$

$$\tilde{B}^V_{u^2} = \exp\left\{-f^\alpha \frac{1}{\sqrt{2u}} c(f_\alpha^{H,N_{Y/X}})\right\} B^V_{u^2} \exp\left\{f^\alpha \frac{1}{\sqrt{2u}} c(f_\alpha^{H,N_{Y/X}})\right\},$$

$$\tilde{N}^V_{u^2} = \exp\left\{-f^\alpha \frac{1}{\sqrt{2u}} c(f_\alpha^{H,N_{Y/X}})\right\} N^V_{u^2} \exp\left\{f^\alpha \frac{1}{\sqrt{2u}} c(f_\alpha^{H,N_{Y/X}})\right\},$$

$$\tilde{A}_T = \tilde{A}_{1,T}, \quad A_T = A_{1,T}.$$

4.2. A proof of Theorems 3.7 and 3.8. In our context, all the constructions of [B5], §7, §8, §9, are G-invariant. The same arguments as in [B4], §9, [B5], §7, §8, §9, give us the proof of Theorems 3.7 and 3.8.

5. The analysis of the two parameters operator $g\exp(-A^2_{u,T})$ in the range $u \in]0,1]$, $T \in \left[0, \frac{1}{u}\right]$

The purpose of this section is to prove Theorem 3.9. This section is the obvious extension of [B4], §11, where we work on the case that S is a point, and of [B5], §11, where Theorem 3.9 was established when G is trivial.

This section is organized as follows. In Section 5.1, we prove (3.20), which is the easy part of Theorem 3.9. In Section 5.2, we show the proof of Theorem 3.9 is local on the fibres X. In Sections 5.3 and 5.4, we construct a coordinate system near W_g and a trivialization of $\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\,\hat{\otimes}\,\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi$. In Section 5.5, following [B5], §11.7, we make a Getzler rescaling [Ge] on the operator $\tilde{A}_{u,T}$. In Section 5.6, we explain the matrix structure of the new rescaled operator $L_{y_0,T}^{3,Z_0/T}$. In Section 5.7, we introduce graded Sobolev spaces with weights. In Section 5.8, we prove Theorem 3.9.

We use the notation and assumptions of Sections 2, 3–4.

5.1. The limit as $u \to 0$ of $\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2)]$.

Proposition 5.1. *Let $T_0 \in [0,+\infty[$. There exists $C > 0$ such that for $u \in]0,1]$, $T \in [0,T_0]$,*

$$(5.1)\qquad \Big|\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2)] - \int_{X_g}\mathrm{Td}_g(TX,h^{TX})\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-C_{T^2}^2)]\Big| \leqq Cu,$$

$$|\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2)] - \Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,0}^2)]| \leqq CT.$$

Proof. By combining the local families index theorem of [B1] and [B4], §2 (cf. [Ma], §2e)), one finds that for any $T \geqq 0$, as $u \to 0$

$$(5.2)\qquad \Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2)] = \int_{X_g}\mathrm{Td}_g(TX,h^{TX})\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-C_{T^2}^2)] + \mathcal{O}(u).$$

Since T only plays the role of a parameter, one obtains the existence of C such that the first inequality in (5.1) holds.

Also

$$(5.3)\qquad \frac{\partial}{\partial T}\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2)] = \frac{\partial}{\partial b}\{\mathrm{Tr}_s[gN_{\mathbf{H}}\exp(-A_{u,T}^2 - b[A_{u,T},V])]\}_{b=0}.$$

Again, by using the techniques of [B1] and [B4], §2 (cf. [Ma], §2e)), one finds that for $u \to 0$, the right-hand side of (5.3) converges boundedly for $T \leqq T_0$. Thus we get the second inequality in (5.1). □

5.2. Localization of the problem. Let d^X, d^Y be the Riemannian distance along the fibre $(X,h^{TX}), (Y,h^{TY})$. Let a^X, a^Y be the infimum of the injectivity radius of the fibres X, Y. We take $\varepsilon_0 > 0$ as in Section 4.1. Let $\varepsilon, \alpha \in \mathbb{R}_+$ be such that $\varepsilon \in \left]0,\frac{1}{2}\inf(a^X,a^Y,\varepsilon_0)\right]$, $\alpha \in]0,\varepsilon/8]$. If $x \in V$, let $B^X(x,\varepsilon)$ be the open ball along the fibre X of centre x and radius ε.

In the sequel, we always assume that given $\varepsilon > 0$, $\alpha > 0$ is chosen small enough so that if $x \in X$, $d^X(g^{-1}x,x) \leqq \alpha$, then $d^X(x,X_g) < \varepsilon/16$, and if $y \in Y$, $d^Y(g^{-1}y,y) \leqq \alpha$, then $d^Y(y,Y_g) \leqq \varepsilon/16$.

Let f be a smooth even function defined on $\mathbb{R}$ with values in $[0,1]$, such that

$$f(t) = 1 \quad \text{for } |t| \leqq \alpha/2, \qquad 0 \quad \text{for } |t| \geqq \alpha. \tag{5.4}$$

Set

$$g(t) = 1 - f(t). \tag{5.5}$$

Definition 5.2. For $u \in]0,1]$, $a \in \mathbb{C}$, set

$$F_u(a) = \int_{-\infty}^{+\infty} \exp(ita\sqrt{2}) \exp\left(\frac{-t^2}{2}\right) f(ut) \frac{dt}{\sqrt{2\pi}},$$
$$G_u(a) = \int_{-\infty}^{+\infty} \exp(ita\sqrt{2}) \exp\left(\frac{-t^2}{2}\right) g(ut) \frac{dt}{\sqrt{2\pi}}. \tag{5.6}$$

The functions $F_u(a), G_u(a)$ are even holomorphic functions. So there exist holomorphic functions $\tilde{F}_u(a), \tilde{G}_u(a)$ such that

$$F_u(a) = \tilde{F}_u(a^2), \quad G_u(a) = \tilde{G}_u(a^2). \tag{5.7}$$

The restrictions of $F_u, G_u, \tilde{F}_u, \tilde{G}_u$ to $\mathbb{R}$ lie in the Schwartz space $S(\mathbb{R})$.

From (5.6), we deduce that

$$\exp(-A_{u,T}^2) = \tilde{F}_u(A_{u,T}^2) + \tilde{G}_u(A_{u,T}^2). \tag{5.8}$$

Theorem 5.3. *There exist* $c > 0$, $C > 0$ *such that for* $u \in]0,1]$, $T \geqq 1$, *then*

$$|\mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{G}_u(A_{u,T}^2)]| \leqq c \exp\left(\frac{-C}{u^2}\right). \tag{5.9}$$

Proof. The same proof of [B5], Theorem 11.3, gives us Theorem 5.3. □

Let $\tilde{F}_u(\tilde{A}_{u,T}^2)(x,x')$ $(x,x' \in X)$ be the smooth kernel of $\tilde{F}_u(\tilde{A}_{u,T}^2)$ with respect to $dv_X(x')/(2\pi)^{\dim X}$. Since $\tilde{A}_{u,T}^2$ is a second order elliptic operator whose principal symbol is given by $u^2|\xi|^2/2$, using finite propagation speed [CP], §7.8, [T], §4.4, and (5.6), we see that for $u \in]0,1]$, if $x \in V$, $\tilde{F}_u(\tilde{A}_{u,T}^2)(x,x')$ vanishes for $x' \notin B^X(x,\alpha)$ and only depends on the restriction of $\tilde{A}_{u,T}^2$ to $B^X(x,\alpha)$. Clearly,

$$\begin{aligned}\mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(A_{u,T}^2)] &= \mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}_{u,T}^2)] \\ &= \int_X \mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}_{u,T}^2)(g^{-1}x,x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \\ &= \int_{x\in X, d(x,X_g)\leqq\varepsilon/8} \mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}_{u,T}^2)(g^{-1}x,x)] \frac{dv_X(x)}{(2\pi)^{\dim X}}.\end{aligned} \tag{5.10}$$

By Theorem 5.3, we find that the proof of Theorem 3.9 has been reduced to a local problem near V_g.

In the rest of this section, we fix $\varepsilon > 0$, $\alpha \in]0, \varepsilon/8]$.

5.3. A rescaling of the coordinate $Z_0 \in N_{Y_g/X_g}$. In the sequel, if $x \in X$, $Z \in (T_{\mathbb{R}}X)_x$, $t \in \mathbb{R} \to x_t = \exp_x^X(tZ) \in X$ denotes the geodesic along the fibre X such that $x_0 = x$, $\left.\frac{dx}{dt}\right|_{t=0} = Z$. A similar notation will be used on Y, X_g.

Let $N_{X_g/X}, N_{Y_g/X_g}$ be the (fibrewise) normal bundles to X_g, Y_g in X, X_g. We identify N_{Y_g/X_g} to the orthogonal bundle to TY_g in TX_g with respect to h^{TX}. As in (1.19), we have the holomorphic orthogonal splitting $TX = TX_g \oplus N_{X_g/X}$. Let $h^{TX_g}, h^{N_{X_g/X}}$ be the metrics on $TX_g, N_{X_g/X}$ induced by h^{TX}. Let $h^{TY_g}, h^{N_{Y_g/X_g}}$ be the metrics on $TY_g, N_{Y_g/X_g}$ induced by h^{TX_g}.

First, we identify a neighbourhood of W_g in V_g to a neighbourhood of W_g in $N_{Y_g/X_g,\mathbb{R}}$ using geodesic coordinates normal to W_g. Since X_g is totally geodesic in X, if $y \in W_g$, $Z \in N_{Y_g/X_g,\mathbb{R},y}$, $|Z| \leqq \varepsilon$, we can identify (y, Z) with $\exp_y^{X_g}(Z)$. We denote by $\mathscr{U}_\varepsilon(Y_g/X_g)$ the corresponding neighbourhood of Y_g in X_g. Also we identify a neighbourhood of V_g in V to a neighbourhood of V_g in $N_{X_g/X,\mathbb{R}}$ using geodesic coordinates normal to X_g in X.

Thus $(y, Z, Z') \in \big(W_g, (N_{Y_g/X_g,\mathbb{R}} \oplus N_{X_g/X,\mathbb{R}})_y\big) \to \exp^X_{\exp_y^{X_g}(Z)}(Z')$ identifies an open neighbourhood of W_g in $N_{Y_g/X,\mathbb{R}}$ to an open neighbourhood of W_g in V. Since X_g is totally geodesic in X, and since g preserves the geodesics in X, the action of g near y is given by

$$g(Z, Z') = (Z, gZ'). \tag{5.11}$$

Let dv_{X_g}, dv_{Y_g} be the Riemannian volume forms on X_g, Y_g with respect to h^{TX_g}, h^{TY_g}. Let $dv_{N_{Y_g/X_g}}, dv_{N_{X_g/X}}$ be the Riemannian volume forms on the fibres on $(N_{Y_g/X_g}, h^{N_{Y_g/X_g}}), (N_{X_g/X}, h^{N_{X_g/X}})$. For $y \in W_g$, $Z \in N_{Y_g/X_g,\mathbb{R},y}$, $Z' \in N_{X_g/X,\mathbb{R},y}$, $|Z|, |Z'| < \frac{\varepsilon}{2}$, let $k(y, Z, Z'), k'(y, Z)$ be defined by

$$\begin{aligned} dv_X(y, Z, Z') &= k(y, Z, Z')\, dv_{X_g}(y, Z)\, dv_{N_{X_g/X}}(Z'); \\ dv_{X_g}(y, Z) &= k'(y, Z)\, dv_{Y_g}(y)\, dv_{N_{Y_g/X_g}}(Z) = k'(y, Z)\, dv_{TX_g,y}(Z). \end{aligned} \tag{5.12}$$

Then $k(y, Z, 0) = 1$, $k'(y, 0) = 1$.

Let $e_1, \dots, e_{2l''}$ be an oriented orthonormal basis of $T_{\mathbb{R}}X_g$, and let $e^1, \dots, e^{2\ell''}$ be the corresponding dual basis of $T^*_{\mathbb{R}}X_g$. If $\beta \in \Lambda(T^*_{\mathbb{R}}V_g)$, let $\beta^{\max}$ be the form in $\Lambda(T^*_{\mathbb{R}}S)$ which factors $e^1 \dots e^{2\ell''}$ in the obvious expansion of β.

Definition 5.4. For $T \geqq 0$, $x \in V_g$, let $\beta_T(x) \in \Lambda(T^*_{\mathbb{R}}S)$ such that

$$\beta_T(x) \frac{1}{(2\pi)^{\dim X_g}} = \{\mathrm{Td}_g(TX, h^{TX}) \Phi \operatorname{Tr}_s[g N_{\mathbf{H}} \exp(-C_{T^2}^2)]\}_x^{\max}. \tag{5.13}$$

The key result of this section is the following extension of [B4], Theorem 11.7, [B5], Theorem 11.5.

Theorem 5.5. *There exists $\gamma \in]0,1]$ such that for any $p \in \mathbb{N}$, there exists $C_p > 0$ such that if $u \in]0,1]$, $T \in \left[1, \frac{1}{u}\right]$, $y_0 \in W_g$, $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon T/2$,*

(5.14)
$$\frac{1}{T^{2\dim N_{Y_g/X_g}}}\Bigg| \int_{\substack{Z \in N_{X_g/X,\mathbb{R},(y_0,Z_0/T)} \\ |Z| \leqq \varepsilon/8}} \mathrm{Tr}_s\left[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T})\left(g^{-1}\left(y_0,\frac{Z_0}{T},Z\right),\left(y_0,\frac{Z_0}{T},Z\right)\right)\right] k\left(y_0,\frac{Z_0}{T},Z\right)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} - \beta_T\left(y_0,\frac{Z_0}{T}\right)\Bigg|$$
$$\leqq C_p(1+|Z_0|)^{-p}\big(u(1+T)\big)^{\gamma}.$$

Remark 5.6. From (5.10), to prove (3.19), we only need to estimate

(5.15)
$$\int_{X_g}\left\{\int_{\substack{Z \in N_{X_g/X,\mathbb{R}} \\ |Z| \leqq \varepsilon/8}} \mathrm{Tr}_s\left[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T})\big(g^{-1}(x,Z),(x,Z)\big)\right]k(x,Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} - \beta_T(x)\right\}\frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g}}.$$

In the same way as in [B4], Remark 11.8, we decompose the above integral as $\int_{\mathscr{U}_{\varepsilon/2}(Y_g/X_g)} + \int_{X_g \backslash \mathscr{U}_{\varepsilon/2}(Y_g/X_g)}$. By Theorem 5.5, we find that $\int_{\mathscr{U}_{\varepsilon/2}(Y_g/X_g)}$ is dominated by $C\big(u(1+T)\big)^{\gamma}$. Using again Theorem 5.5 for $W_g = \emptyset$, we get a similar estimate for $\int_{X_g \backslash \mathscr{U}_{\varepsilon/2}(Y_g/X_g)}$. Using now Proposition 5.1, we have thus proved Theorem 3.9.

5.4. A local coordinate system near W_g and a trivialization of $\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$. Let $\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}$ be the connection on

$$\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)$$

along the fibres X, which is induced by $\nabla^{\Lambda(T^{*(0,1)}X)}$. Let

$${}^1\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}, \quad {}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}$$

be the connections on $\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)$, along the fibres X defined in [B5], Definition 11.7.

For $u > 0$, let $\psi_u : \Lambda(T_{\mathbb{R}}^*S) \to \Lambda(T_{\mathbb{R}}^*S)$ be the map

(5.16)
$$\alpha \in \Lambda(T_{\mathbb{R}}^*S) \to u^{-\deg\alpha}\alpha \in \Lambda(T_{\mathbb{R}}^*S).$$

For $u > 0$, let ${}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X),u}$ be the connection on $\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)$ along the fibres X (cf. [B5], Definition 11.9)

(5.17)
$${}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X),u} = \psi_u\,{}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}\psi_u^{-1}.$$

In the sequel, we will use trivializations with respect to the connection ${}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X),u}$. It will be often more convenient to trivialize with respect to ${}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}$, and to apply afterwords the operator ψ_u.

In [B4], §9, [BL], §8f), a G-invariant orthogonal splitting of $\mathbb{Z}$-graded vector bundles $\xi = \xi^+ \oplus \xi^-$ of ξ near W was obtained. Let $P^{\xi^\pm}$ be the orthogonal projection operators from ξ on $\xi^\pm$. Let $\tilde{\nabla}^{\xi^\pm}$ be the connection on $\xi^\pm$, which is the orthogonal projection of ∇^ξ on $\xi^\pm$. Set $\tilde{\nabla}^\xi = \tilde{\nabla}^{\xi^+} \oplus \tilde{\nabla}^{\xi^-}$. Then $\tilde{\nabla}^\xi$ is G-invariant.

Take $y_0 \in W_g$. Let $B_{y_0}^{TX}(0,\varepsilon)$ be the open ball in $(T_{\mathbb{R}}X)_{y_0}$ of centre 0 and of radius ε. The ball $B_{y_0}^{TX}(0,\varepsilon)$ is then identified to $B^X(y_0,\varepsilon)$ using the map $\exp_{y_0}^X$.

We fix $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon/2$. Take $Z \in (T_{\mathbb{R}}X)_{y_0}$, $|Z| \leqq \varepsilon/2$. The curve $t \in [0,1] \to Z_0 + tZ$ lies in $B_{y_0}^{TX}(0,\varepsilon)$. We identify $\big(\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\big)_{Z_0+Z}$ to $\big(\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\big)_{Z_0}$ (resp. ξ_{Z_0+Z} to ξ_{Z_0}) by parallel transport with respect to the connection ${}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X),u}$ (resp. $\tilde{\nabla}^\xi$) along $t \in [0,1] \to Z_0 + tZ$.

When $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon/2$ is allowed to vary, we identify

$$\big(\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\big)_{Z_0}$$

(resp. $(T_{\mathbb{R}}X)_{Z_0}, \xi_{Z_0}$) to $\big(\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\big)_{y_0}$ (resp. $(T_{\mathbb{R}}X)_{y_0}, \xi_{y_0}$) by parallel transport with respect to $\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}$ (resp. $\nabla^{T_{\mathbb{R}}X}, \tilde{\nabla}^\xi$) along $t \in [0,1] \to tZ_0$. Therefore the fibres of $\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)$ at $Z_0 + Z$ and y_0 are identified by parallel transport along the broken curve $t \in [0,1] \to 2tZ_0$, $0 \leqq t \leqq 1/2$, $Z_0 + (2t-1)Z$, $\frac{1}{2} \leqq t \leqq 1$.

Let $\mathbf{H}_{y_0}$ be the vector space of smooth sections of $\big(\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X) \hat{\otimes} \xi\big)_{y_0}$ over $(T_{\mathbb{R}}X)_{y_0}$. Let Δ^{TX} be the ordinary flat Laplacian of $T_{\mathbb{R}}X$. Then Δ^{TX} acts naturally on $\mathbf{H}_{y_0}$. Let γ be a smooth function defined on $\mathbb{R}$ considered in (4.2). If $Z \in (T_{\mathbb{R}}X)_{y_0}$, put

$$\rho(Z) = \gamma\left(\frac{|Z|}{4\varepsilon}\right). \tag{5.18}$$

We now fix $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon/2$. Recall that the considered trivialization of $\pi_V^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ depends on Z_0. Therefore the action of D^X also depends on Z_0.

Definition 5.7. For $u > 0$, $T \geqq 0$, let $L_{u,T}^{1,Z_0}$, M_u^{1,Z_0} be the operators acting on $\mathbf{H}_{y_0}$,

$$L_{u,T}^{1,Z_0} = \big(1 - \rho^2(Z)\big)\left(\frac{-u^2}{2}\Delta^{TX} + T^2 P^{\xi_{y_0}^+}\right) + \rho^2(Z)\tilde{A}_{u,T}^2(Z_0 + Z), \tag{5.19}$$

$$M_u^{1,Z_0} = -\frac{u^2}{2}\big(1 - \rho^2(Z)\big)\Delta^{TX} + \rho^2(Z)\tilde{B}_{u^2}^{V,2}(Z_0 + Z).$$

Let $\tilde{F}_u(L_{u,T}^{1,Z_0})(Z,Z')$ $(Z, Z' \in (T_{\mathbb{R}}X)_{y_0}, |Z'| < \varepsilon/2)$ be the smooth kernel associated to $\tilde{F}_u(L^{1,Z_0}_{u,T})$ with respect to $dv_X(Z_0+Z')/(2\pi)^{\dim X}$. By using propagation finite speed [CP], §7.8, [T], §4.4, we see that for any $y_0 \in W_g$, $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon/2$, $Z \in (T_{\mathbb{R}}X)_{y_0}$, $|Z| \leqq \varepsilon/2$,

$$\tilde{F}_u(\tilde{A}^2_{u,T})\big((y_0,Z_0,g^{-1}Z),(y_0,Z_0,Z)\big) = \tilde{F}_u(L^{1,Z_0}_{u,T})(g^{-1}Z,Z). \tag{5.20}$$

5.5. Rescaling of the variable Z and of the Clifford variables. For $u>0$, let F_u be the linear map

$$h \in \mathbf{H}_{y_0} \to F_u h \in \mathbf{H}_{y_0}; \quad F_u h(Z) = h(Z/u). \tag{5.21}$$

For $u>0$, $T \geqq 0$, set

$$\begin{aligned} L^{2,Z_0}_{u,T} &= F_u^{-1} L^{1,Z_0}_{u,T} F_u, \\ M^{2,Z_0}_u &= F_u^{-1} M^{1,Z_0}_u F_u. \end{aligned} \tag{5.22}$$

Let $e_1,\dots,e_{2l'}$ be an orthonormal basis of $(T_{\mathbb{R}}Y_g)_{y_0}$, let $e_{2l'+1},\dots,e_{2l''}$ be an orthonormal basis of $N_{Y_g/X_g,\mathbb{R},y_0}$, let $e_{2l''+1},\dots,e_{2l}$ be an orthonormal basis of $N_{X_g/X,\mathbb{R},y_0}$. Then $e_1,\dots,e_{2l}$ is an orthonormal basis of $(T_{\mathbb{R}}X)_{y_0}$. Let $e^1,\dots,e^{2l}$ be its dual basis of $(T^*_{\mathbb{R}}X)_{y_0}$. For $U \in (T_{\mathbb{R}}X)_{Z_0}$, let $\tau U^{Z_0}(Z)$ be the parallel transport of U with respect to ∇^{TX} along the curve $t \in [0,1] \to Z_0 + tZ$. For $1 \leqq i \leqq 2l$, put

$$\dot{e}_i = \tau e_i^0(Z_0), \quad \tau^{Z_0}\dot{e}_i(Z_0+Z) = \tau\dot{e}_i^{Z_0}(Z). \tag{5.23}$$

As X_g is totally geodesic along X, it is important to observe that under the considered identification of $(T^*_{\mathbb{R}}X)_{Z_0}$ with $(T^*_{\mathbb{R}}X)_{y_0}$, at $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$ which represents an element of X_g, $\dot{e}_1,\dots,\dot{e}_{2l''}$ (resp. $\dot{e}_{2l''+1},\dots,\dot{e}_{2l}$) is an orthonormal basis of $(T_{\mathbb{R}}X_g)_{Z_0}$ (resp. $(N_{X_g/X,\mathbb{R}})_{Z_0}$).

Definition 5.8. For $u>0$, $T>0$, set

$$\begin{aligned} c_{u,T}(e_j) &= \frac{\sqrt{2}e^j}{u} \wedge -\frac{u}{\sqrt{2}} i_{e_j}, \quad 1 \leqq j \leqq 2l', \\ c_{u,T}(e_j) &= \frac{\sqrt{2}e^j}{uT} \wedge -\frac{uT}{\sqrt{2}} i_{e_j}, \quad 2l'+1 \leqq j \leqq 2l''. \end{aligned} \tag{5.24}$$

Let Op be the set of scalar differential operators acting on smooth functions on $(T_{\mathbb{R}}X)_{y_0}$.

Definition 5.9. For $u>0$, $T>0$, let

$$L^{3,Z_0}_{u,T}, M^{3,Z_0}_{u,T} \in \big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S) \mathbin{\hat{\otimes}} \operatorname{End}\big(\Lambda(T^*_{\mathbb{R}}X_g) \mathbin{\hat{\otimes}} \xi\big) \mathbin{\hat{\otimes}} c(N_{X_g/X,\mathbb{R}})\big)_{y_0} \otimes \mathrm{Op}$$

be the operator obtained from $L^{2,Z_0}_{u,T}, M^{2,Z_0}_u$ by replacing the Clifford variables $c(e_j)$ $(1 \leqq j \leqq 2l'')$ by the operator $c_{u,T}(e_j)$, while leaving unchanged the $c(e_j)$ $(2l''+1 \leqq j \leqq 2l)$.

The complicating fact with respect to [B5], §11.7, is that the $c(e_j)$ $(2l''+1 \leqq j \leqq 2l)$ are not rescaled. However, the rescaling is the same as in [B4], Definition 11.10.

Let $\tilde{F}_u(L^{3,Z_0}_{u,T})(Z,Z')$ $(Z,Z' \in (T_{\mathbb{R}}X)_{y_0}, |Z'| < \varepsilon/2)$ be the smooth kernel associated to $\tilde{F}_u(L^{3,Z_0}_{u,T})$ calculated with respect to $k'(y_0,Z_0)\,dv_{(TX)_{y_0}}(Z')/(2\pi)^{\dim X}$. Note that, at $Z'=0$ (representing (y_0,Z_0)), this last density coincides with $dv_X/(2\pi)^{\dim X}$. Here $\tilde{F}_u(L^{3,Z_0}_{u,T})(Z,Z')$ lies in $\big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\mathrm{End}\big(\Lambda(T^*_{\mathbb{R}}X_g)\,\hat{\otimes}\,\xi\big)\,\hat{\otimes}\,c(N_{X_g/X,\mathbb{R}})\big)_{y_0}$. Moreover g acts naturally on $\big(\Lambda(\overline{N}^*_{X_g/X})\,\hat{\otimes}\,\xi\big)_{y_0}$ as an element of $\big(c(N_{X_g/X,\mathbb{R}})\,\hat{\otimes}\,\mathrm{End}(\xi)\big)_{y_0}$. So $g\tilde{F}_u(L^{3,Z_0}_{u,T})(Z,Z')$ lies in $\big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\mathrm{End}\big(\Lambda(T^*_{\mathbb{R}}X_g)\big)\,\hat{\otimes}\,c(N_{X_g/X,\mathbb{R}})\,\hat{\otimes}\,\mathrm{End}(\xi)\big)_{y_0}$.

Now we use the notation of [B5], (11.57). Namely, $\tilde{F}_u(L^{3,Z_0}_{u,T})(g^{-1}Z,Z)$ can be expanded in the form

(5.25)
$$\tilde{F}_u(L^{3,Z_0}_{u,T})(g^{-1}Z,Z) = \sum_{\substack{1\leqq i_1<\cdots<i_p\leqq 2l''\\ 1\leqq j_1<\cdots<j_q\leqq 2l''}} e^{i_1}\wedge\cdots\wedge e^{i_p}\wedge i_{e_{j_1}}\cdots i_{e_{j_q}}\,\hat{\otimes}\,Q^{i_1\cdots i_p}_{j_1\cdots j_q}(g^{-1}Z,Z),$$
$$Q^{i_1\cdots i_p}_{j_1\cdots j_q}(g^{-1}Z,Z) \in \big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,c(N_{X_g/X,\mathbb{R}})\,\hat{\otimes}\,\mathrm{End}(\xi)\big)_{y_0}.$$

Set

$$[\tilde{F}_u(L^{3,Z_0}_{u,T})(g^{-1}Z,Z)]^{\max} = Q_{1,\ldots,2l''}(g^{-1}Z,Z) \in \big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,c(N_{X_g/X,\mathbb{R}})\,\hat{\otimes}\,\mathrm{End}(\xi)\big)_{y_0}. \tag{5.26}$$

The following theorem extends [B4], Proposition 11.12, [B5], Proposition 11.16:

Proposition 5.10. *If $Z \in N_{X_g/X,\mathbb{R},y_0}$, the following identity holds*:

$$\frac{1}{T^{2\dim N_{Y_g/X_g}}}\,\mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(L^{1,Z_0}_{u,T})(g^{-1}Z,Z)]k(y_0,Z_0,Z) = (-i)^{\dim X_g}\frac{1}{u^{2\dim N_{X_g/X}}}\,\mathrm{Tr}_s\left[gN_{\mathbf{H}}\left[\tilde{F}_u(L^{3,Z_0}_{u,T})\left(\frac{g^{-1}Z}{u},\frac{Z}{u}\right)\right]^{\max}\right]. \tag{5.27}$$

Proof. As in the proof of [B4], Proposition 11.12, note that since g preserves the geodesics and the obvious connections on

$$\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi \simeq \big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi\big)_{y_0},$$

g just acts as the obvious constant linear map on $\big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi\big)_{y_0}$. Since g acts like the identity on $\Lambda(T^{*(0,1)}X_g)$, $g \in c(N_{X_g/X,\mathbb{R}})_{y_0}$. Therefore the rescaling of the Clifford variables in (5.24) has no effect on g. Identity (5.27) is now a trivial consequence of [BL], Proposition 11.2. □

By (5.20), (5.27), we find that for $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon T/2$,

(5.28)

$$\frac{1}{T^{2\dim N_{Y_g/X_g}}}\int_{\substack{|Z|\leqq\varepsilon/8\\ Z\in N_{X_g/X,\mathbb{R},(y_0,Z_0/T)}}}\mathrm{Tr}_s\Big[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T})\Big(g^{-1}\big(y_0,\frac{Z_0}{T},Z\big),\big(y_0,\frac{Z_0}{T},Z\big)\Big)\Big] k\Big(y_0,\frac{Z_0}{T},Z\Big)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}}$$

$$= (-i)^{\dim X_g}\int_{\substack{|Z|\leqq\varepsilon/8u\\ Z\in N_{X_g/X,\mathbb{R},(y_0,Z_0/T)}}}\mathrm{Tr}_s[gN_{\mathbf{H}}[\tilde{F}_u(L^{3,Z_0/T}_{u,T})(g^{-1}Z,Z)]^{\max}]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}}.$$

Let $N^{N_{Y_g/X_g}}$ be the number operator of $\Lambda(N^*_{Y_g/X_g})$. Then $N^{N_{Y_g/X_g}}$ acts naturally on $\Lambda(T^*_{\mathbb{R}}X)|_{W_g}$. For $U\in(T_{\mathbb{R}}X)_{y_0}$, let ∇_U be the standard differential operator acting on smooth functions on $(T_{\mathbb{R}}X)_{y_0}$. Set

$$R'^{\xi}=R^{\xi}+\frac{1}{2}\,\mathrm{Tr}[R^{TX}]. \tag{5.29}$$

Let C be a smooth section of $T^*_{\mathbb{R}}X\,\hat{\otimes}\,\mathrm{End}\big(\pi^*_V\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi\big)$. We use the notation

$$\big(\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}_{\dot{e}_i}+C(\dot{e}_i)\big)^2=\sum_{i=1}^{2l}\big(\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}_{\dot{e}_i}+C(\dot{e}_i)\big)^2 - \nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}_{\sum_{i=1}^{2l}\nabla^{TX}_{\dot{e}_i}\dot{e}_i}-C\Big(\sum_{i=1}^{2l}\nabla^{TX}_{\dot{e}_i}\dot{e}_i\Big).$$

Let

$$R^{TX}_{Z_0}|_{V_g}\in\Lambda^2(TV_g)\otimes\mathrm{End}(TX),\quad R'^{\xi}_{Z_0}|_{V_g}\in\Lambda^2(TV_g)\otimes\mathrm{End}(\xi),$$
$$\big(\nabla^{\xi}V(Z_0)\big)|_{V_g}\in T^*V_g\otimes\mathrm{End}(\xi)$$

be the restrictions of $R^{TX}_{Z_0},R'^{\xi}_{Z_0},\nabla^{\xi}V(Z_0)$ in the direction V_g. By using [B5], Proposition 11.8 and Theorem 11.11, $L^{3,Z_0}_{u,T}$ can be extended by continuity at $u=0$. As in [B5], (11.60)–(11.65), we have the formula,

$$L^{3,Z_0}_{0,T}(Z)=T^{-N^{N_{Y_g/X_g}}}\Big\{-\frac{1}{2}\sum_{i=1}^{2l}\Big(\nabla_{\dot{e}_i}+\frac{1}{2}\langle R^{TX}_{Z_0}|_{V_g}Z,\dot{e}_i\rangle\Big)^2 + R'^{\xi}_{Z_0}|_{V_g}+T\big(\nabla^{\xi}V(Z_0)\big)|_{V_g}+T^2V^2(Z_0)\Big\}T^{N^{N_{Y_g/X_g}}}. \tag{5.30}$$

By [B3], (3.16)–(3.21), one finds easily that

(5.31)

$$(-i)^{\dim X_g}\int_{Z\in N_{X_g/X,\mathbb{R}}}\mathrm{Tr}_s[gN_{\mathbf{H}}[\exp(-L^{3,Z_0/T}_{0,T})(g^{-1}Z,Z)]^{\max}]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}}=\beta_T(y_0,Z_0/T).$$

In view of (5.27), (5.28) and (5.31), Theorem 5.5 follows from the following result.

Theorem 5.11. *There exist* $\gamma \in]0,1]$, $C > 0$ *such that for any* $p \in \mathbb{N}$, *there is* $C_0 > 0$, $r \in \mathbb{N}$, *such that for* $u \in]0,1]$, $T \in [1, 1/u]$, $y_0 \in W_g$, $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon T/2$, $Z, Z' \in (T_{\mathbb{R}}X)_{y_0}$, $|Z|, |Z'| \leqq \varepsilon/8u$, *then*

$$\tag{5.32} \begin{aligned} &\big|\big(\tilde{F}_u(L^{3,Z_0/T}_{u,T}) - \exp(-L^{3,Z_0/T}_{0,T})\big)(Z,Z')\big| \\ &\qquad \leqq C_0(1+|Z_0|)^{-p}(1+|Z|+|Z'|)^{2r}\exp(-C|Z-Z'|^2)\big(u(1+T)\big)^{\gamma}. \end{aligned}$$

Proof. The remainder of the section is devoted to the proof of Theorem 5.11, which is similar to [B4], Theorem 11.13. □

5.6. The matrix structure of the operator $L^{3,Z_0/T}_{u,T}$. As in [B5], §11.8, we calculate the asymptotic expansion of the operator $L^{3,Z_0/T}_{u,T}$ as $u \to 0$. The basic difference is that here, the operators $c(e_j)$ $(2l''+1 \leqq j \leqq 2l)$ are not rescaled. This does not create any difficulty. To the contrary while the rescaled operators $c_{u,T}(e_j)$, $1 \leqq j \leqq 2l''$ are not uniformly bounded as $u \to 0$, the operators $c(e_j)$, $2l''+1 \leqq j \leqq 2l$ remain constant. These operators improve the estimates with respect to [B5], §11.8. In the limit as $u \to 0$, they disappear, as is made clear in equation (5.30).

If $C \in \big(\pi_V^*\Lambda(T^*_{\mathbb{R}}S) \,\hat{\otimes}\, c(T^*_{\mathbb{R}}X) \,\hat{\otimes}\, \mathrm{End}(\xi)\big)_{Z_0+Z}$, let

$$C^{(3)}_{u,T} \in \big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S) \,\hat{\otimes}\, \mathrm{End}\big(\Lambda(T^*_{\mathbb{R}}X_g) \,\hat{\otimes}\, \xi\big) \,\hat{\otimes}\, c(N_{X_g/X,\mathbb{R}})\big)_{y_0}$$

be the operator obtained from C by the trivialization indicated in Section 5.4, and by making the Getzler rescaling in Definition 5.8. By [B5], (11.66), as in [B5], (11.67), we get

$$\tag{5.33} \begin{aligned} L^{3,Z_0/T}_{u,T} &= M^{3,Z_0/T}_{u,T} \\ &+ \rho^2(uZ)\Big\{\Big(T^2V^2 + Tf^{\alpha}\nabla^{\xi}_{f^{H,W}_{\alpha}}V + \sum_{i=1}^{2l} uTc(\tau^{Z_0/T}\dot{e}_i)\nabla^{\xi}_{\tau^{Z_0/T}\dot{e}_i}V\Big)(Z_0/T+uZ)\Big\}^{(3)}_{u,T} \\ &+ T^2\big(1-\rho^2(uZ)\big)P^{\xi^+_{y_0}}. \end{aligned}$$

Comparing with [B5], §11.8, there is an extra term

$$\Big\{\sum_{2l''+1}^{2l} uTc(\tau^{Z_0/T}\dot{e}_i)(\nabla^{\xi}_{\tau^{Z_0/T}\dot{e}_i}V)(Z_0/T+uZ)\Big\}^{(3)}_{u,T},$$

but it does not introduce any extra difficulty, because $uT \leqq 1$.

5.7. A family of Sobolev spaces with weights. Set

$$\tag{5.34} \Lambda^{p,q}(T^*_{\mathbb{R}}X_g)_{y_0} = \Lambda^p(T^*_{\mathbb{R}}Y_g)_{y_0} \,\hat{\otimes}\, \Lambda^q(N^*_{Y_g/X_g,\mathbb{R}})_{y_0}.$$

Let $\mathbf{I}_{y_0}$ be the set of smooth sections of $\big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S) \,\hat{\otimes}\, \Lambda(T^*_{\mathbb{R}}X_g) \,\hat{\otimes}\, \Lambda(\overline{N}^*_{X_g/X}) \,\hat{\otimes}\, \xi\big)_{y_0}$ over $(T_{\mathbb{R}}X)_{y_0}$. Let $\mathbf{I}_{(p,q,r),y_0}$ be the set of smooth sections of

$$\left(\pi_W^* \Lambda^r(T_{\mathbb{R}}^* S) \hat{\otimes} \Lambda^{p,q}(T_{\mathbb{R}}^* X_g) \hat{\otimes} \Lambda(\overline{N}_{X_g/X}^*) \hat{\otimes} \xi\right)_{y_0}$$

over $(T_{\mathbb{R}}X)_{y_0}$. As in [B5], §11.9, we introduce a family of Sobolev spaces with weights. These weights are strictly similar to the corresponding weights in [B5], Definition 11.17. The results contained in [BL], Proposition 11.24–Theorem 11.30, remain valid, essentially because the operator $L_{u,T}^{3,Z_0/T}$ which is considered here has the same structure as in [B5], §11.8.

5.8. Proof of Theorem 5.11. We have the following analogue of [B4], Theorem 11.14, in our context.

Theorem 5.12. *There is $C > 0$ such that for $p \in \mathbb{N}$, $p' \in \mathbb{N}$, there exist $C_0 > 0$, $r \in \mathbb{N}$ such that for any $u \in]0,1]$, $T \in [1, 1/u]$, $y_0 \in W_g$, $Z_0 \in N_{Y_g/X_g,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon T/2$, $Z, Z' \in (T_{\mathbb{R}}X)_{y_0}$, $|Z|, |Z'| \leqq \varepsilon/6u$, then*

$$(5.35) \qquad (1+|Z_0|)^p \sup_{|\alpha_0|,|\alpha_1| \leqq p'} \left| \frac{\partial^{|\alpha_0|+|\alpha_1|}}{\partial Z^{\alpha_0} \partial Z'^{\alpha_1}} \tilde{F}_u(L_{u,T}^{3,Z_0/T})(Z,Z') \right|$$
$$\leqq C_0(1+|Z|+|Z'|)^r \exp(-C|Z-Z'|^2).$$

Proof. At least formally, the problem treated here is the obvious analogue of the problem considered in [B4], §11h), with extra Grassmann variables f^α. One can then proceed formally as in [B4], §11h) and obtain (5.35). As in [B4], §11h), the Sobolev norms in Section 5.7 play a key role in proving the required estimates. Of course, here we deal with the kernel of $\tilde{F}_u(L_{u,T}^{3,Z_0/T})$, while in [B4], §11h), the kernel $\exp(-L_{u,T}^{3,Z_0/T})$ was considered. For $c > 0$, set

$$(5.36) \qquad V_c = \left\{ \lambda \in \mathbb{C}, \operatorname{Re}\lambda \geqq \frac{\operatorname{Im}^2\lambda}{4c^2} - c^2 \right\}.$$

Then $V_c = \{\lambda^2, |\operatorname{Im}\lambda| \leqq c\}$. Now from (5.6), for $m, m' \in \mathbb{N}$, there exists $C_{m,m'} > 0$ such that for $a \in \mathbb{C}$, $|\operatorname{Im} a| \leqq c$,

$$(5.37) \qquad |a|^m |F_u^{(m')}(a)| \leqq C_{m,m'}.$$

So given $k \in \mathbb{N}$, there is a unique holomorphic function $\tilde{F}_{u,k}(\lambda)$ defined on a neighbourhood of V_c such that $\tilde{F}_{u,k}(\lambda) \to 0$ as $\lambda \to +\infty$ and $\tilde{F}_u(\lambda) = \tilde{F}_{u,k}^{(k-1)}(\lambda)/(k-1)!$. Then by (5.37),

$$(5.38) \qquad \sup_{\lambda \in V_c} |\lambda|^m |\tilde{F}_{u,k}^{(m')}(\lambda)| \leqq C_{m,m'}.$$

Then $\tilde{F}_u(L_{u,T}^{3,Z_0/T})$ can be interpreted as a contour integral similar to [BL], (11.117). By the argument in [B4], p. 125, we get (5.35) with $C = 0$.

For $q \in \mathbb{N}^*$, set

$$(5.39) \qquad K_{u,q}(a) = \int_{-\infty}^{+\infty} \exp(it\sqrt{2}a) \exp\left(\frac{-t^2}{2}\right) f(ut) g\left(\frac{t}{q}\right) dt.$$

There is a holomorphic function $\tilde{K}_{u,q}(a)$ such that $K_{u,q}(a) = \tilde{K}_{u,q}(a^2)$. Now as in [B4], (11.53), we find that for any $c > 0$, there exists $C > 0$ such that for $m, m' \in \mathbb{N}$, there exists $C' > 0$ such that for $q \geqq 1$,

$$\sup_{\lambda \in V_c} |\lambda|^m |\tilde{K}_{u,q}^{(m')}(\lambda)| \leqq C' \exp(-Cq^2). \tag{5.40}$$

By proceeding as in [B4], p. 127, we get (5.35) with $C > 0$. □

Observe that $\tilde{F}_0(a) = \exp(-a^2)$. Moreover by [B5], (11.82), for any $p \in \mathbb{N}$, $u \in]0,1]$,

$$\sup_{|\mathrm{Im}(a)| \leqq c} |u|^p |F_u(a) - \exp(-a^2)| \leqq c' \exp\left(\frac{-C}{u^2}\right). \tag{5.41}$$

So by (5.41) and by the analogue of [BL], Theorem 11.36, we get the analogue of [B4], (11.62) for the estimate of a natural norm of $\tilde{F}_u(L_{u,T}^{3,Z_0}) - \exp(-L_{0,T}^{3,Z_0})$. By proceeding as in [B4], §11i), when $u \to 0$,

$$\tilde{F}_u(L_{u,T}^{3,Z_0/T})(Z,Z') \to \exp(-L_{0,T}^{3,Z_0/T})(Z,Z') \tag{5.42}$$

uniformly for Z, Z' in any compact set. By (5.42), (5.35) is also true for $u = 0$. By using again Theorem 5.12, as same as in [B4], §11i), we get Theorem 5.11. □

6. The analysis of the kernel of $g\tilde{F}_u(A_{u,T/u}^2)$ for $T > 0$ as $u \to 0$

The purpose of this section is to prove Theorem 3.10. This section is the obvious extension of [B5], §12, where Theorem 3.10 was established when G is trivial, of [B4], §12, where the case where S is a point was treated.

This section is organized as follows. In Section 6.1, we show that the proof of Theorem 3.10 is local on X. In Section 6.2, we rescale the coordinate Z in $(T_{\mathbb{R}}X)_{y_0}$ and also the Clifford variables. In Section 6.3, we calculate the asymptotics of the operator $L_{u,T/u}^{3,y_0}$ which was obtained from $A_{u,T/u}^2$ by a rescaling. In Section 6.4, we prove Theorem 3.10.

We use the assumptions and notation of Sections 2, 3–5.

6.1. Localization of the problem. By (5.8), and Theorem 5.3, we see that to establish Theorem 3.10, we just need to show that as $u \to 0$,

$$\Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(A_{u,T/u}^2)] \to \int_{Y_g} \Phi\,\mathrm{Tr}_s[gN_{\mathbf{H}} \exp(-\mathscr{B}_{T^2}^2)]\,\mathrm{ch}_g(\eta, h^\eta). \tag{6.1}$$

As in Section 5.2, using finite propagation speed, the proof of Theorem 3.10 has been reduced to a local problem near X_g. As in (5.10),

$$\mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(A_{u,T/u}^2)] = \int_X \mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}_{u,T/u}^2)(g^{-1}x, x)]\,dv_X(x)/(2\pi)^{\dim X}. \tag{6.2}$$

Take $y_0 \in W_g$. If $Z \in (T_{\mathbb{R}}X)_{y_0}$, $|Z| < \varepsilon$, we identify $Z \in (T_{\mathbb{R}}X)_{y_0}$ with $\exp^X_{y_0}(Z) \in X$. Take $u > 0$, if $|Z| < \varepsilon$, we identify $\big(\pi_V^* \Lambda(T_{\mathbb{R}}^* S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\big)_Z, \xi_Z$ to

$$\big(\pi_W^* \Lambda(T_{\mathbb{R}}^* S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\big)_{y_0}, \xi_{y_0}$$

by parallel transport with respect to the connection ${}^2\nabla^{\pi_V^*\Lambda(T_{\mathbb{R}}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X),u}, \tilde{\nabla}^\xi$ along the curve $t \in [0,1] \to tZ$.

If $U \in (T_{\mathbb{R}}X)_{y_0}$, $\tau U(Z) \in (T_{\mathbb{R}}X)_Z$ denotes the parallel transport of U along the curve $t \in [0,1] \to tZ$ with respect to ∇^{TX}.

6.2. Rescaling of the variable Z and of the horizontal Clifford variables. We use the notation of Definition 5.7.

Definition 6.1. For $u > 0$, $T > 0$, $y_0 \in W_g$, set

$$L^{1,y_0}_{u,T/u} = L^{1,0}_{u,T/u}, \quad M^{1,y_0}_u = M^{1,0}_u; \tag{6.3}$$

$$L^{2,y_0}_{u,T/u} = L^{2,0}_{u,T/u}, \quad M^{2,y_0}_u = M^{2,0}_u.$$

Let $\tilde{F}_u(L^{1,y_0}_{u,T/u})(Z,Z')$ $(Z,Z' \in (T_{\mathbb{R}}X)_{y_0})$ be the smooth kernel associated to $\tilde{F}_u(L^{1,y_0}_{u,T/u})$ calculated with respect to $dv_{(TX)_{y_0}}(Z')/(2\pi)^{\dim X}$.

Let $N_{Y_g/X}, N_{Y_g/Y}$ be the normal bundles of Y_g in X, Y. Then we have the holomorphic orthogonal splitting $TY = TY_g \oplus N_{Y_g/Y}$. We identify $N_{Y_g/X}$ to the orthogonal bundle to TY_g in (TX, h^{TX}). Let $k''(y_0,Z)$ be such that for $Z \in N_{Y_g/X,\mathbb{R}}$, $|Z| < \varepsilon$,

$$dv_X(y_0,Z) = k''(y_0,Z)\, dv_{(TX)_{y_0}}(Z). \tag{6.4}$$

Then as (5.20), for $|Z| < \varepsilon$

$$\tilde{F}_u(\tilde{A}^2_{u,T/u})\big((y_0,g^{-1}Z),(y_0,Z)\big)k''(y_0,Z) = \tilde{F}_u(L^{1,y_0}_{u,T/u})(g^{-1}Z,Z). \tag{6.5}$$

Let $e_1,\dots,e_{2l'}$ be an orthonormal basis of $(T_{\mathbb{R}}Y_g)_{y_0}$. Let $e_{2l'+1},\dots,e_{2l'''}$ be an orthonormal basis of $N_{Y_g/Y,\mathbb{R},y_0}$. Let $e_{2l'''+1},\dots,e_{2l}$ be an orthonormal basis of $N_{Y/X,\mathbb{R},y_0}$. Then $e_1,\dots,e_{2l}$ is an orthonormal basis of $(T_{\mathbb{R}}X)_{y_0}$. Let $e^1,\dots,e^{2l'},e^{2l'+1},\dots,e^{2l}$ be the corresponding dual bases of $(T_{\mathbb{R}}^*Y_g)_{y_0}, N^*_{Y_g/X,\mathbb{R},y_0}$.

Definition 6.2. For $u > 0$, set

$$c_u(e_j) = \frac{\sqrt{2}e^j}{u} \wedge -\frac{u}{\sqrt{2}} i_{e_j}, \quad 1 \leqq j \leqq 2l'. \tag{6.6}$$

For $u > 0$, $T > 0$, let

$$L^{3,y_0}_{u,T/u}, M^{3,y_0}_u \in \big(\pi_W^*\Lambda(T_{\mathbb{R}}^*S) \hat{\otimes} \operatorname{End}\big(\Lambda(T_{\mathbb{R}}^*Y_g) \hat{\otimes} \xi\big) \hat{\otimes} c(N_{Y_g/X,\mathbb{R}})\big)_{y_0} \otimes \operatorname{Op}$$

be the operators obtained from $L^{2,y_0}_{u,T/u}, M^{2,y_0}_u$ by replacing the Clifford variables $c(e_j)$ $(1 \leqq j \leqq 2l')$ by the operator $c_u(e_j)$.

Let $\tilde{F}_u(L^{3,y_0}_{u,T/u})(Z,Z')$ $(Z,Z' \in (T_{\mathbb{R}}X)_{y_0})$ be the smooth kernel associated to $\tilde{F}_u(L^{3,y_0}_{u,T/u})$ calculated with respect to $dv_{(TX)_{y_0}}(Z')/(2\pi)^{\dim X}$. We can still expand $\tilde{F}_u(L^{3,y_0}_{u,T/u})(g^{-1}Z,Z)$ as in (5.25), the difference being that in the right hand side of (5.25), l'' is replaced by l', and $N_{X_g/X}$ by $N_{Y_g/X}$. We define

$$[\tilde{F}_u(L^{3,y_0}_{u,T/u})(g^{-1}Z,Z)]^{\max} \in \big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S) \mathbin{\hat{\otimes}} c(N_{Y_g/X,\mathbb{R}}) \mathbin{\hat{\otimes}} \mathrm{End}(\xi)\big)_{y_0}$$

as in (5.26), l'' being replaced by l'.

Also $c(N_{Y_g/X,\mathbb{R}}) \mathbin{\hat{\otimes}} \mathrm{End}(\xi)$ acts on $\big(\Lambda(\overline{N}^*_{Y_g/X}) \mathbin{\hat{\otimes}} \xi\big)_{y_0}$, and so the supertrace of elements in this algebra is well defined.

We now extend [B4], Proposition 12.7, [B5], Proposition 12.4:

Theorem 6.3. *For any $u > 0$, $T > 0$, $y_0 \in W_g$, $Z \in N_{Y_g/X,\mathbb{R},y_0}$, $Z \leqq \frac{\varepsilon}{8u}$, the following identity holds*:

$$u^{2\dim N_{Y_g/X}} \mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})(g^{-1}uZ,uZ)]k''(y_0,uZ) = (-i)^{\dim Y_g}\,\mathrm{Tr}_s[gN_{\mathbf{H}}[\tilde{F}_u(L^{3,y_0}_{u,T/u})(g^{-1}Z,Z)]^{\max}]. \tag{6.7}$$

Proof. Observe that since g acts as the identity on TY_g, applying Getzler rescaling on g does not change g. By using (6.5), our theorem is a trivial consequence of [BL], Proposition 11.2. □

6.3. The asymptotics of the operator $L^{3,y_0}_{u,T/u}$ as $u \to 0$. If

$$C \in \big(\pi_V^*\Lambda(T^*_{\mathbb{R}}S) \mathbin{\hat{\otimes}} c(T^*_{\mathbb{R}}X) \mathbin{\hat{\otimes}} \mathrm{End}(\xi)\big)_{Z},$$

let $C^{(3)}_u \in \big(\pi_V^*\Lambda(T^*_{\mathbb{R}}S) \mathbin{\hat{\otimes}} \mathrm{End}\big(\Lambda(T^*_{\mathbb{R}}Y_g) \mathbin{\hat{\otimes}} \xi\big) \mathbin{\hat{\otimes}} c(N_{Y_g/X,\mathbb{R}})\big)_{y_0}$ be the operator obtained from C by the trivialization indicated in Section 5.4, and by making the Getzler rescaling in Definition 6.2. By as in [B5], (11.67), and (5.33), we get

$$L^{3,y_0}_{u,T/u} = M^{3,y_0}_u + \rho^2(uZ)\bigg\{\bigg(\frac{T^2}{u^2}V^2 + \frac{T}{u}f^\alpha\nabla^\xi_{f^{H,W}_\alpha}V + \sum_{i=1}^{2l} Tc(\tau e_i)\nabla^\xi_{\tau e_i}V\bigg)(uZ)\bigg\}^{(3)}_u + \frac{T^2}{u^2}\big(1-\rho^2(uZ)\big)P^{\xi^+_{y_0}}. \tag{6.8}$$

Let $i_g : W_g \to V$ be the embedding. Then we have the obvious extension of [B5], Theorem 12.6:

Theorem 6.4. *As $u \to 0$,*

$$M^{3,y_0}_u \to M^{3,y_0}_0 = -\frac{1}{2}\sum_{i=1}^{2l}\bigg(\nabla_{e_i} + \frac{1}{2}\langle i_g^*R^{TX}_{y_0}Z, e_i\rangle\bigg)^2 + i_g^*R'^{\xi}_{y_0}. \tag{6.9}$$

Proof. We proceed as in the proof of (5.30). The main difference is that because the Clifford variables $c(e_i)$ $(2l'+1 \leqq i \leqq 2l)$ are not rescaled, they ultimately disappear in the limit. Still, we use [B5], (11.61). □

Then we have an obvious extension of [B5], Theorem 12.7, by replacing i^* there by i_g^*, as we only rescale the Clifford variable $c(e_i)$ $(1 \leqq i \leqq 2l')$.

6.4. Proof of Theorem 3.10. Recall that we reduced the proof of Theorem 3.10 to the proof of (6.1).

We have an identification of smooth vector bundles on W_g

$$TX = TY_g \oplus N_{Y_g/X}, \quad N_{Y_g/X} = N_{Y_g/Y} \oplus N_{Y/X}.$$

Let $P^{N_{Y_g/Y}}$ be the orthogonal projection $TX \to N_{Y_g/Y}$. Let $R^{\Lambda(N^*_{Y/X})}$, R^η be the curvatures of the holomorphic Hermitian connections on $\left(\Lambda(N^*_{Y/X}), h^{\Lambda(N^*_{Y/X})}\right)$, (η, h^η).

We claim that using Theorems 6.4 and the corresponding extension of [B5], Theorem 12.7, the proof of (6.1) is essentially identical to the proof of [B4], Theorem 8.7, given in [B4], §12. By using the arguments of Section 5.8, the obvious analogue of [B4], Theorem 12.11, holds. Namely, we obtain uniform estimates on the kernel $\tilde{F}_u(L^{3,y_0}_{u,T/u})(Z,Z')$ and its derivatives.

Theorem 6.5. *There exists $C > 0$ such that for $p \in \mathbb{N}$, there exist $C' > 0$, $r \in \mathbb{N}$, for which if $u \in]0,1]$, $y_0 \in W_g$, $Z, Z' \in N_{Y_g/X,\mathbb{R},y_0}$, $|Z|, |Z'| \leqq \varepsilon/8u$, then*

$$|\tilde{F}_u(L^{3,y_0}_{u,T/u})(Z,Z')| \leqq C'(1+|P^{N_{Y/X}}Z|)^{-p} (1+|P^{N_{Y_g/Y}}Z|)^r \exp(-C|Z-Z'|^2). \tag{6.10}$$

For $M > 0$, $p' \in \mathbb{N}$, there exists $C'' > 0$ such that for $u \in]0,1]$, $y_0 \in W_g$, $Z, Z' \in (T_\mathbb{R}X)_{y_0}$, $|Z|, |Z'| \leqq M$,

$$\sup_{|\alpha|,|\alpha'| \leqq p'} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \tilde{F}_u(L^{3,y_0}_{u,T/u})(Z,Z') \right| \leqq C''. \tag{6.11}$$

Let $\mathbf{I}^\pm_{y_0}$ be the vector spaces of Section 5.7 which are associated to $\xi^\pm$ instead of ξ. We write the operator $L^{3,y_0}_{u,T/u}$ in matrix form with respect to the splitting $\mathbf{I}_{y_0} = \mathbf{I}^-_{y_0} \oplus \mathbf{I}^+_{y_0}$ so that $L^{3,y_0}_{u,T/u} = \begin{bmatrix} L_{u,1} & L_{u,2} \\ L_{u,1} & L_{u,2} \end{bmatrix}$. Then by Theorems 6.4 and the corresponding extension of [B5], Theorem 12.7, we obtain the analogue of [BL], (12.95). Namely, as $u \to 0$,

$$L_{u,1} \to \mathscr{B}^{2,y_0}_{T^2} - i_g^* R^{\Lambda(N^*_{Y/X})}_{y_0} + i_g^* P^{\xi^-_{y_0}} R^\xi_{y_0} P^{\xi^-_{y_0}}. \tag{6.12}$$

The precise sense in which (6.12) holds is made explicit in [BL]. By the argument in [BL], §12f), the analogue of [BL], Theorem 12.16, holds. Namely for $T > 0$, $y_0 \in W_g$, $\lambda \in U = \{\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) \leqq \delta \operatorname{Im}^2(\lambda) - A\}$, if $A > 0$ is large enough, and if $\delta > 0$ is small enough, as $u \to 0$, in the sense of distributions,

$$(6.13)\qquad (\lambda - L^{3,y_0}_{u,T/u})^{-1} \to P^{\xi^-_{y_0}}(\lambda - \mathscr{B}^{2,y_0}_{T^2} - i_g^* R^{\eta}_{y_0})^{-1} P^{\xi^-_{y_0}}.$$

Let $Q^{y_0}_{T^2}(Z,Z')$ $(Z,Z' \in (T_{\mathbb{R}}X)_{y_0})$ be the smooth kernel associated to $\exp(-\mathscr{B}^{2,y_0}_{T^2})$ with respect to $dv_{TX}(Z')/(2\pi)^{\dim X}$. By (5.41), Theorem 6.5 and (6.11), as in [B4], §12h), we find that as $u \to 0$, uniformly over compact sets in $(T_{\mathbb{R}}X)_{y_0} \times (T_{\mathbb{R}}X)_{y_0}$

$$(6.14)\qquad \tilde{F}_u(L^{3,y_0}_{u,T/u})(Z,Z') \to Q^{y_0}_{T^2}(Z,Z')\exp(-R^{\eta}_{y_0}).$$

We decompose the integral in (6.2) by $\int\limits_{\substack{x\in X\\ d(x,Y_g)\leqq\varepsilon/8}} + \int\limits_{\substack{x\in X\\ d(x,Y_g)\geqq\varepsilon/8}}$. As in Remark 5.5, the first integral converges to (6.1). If we apply the above argument to $W_g = \emptyset$, we get $\xi^- = 0$ and the right hand side of (6.13) is 0. Thus the second integral converges to 0 as $u \to 0$.

The proof of (6.1) is completed. □

7. The analysis of the two parameter operator $g\exp(-A^2_{u,T})$ in the range $u \in]0,1]$, $T \geqq 1/u$

The purpose of this section is to prove Theorem 3.11. This section is the extension of [B5], §13, where Theorem 3.11 established when G is trivial, and of [B4], §13, where it was considered when S is a point.

This section is organized as follows. In Section 7.1, we show that our problem is localized globally near W, and we prove Theorem 3.11 by using Theorem 7.2. In Section 7.2, we construct a coordinate system and a trivialization of $\pi_V^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi$. In Section 7.3, we rescale the coordinate $Z \in (T_{\mathbb{R}}X)_{y_0}$, and we use a Getzler rescaling on certain Clifford variables. The operator $\tilde{A}^2_{u,T/u}$ is then replaced by an operator $\mathscr{L}^{3,y_0}_{u,T}$. In Sections 7.4–7.6, we summarize very briefly the content of key subsections of [B5], §13.6–13.8, and we indicate the difference here. In Section 7.7, we establish key estimates on the kernel of $\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})$, and we prove Theorem 7.2.

We use the notation and assumptions of Sections 2, 3–6.

7.1. The problem is localizable near W. We use the notation of Section 5.2. Recall that $\varepsilon, \alpha > 0$ are constants taken as in Section 5.2. Now we fix $\varepsilon > 0$. The precise value of α will be determined in Section 7.2.

If $y \in Y$, $U \in (T_{\mathbb{R}}Y)_y$, recall that $t \in \mathbb{R} \to y_t = \exp^Y_y(tU) \in Y$ is the geodesic in Y such that $y_0 = y$, $\left.\dfrac{dy}{dt}\right|_{t=0} = U$. If $U' \in N_{Y/X,\mathbb{R},y}$, we still denote by $U' \in N_{Y/X,\mathbb{R},\exp^Y_y(U)}$ the parallel transport of U' with respect to $\nabla^{N_{Y/X}}$ along $t \in [0,1] \to y_t \in Y$.

We have the following extension of [B5], Theorem 13.1:

Theorem 7.1. *There exist $c > 0$, $C > 0$, $\delta \in]0,1]$ such that for $u \in]0,1]$, $T \geqq 1$,*

$$(7.1)\qquad \left|\Phi\operatorname{Tr}_s[gN_{\mathbf{H}}\tilde{G}_u(A^2_{u,T/u})] - \frac{1}{2}\dim N_{Y/X}\Phi\operatorname{Tr}_s[g\tilde{G}_u(B^{W,2}_{u^2})]\right| \leqq \frac{c}{T^{\delta}}\exp\left(\frac{-C}{u^2}\right).$$

Proof. The proof of our theorem is essentially the same as the proof of [B5], Theorem 13.1, as we pointed out in Section 4 that each step in [B5], §9, is G-invariant. □

In view of Theorem 7.1, to prove Theorem 3.11, we only need to show that there exist $C > 0$, $\delta > 0$ such that for $u \in]0,1]$, $T \geqq 1$,

$$\left| \Phi \operatorname{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(A^2_{u,T/u})] - \frac{1}{2}\dim N_{Y/X}\Phi \operatorname{Tr}_s[g\tilde{F}_u(B^{W,2}_{u^2})] \right| \leqq \frac{C}{T^\delta}. \tag{7.2}$$

By (6.2), we will use instead the operator $\tilde{F}_u(\tilde{A}^2_{u,T/u})$. By the results of Section 5.2, we know that $\tilde{F}_u(\tilde{A}^2_{u,T/u})(x,x')$ vanishes for $x' \notin B^X(x,\alpha)$ and only depends on the restriction of $\tilde{A}^2_{u,T/u}$ to $B^X(x,\alpha)$.

Recall that $P^{N_{X_g/X}}, P^{N_{Y_g/Y}}, P^{N_{Y/X}}$ are the orthogonal projections from TX on $N_{X_g/X}, N_{Y_g/Y}, N_{Y/X}$. Let $\tilde{N}$ be the excess normal bundle $\tilde{N} = \dfrac{TX|_{W_g}}{TX_g|_{W_g} + TY|_{W_g}}$ on W_g. We have the exact sequence of holomorphic Hermitian vector bundles on W_g,

$$0 \to N_{Y_g/X_g} \oplus N_{Y_g/Y} \to N_{Y_g/X} \to \tilde{N} \to 0. \tag{7.3}$$

Moreover, N_{Y_g/X_g} and $N_{Y_g/Y}$ are mutually orthogonal in $N_{Y_g/X}$. As usual, we identify $\tilde{N}$ (as a smooth vector bundle) to the orthogonal bundle to $N_{Y_g/X_g} \oplus N_{Y_g/Y}$ in $N_{Y_g/X}$. So we have an identification of smooth vector bundles,

$$N_{Y_g/X} = N_{Y_g/X_g} \oplus N_{Y_g/Y} \oplus \tilde{N}. \tag{7.4}$$

Let $\nabla^{N_{Y_g/X_g}}, \nabla^{N_{Y_g/Y}}, \nabla^{\tilde{N}}$ be the holomorphic Hermitian connections on $N_{Y_g/X_g}, N_{Y_g/Y}, \tilde{N}$.

Take $y_0 \in W_g$. Set

$$\mathscr{U}_\varepsilon = \{(y_0, Z_0) \in N_{Y_g/X,\mathbb{R}}, |P^{N_{Y_g/Y}}Z_0| < \varepsilon, |P^{N_{Y/X}}Z_0| < \varepsilon\}. \tag{7.5}$$

We identify $(y_0, Z_0) \in \mathscr{U}_\varepsilon$ to $\exp^X_{\exp^Y_{y_0}(P^{N_{Y_g/Y}}Z_0)}(P^{N_{Y/X}}Z_0)$.

Let $\kappa(y_0, Z_0)$ $((y_0, Z_0) \in N_{Y_g/X,\mathbb{R}})$, $\kappa'(y_0, Z'_0)$ $((y_0, Z'_0) \in N_{Y_g/Y,\mathbb{R}})$ be the smooth functions defined by

$$\begin{aligned} dv_X(y_0, Z_0) &= \kappa(y_0, Z_0)\, dv_{Y_g}(y_0) dv_{N_{Y_g/X}}(Z_0), \\ dv_Y(y_0, Z'_0) &= \kappa'(y_0, Z'_0)\, dv_{Y_g}(y_0)\, dv_{N_{Y_g/Y}}(Z'_0). \end{aligned} \tag{7.6}$$

Then $\kappa(y_0, 0) = \kappa'(y_0, 0) = 1$, and κ' is the restriction of κ to $N_{Y_g/Y,\mathbb{R}}$. Let $\tilde{F}_u(B^{W,2}_{u^2})(y, y')$ $(y, y' \in Y)$ be the smooth kernel associated to $\tilde{F}_u(B^{W,2}_{u^2})$ with respect to the volume element $dv_Y(y')/(2\pi)^{\dim Y}$. Clearly,

$$\int_{\mathscr{U}_{\varepsilon/8}} \operatorname{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})(g^{-1}x,x)]\frac{dv_X(x)}{(2\pi)^{\dim X}} \tag{7.7}$$

$$= \left(\frac{1}{2\pi}\right)^{\dim X}\int_{Y_g} dv_{Y_g}(y_0) \int_{\substack{Z_0\in N_{Y_g/X,\mathbb{R}}\\ |P^{N_{Y_g/Y}}Z_0|<\varepsilon/8u\\ |P^{N_{Y/X}}Z_0|<\varepsilon\sqrt{T}/8u}} \frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}}$$

$$\operatorname{Tr}_s\left[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})\left(g^{-1}\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right),\right.\right.$$

$$\left.\left.\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right)\right)\right]$$

$$\kappa\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right)dv_{N_{Y_g/X}}(Z_0).$$

Now we state an extension of [B4], Theorem 13.6, [B5], Theorem 13.2.

Theorem 7.2. *If ε, α are small enough, for any $p\in\mathbb{N}$, there exists $C>0$ such that for $u\in\,]0,1]$, $T\geqq 1$, $y_0\in W_g$, $Z_0\in(N_{Y_g/X,\mathbb{R}})_{y_0}$, $|P^{N_{Y/X}}Z_0|\leqq\dfrac{\varepsilon\sqrt{T}}{8u}$, $|P^{N_{Y_g/Y}}Z_0|\leqq\dfrac{\varepsilon}{8u}$, then*

$$\frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}}\left|\operatorname{Tr}_s\left[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})\left(g^{-1}\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right),\right.\right.\right. \tag{7.8}$$

$$\left.\left.\left.\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right)\right)\right]\right|$$

$$\leqq C'(1+|P^{N_{Y_g/X_g}}Z_0|)^{-p}\exp(-C|P^{N_{X_g/X}}Z_0|^2).$$

There exist $C''>0$, $\delta'\in\,]0,1/2]$ such that under the same conditions as before, we have

(7.9)

$$\left|\left(\frac{1}{2\pi}\right)^{\dim X}\frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}}\operatorname{Tr}_s\left[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})\left(g^{-1}\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right),\right.\right.\right.$$

$$\left.\left.\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right)\right)\right]\kappa\left(y_0, uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right)$$

$$-\,u^{\dim N_{Y_g/Y}}\frac{\exp(-|P^{N_{Y/X}}Z_0|^2)}{\pi^{\dim N_{Y/X}}}\frac{\dim N_{Y/X}}{2}\left(\frac{1}{2\pi}\right)^{\dim Y}$$

$$\left.\operatorname{Tr}_s[g\tilde{F}_u(B^{W,2}_{u^2})(g^{-1}(y_0,uP^{N_{Y_g/Y}}Z_0),(y_0,uP^{N_{Y_g/Y}}Z_0))]\kappa'(y_0,uP^{N_{Y_g/Y}}Z_0)\right|$$

$$\leqq\frac{C''}{T^{\delta'}}.$$

Proof. The remainder of this section is devoted to the proof of Theorem 7.2. □

Proof of Theorem 3.11. Using (7.7) and Theorem 7.2, it is clear that there exists $C > 0$ such that for $u \in]0,1]$, $T \geqq 1$,

$$\left| \int_{\mathscr{U}_{\varepsilon/8}} \mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})(g^{-1}x,x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} - \frac{\dim N_{Y/X}}{2} \int_{\mathscr{U}_{\varepsilon/8}\cap W} \mathrm{Tr}_s[g\tilde{F}_u(B^{W,2}_{u^2})(g^{-1}y,y)] \frac{dv_Y(y)}{(2\pi)^{\dim Y}} \right| \leqq \frac{C}{T^{\delta'/2}}. \tag{7.10}$$

As before, the integrals in (7.10) are the integrals along the fibre on S. Observe that for $y \in W$, if $y \notin \mathscr{U}_{\varepsilon/8} \cap W$, then $d^Y(g^{-1}y, y) \geqq \alpha$. But by using again finite propagation speed, it is clear that

$$g\tilde{F}_u(B^{W,2}_{u^2})(g^{-1}y,y) = 0 \quad \text{if } d^Y(g^{-1}y,y) \geqq \alpha. \tag{7.11}$$

By applying Theorem 7.2 to the case where $Y = \emptyset$, we find that

$$\left| \int_{V\setminus\mathscr{U}_{\varepsilon/8}} \mathrm{Tr}_s[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})(g^{-1}x,x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \right| \leqq \frac{C}{T^{\delta'/2}}. \tag{7.12}$$

By (6.2), (7.10), (7.11), (7.12), we get (7.2). The proof of Theorem 3.11 is completed. □

7.2. A local coordinate system near $y_0 \in W_g$ and a trivialization of $\pi_V^*\Lambda(T^*_{\mathbb{R}}S) \hat{\otimes} \Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$. In [B5], Definition 13.4, by parallel transport along the geodesics normal to Y with respect to the connection ∇^{TX}, from the smooth splitting $TX|_W = TY \oplus N_{Y/X}$ on W, we get a smooth orthogonal splitting $TX = TX^1 \oplus TX^2$ near W. Let P^{TX^1}, P^{TX^2} be the orthogonal projections from TX on TX^1, TX^2.

Also a connection ${}^0\nabla^{TX} = \nabla^{TX^1} \oplus \nabla^{TX^2}$ on $TX = TX^1 \oplus TX^2$ is constructed in [B5], §13.2, by projecting orthogonally ∇^{TX} on TX^1, TX^2. On W, $\nabla^{TX^1}, \nabla^{TX^2}$ restrict to the holomorphic Hermitian connections $\nabla^{TY}, \nabla^{N_{Y/X}}$ on $(TY, h^{TY}), (N_{Y/X}, h^{N_{Y/X}})$. For details, we refer to [B5], §13.2.

Take $y_0 \in W_g$. Recall that Y_g is totally geodesic in Y. So if $Z'' \in (T_{\mathbb{R}}Y_g)_{y_0}$, then $t \to y_t = \exp^Y_{y_0}(tZ'') \in Y_g$ is the geodesic in Y_g such that $y|_{t=0} = y_0$, $\left.\frac{dy}{dt}\right|_{t=0} = Z''$.

If $Z'' \in (T_{\mathbb{R}}Y_g)_{y_0}$, $Z'_0 \in N_{Y_g/X,\mathbb{R},y_0}$, we still denote by $Z'_0 \in N_{Y_g/X,\mathbb{R},\exp^Y_{y_0}(Z'')}$ the parallel transport of Z'_0 along the curve $t \in [0,1] \to \exp^Y_{y_0}(tZ'')$ with respect to the connection $\nabla^{N_{Y_g/X_g}} \oplus \nabla^{N_{Y_g/Y}} \oplus \nabla^{\tilde{N}}$.

If $y \in W$, $Z \in (T_{\mathbb{R}}Y)_y$, $Z' \in N_{Y/X,\mathbb{R},y}$, we still denote by $Z' \in N_{Y/X,\mathbb{R},\exp^Y_y(Z)}$ the parallel transport of Z' with respect to $\nabla^{N_{Y/X}}$ along the curve $t \in [0,1] \to \exp^Y_y(tZ)$.

Ultimately, if $Z \in (T_{\mathbb{R}}X)_{y_0}$, $|Z| < \varepsilon$, we identify Z to

$$\exp^X_{\exp^Y_{\exp^Y_{y_0}(P^{TY_g}Z)}(P^{N_{Y_g/Y}}Z)}(P^{N_{Y/X}}Z) \in X.$$

Let $\mathscr{W}_\varepsilon(y_0)$ be the open neighbourhood of y_0 in X, given by

$$\mathscr{W}_\varepsilon(y_0) = \{Z \in (T_\mathbb{R}X)_{y_0}, |P^{TY}Z| < \varepsilon, |P^{N_{Y/X}}Z| \leqq \varepsilon\}.$$

Clearly, there exists $\alpha_0(\varepsilon) > 0$ such that for $y_0 \in W_g$, $Z_0 \in N_{Y/X,\mathbb{R},y_0}$, $|Z_0| < \varepsilon/8$, the open Riemannian ball in X, $B^X(Z_0, \alpha_0(\varepsilon))$, is contained in $\mathscr{W}_{\varepsilon/2}(y_0)$. In particular, $0 < \alpha_0(\varepsilon) \leqq \varepsilon/2 \leqq a^Y/4$. We fix $\alpha \in \left]0, \inf(\alpha_0(\varepsilon), \varepsilon/8)\right]$ small enough.

Let $\kappa''(Z_0)$, $Z_0 \in (T_\mathbb{R}X)_{y_0}$, $|Z_0| < \varepsilon$, $\kappa'''(Z_0')$, $Z_0' \in (T_\mathbb{R}Y)_{y_0}$, $|Z_0'| < \varepsilon$ be the functions defined by

$$\begin{aligned} dv_X(Z_0) &= \kappa''(Z_0)\, dv_{TX}(Z_0), \\ dv_Y(Z_0') &= \kappa'''(Z_0')\, dv_{TY}(Z_0'). \end{aligned} \tag{7.13}$$

Then by (7.6), (7.13), one easily verifies that if $Z_0 \in N_{Y_g/X,\mathbb{R},y_0}$, $Z_0' \in N_{Y_g/Y,\mathbb{R},y_0}$,

$$\kappa''(Z_0) = \kappa(y_0, Z_0), \quad \kappa'''(Z_0') = \kappa'(y_0, Z_0'). \tag{7.14}$$

As in Section 6.2, let $e_1, \dots, e_{2l'}, e_{2l'+1}, \dots, e_{2l'''}$ and $e_{2l'''+1}, \dots, e_{2l}$ be orthonormal bases of $(T_\mathbb{R}Y_g)_{y_0}, N_{Y_g/Y,\mathbb{R},y_0}$ and $N_{Y/X,\mathbb{R},y_0}$. Then $e_1, \dots, e_{2l}$ is an orthonormal basis of $(T_\mathbb{R}X)_{y_0}$. Let ${}^3\nabla^{\pi_V^*\Lambda(T_\mathbb{R}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}$ be the connection on $\pi_V^*\Lambda(T_\mathbb{R}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)$ along the fibres X over $\mathscr{U}_\varepsilon$ defined in [B5], Definition 13.5. Put (cf. (5.16))

$${}^3\nabla^{\pi_V^*\Lambda(T_\mathbb{R}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X),u} = \psi_u\, {}^3\nabla^{\pi_V^*\Lambda(T_\mathbb{R}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X)}\psi_u^{-1}. \tag{7.15}$$

Take $u \in]0,1]$. If $Z \in (T_\mathbb{R}X)_{y_0}$, we identify $\left(\pi_V^*\Lambda(T_\mathbb{R}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\right)_Z$ (resp. ξ_Z) to $\left(\pi_V^*\Lambda(T_\mathbb{R}^*S) \hat{\otimes} \Lambda(T^{*(0,1)}X)\right)_{y_0}$ (resp. ξ_{y_0}) by parallel transport with respect to the connection ${}^3\nabla^{\pi_V^*\Lambda(T_\mathbb{R}^*S)\hat{\otimes}\Lambda(T^{*(0,1)}X),u}$ (resp. $\tilde{\nabla}^\xi$) along the path

$$\begin{aligned} t \in [0,3] \to tP^{TY_g}Z, &\quad 0 \leqq t \leqq 1; \\ P^{TY_g}Z + (t-1)P^{N_{Y_g/Y}}Z, &\quad 1 \leqq t \leqq 2; \\ P^{TY}Z + (t-2)P^{N_{Y/X}}Z, &\quad 2 \leqq t \leqq 3. \end{aligned} \tag{7.16}$$

Remark that by [B5], §13.2, for $2 \leqq t \leqq 3$, parallel transport with respect to ${}^0\nabla^{TX}$ coincides with parallel transport with respect to ∇^{TX}.

If $U \in (T_\mathbb{R}X)_{y_0}$, $Z \in \mathscr{W}_\varepsilon(y_0)$, let ${}^0\tau U(Z)$ be the parallel transport of U along the curve (7.16) with respect to ${}^0\nabla^{TX}$.

Let $\mathscr{L}_{u,T}^{1,y_0}, \mathscr{M}_{u,T}^{1,y_0}$ be the operators acting on $\mathbf{H}_{y_0}$ defined in [B5], Definition 13.7. Let $\tilde{F}_u(\mathscr{L}_{u,T}^{1,y_0})(Z,Z')$ $(Z,Z' \in (T_\mathbb{R}X)_{y_0})$ be the smooth kernel associated to $\tilde{F}_u(\mathscr{L}_{u,T}^{1,y_0})$, calculated with respect to $dv_{TX}(Z')/(2\pi)^{\dim X}$. By using finite propagation speed, it is clear that if $Z_0 \in N_{Y_g/X,\mathbb{R},y_0}$, $|Z_0| \leqq \varepsilon/8$, as in (5.20),

$$(7.17) \qquad \operatorname{Tr}_s\big[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})\big(g^{-1}(y_0,Z_0),(y_0,Z_0)\big)\big]\kappa''(Z_0) = \operatorname{Tr}_s\big[gN_{\mathbf{H}}\tilde{F}_u(\mathscr{L}^{1,y_0}_{u,T})\big(g^{-1}(y_0,Z_0),(y_0,Z_0)\big)\big].$$

7.3. Rescaling of the variable *Z* and of the Clifford variables. For $u>0$, $T>0$, let $G_{u,T}$ be the linear map $h\in\mathbf{H}_{y_0}\to G_{u,T}h\in\mathbf{H}_{y_0}$ such that if $Z\in(T_{\mathbb{R}}X)_{y_0}$,

$$(7.18) \qquad G_{u,T}h(Z) = h\left(\frac{P^{TY}Z}{u}+\frac{\sqrt{T}}{u}P^{N_{Y/X}}Z\right).$$

Set

$$(7.19) \qquad \mathscr{L}^{2,y_0}_{u,T} = G^{-1}_{u,T}\mathscr{L}^{1,y_0}_{u,T}G_{u,T}, \qquad \mathscr{M}^{2,y_0}_{u,T} = G^{-1}_{u,T}\mathscr{M}^{1,y_0}_{u}G_{u,T}.$$

Definition 7.3. For $u>0$, $T>0$, let

$$\mathscr{L}^{3,y_0}_{u,T}, \mathscr{M}^{3,y_0}_{u,T} \in \big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\operatorname{End}\big(\Lambda(T^*_{\mathbb{R}}Y_g)\,\hat{\otimes}\,\xi\big)\,\hat{\otimes}\,c(N_{Y_g/X,\mathbb{R}})\big)_{y_0}\otimes\operatorname{Op}$$

be the operator obtained from $\mathscr{L}^{2,y_0}_{u,T}, \mathscr{M}^{2,y_0}_{u,T}$ by replacing the Clifford variables $c(e_j)$ $(1\leqq j\leqq 2l')$ by the operators $c_u(e_j)$ in Definition 6.2, while leaving unchanged the $c(e_j)$ $(2l'+1\leqq j\leqq 2l)$.

Let $\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z,Z')$ $(Z,Z'\in(T_{\mathbb{R}}X)_{y_0})$ be the smooth kernel associated to $\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})$ calculated with respect to $dv_{(TX)_{y_0}}(Z')/(2\pi)^{\dim X}$. We still define $[\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(g^{-1}Z,Z)]^{\max}$ as in Section 6.2.

Proposition 7.4. *For any $u>0$, $T>0$, $y_0\in W_g$, $Z_0\in N_{Y_g/X,\mathbb{R},y_0}$, $|P^{TY}Z_0|\leqq\varepsilon/8u$, $|P^{N_{Y/X}}Z|\leqq\varepsilon\sqrt{T}/8u$, the following identity holds*:

$$(7.20) \qquad \frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}}\operatorname{Tr}_s\left[gN_{\mathbf{H}}\tilde{F}_u(\tilde{A}^2_{u,T/u})\left(g^{-1}\left(y_0,uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right),\left(y_0,uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right)\right)\right]\kappa''\left(uP^{N_{Y_g/Y}}Z_0+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right) = (-i)^{\dim Y_g}\operatorname{Tr}_s[gN_{\mathbf{H}}[\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(g^{-1}Z_0,Z_0)]^{\max}].$$

Proof. Since g preserves the geodesics in X and Y and also the connections on the vector bundles considered before, it is clear that g acts linearly in the coordinate Z_0. Observe also that since g acts as identity on TY_g, applying Getzler rescaling on g does not change g. Our theorem is now a trivial consequence of [BL], Proposition 13.17. □

7.4. A formula for $\mathscr{L}^{3,y_0}_{u,T}$. If $C\in\big(\pi_V^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,c(T^*_{\mathbb{R}}X)\,\hat{\otimes}\,\operatorname{End}(\xi)\big)_Z$ $(Z\in(T_{\mathbb{R}}X)_{y_0})$, let $C^3_u\in\big(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\,\hat{\otimes}\,\operatorname{End}\big(\Lambda(T^*_{\mathbb{R}}Y_g)\,\hat{\otimes}\,\xi\big)\,\hat{\otimes}\,c(N_{Y_g/X,\mathbb{R}})\big)_{y_0}$ be the operator obtained from C by the trivialization indicated in Section 7.2, and be making the Getzler rescaling in Definition 7.3.

The discussion in [B5], §13.6, applies with minor modifications. The main difference is that the Clifford variables $c(e_i)$ $(2l'+1 \leqq i \leqq 2l''')$ are not rescaled, while they are rescaled in [B5], §13. However this just introduces fewer diverging terms than in [B5], §13. In particular, [B5], Theorem 13.10, for $\mathscr{M}^{2,y_0}_{u,T}$ still holds. We should modify the second equation of [B5], (13.41), as following:

(7.21)

$$\begin{aligned} T\sum_{i=2l'+1}^{2l}\left\{\frac{c({}^0\tau e_i)}{\sqrt{2}}\right\}_u^3 \nabla^\xi_{{}^0\tau e_i} V\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N_{Y/X}}Z\right) \\ = T\sum_{i=2l'''+1}^{2l}\frac{c(e_i)}{\sqrt{2}}(\nabla^\xi_{{}^0\tau e_i}V)(uP^{TY}Z)+u\sqrt{T}\sum_{i=2l'''+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\tilde{\nabla}^\xi_{P^{N_{Y/X}}Z}\nabla^\xi_{{}^0\tau e_i}V(uP^{TY}Z) \\ + T\sum_{i=2l'+1}^{2l'''}\left\{\frac{c({}^0\tau e_i)}{\sqrt{2}}\right\}_u^3(\nabla^\xi_{{}^0\tau e_i}V)(uP^{TY}Z) \\ + u\sqrt{T}\sum_{i=2l'+1}^{2l'''}\left\{\frac{c({}^0\tau e_i)}{\sqrt{2}}\right\}_u^3\tilde{\nabla}^\xi_{P^{N_{Y/X}}Z}\nabla^\xi_{{}^0\tau e_i}V(uP^{TY}Z)+\mathscr{O}(u|P^{N_{Y/X}}Z|^2). \end{aligned}$$

After this modification, the analogues of [B5], Theorem 13.11, still holds.

7.5. The algebraic structure of the operator $\mathscr{L}^{3,y_0}_{u,T}$ as $u\to 0$. By replacing i^* by i_g^* in the limits, the analogue of [B5], §13.7, still holds. In particular, by (4.3), as in [B5], (13.63), as $u\to 0$,

$$\begin{aligned} (7.22)\qquad & \{u^2\psi_u{}^3\nabla^{\pi_V^*\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^{*(0,1)}X),2}(Z,e_i)\psi_u^{-1}\}_u^3 \\ & \to \langle i_g^* R^{TX}_{y_0}Z,e_i\rangle-\langle i_g^* A^2_{y_0}P^{TY}Z,P^{TY}e_i\rangle. \end{aligned}$$

Here A_{y_0} is the second fundamental form of $TY\subset TX$ as in [B5], (1.32). Thus as $u\to 0$, the operator $\mathscr{M}^{3,y_0}_{u,T}$ converges to an operator $\mathscr{M}^{3,y_0}_{0,T}$ as in [B5], (13.64).

7.6. The algebraic structure of the operator $\mathscr{L}^{3,y_0}_{u,T}$ as $T\to+\infty$. For a fixed $u>0$, the analysis of the matrix structure of $\mathscr{L}^{3,y_0}_{u,T}$ as $T\to+\infty$ is the same as in [B5], §13.8. Of course the rescaling on the Clifford variables, which depends on $u>0$, is different, but again, this improves the situation, since there are fewer diverging terms. In particular, the matrix structure of the operator is unchanged with respect to [B5], Theorem 13.14.

We still define the function $g_{u,T}(Z),\tilde{g}_u(U)$ as in [B5], Definition 13.18. The algebra $\left(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^*_{\mathbb{R}}Y_g)\right)_{y_0}$ splits into

$$\begin{aligned} (7.23)\qquad \left(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^*_{\mathbb{R}}Y_g)\right)_{y_0} &= \bigoplus_r\Big(\bigoplus_{p+q=r}\left(\pi_W^*\Lambda^p(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda^q(T^*_{\mathbb{R}}Y_g)\right)_{y_0}\Big) \\ &= \bigoplus_r\left(\pi_W^*\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^*_{\mathbb{R}}Y_g)\right)^r_{y_0}. \end{aligned}$$

Then we introduce the obvious modification of the system of norms $|\ |_{u,T,y_0,j}$, $j=-1,0,1$ of [B5], Definitions 13.19 and 13.20, adapted to the splitting (7.23).

In view of Sections 7.4, 7.5, it should now be clear that the functional analytic arguments of [BL], §13k)–13o), can be used without any change. In particular, we choose $T_0 \geqq 1$ as in [BL], Theorem 13.27.

7.7. Uniform estimates of the kernel of $\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})$. We now state an extension of [B4], Theorem 13.14:

Theorem 7.5. *There exists $C > 0$ such that for any $p \in \mathbb{N}$, there exists $C' > 0$ such that if $u \in]0,1]$, $T \geqq T_0$, $y_0 \in W_g$, $Z_0, Z'_0 \in (T_{\mathbb{R}}X)_{y_0}$, $|P^{N_{Y/X}}Z_0|, |P^{N_{Y/X}}Z'_0| \leqq \varepsilon\sqrt{T}/4u$, $|P^{TY}Z_0|, |P^{TY}Z'_0| \leqq \varepsilon/4u$, then*

$$
(7.24)\qquad |\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z_0,Z'_0)| \leqq C'(1+|P^{N_{Y/X}}Z_0|)^{-p} \times (1+|P^{TY}Z_0|)^{2l}\exp(-C|Z_0-Z'_0|^2).
$$

There exists $C > 0$ such that if $p' \in \mathbb{N}$, there exists $C' > 0$ such that if $|\alpha|, |\alpha'| \leqq p'$, $u \in]0,1]$, $T \geqq T_0$, $y_0 \in W_g$, $Z_0, Z'_0 \in (T_{\mathbb{R}}X)_{y_0}$,

$$
(7.25)\qquad \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_0^{\alpha}\partial Z_0'^{\alpha'}}\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z_0,Z'_0)\right| \leqq C'(1+|Z_0|)^{2l}\exp(-C|Z_0-Z'_0|^2).
$$

Proof. The bounds in (7.24), (7.25) with $C = 0$ are easily obtained by proceeding as in the proof of [BL], Theorem 13.32. To get the required $C > 0$, we proceed as in the proof of Theorem 5.12. Using finite propagation speed, we see that there is $C'' > 0$ such that if $|Z_0 - Z'_0| \geqq C''q$, then

$$
(7.26)\qquad \tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z_0,Z'_0) = \tilde{K}_{u,q}(\mathscr{L}^{3,y_0}_{u,T})(Z_0,Z'_0).
$$

By (5.40), as in Section 5.8 and [B4], §11h), we get (7.24), (7.25). □

Let $\Xi^{y_0}_u$ be the analogue of the elliptic second order differential operator considered in [B5], Definition 13.21.

Let $\nabla^{\pi^*_W\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^{*(0,1)}Y)}$ be the connection on $\pi^*_W\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^{*(0,1)}Y)$ along the fibre Y defined in [B5], (11.32), for the fibration $\pi_W : W \to S$.

If $U \in B^{TY}_{y_0}(0,\varepsilon)$, we identify

$$
\big(\pi^*_W\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^{*(0,1)}Y)\big)_U, \eta_U \quad \text{with } \big(\pi^*_W\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^{*(0,1)}Y)\big)_{y_0}, \eta_{y_0}
$$

by parallel transport with respect to the connection $\psi_u\nabla^{\pi^*_W\Lambda(T^*_{\mathbb{R}}S)\hat{\otimes}\Lambda(T^{*(0,1)}Y)}\psi_u^{-1}$, ∇^{η} along the path

$$
(7.27)\qquad t \in [0,2] \to tP^{TY_g}Z, \quad 0 \leqq t \leqq 1; \qquad P^{TY_g}Z + (t-1)P^{N_{Y_g/Y}}Z, \quad 1 \leqq t \leqq 2.
$$

Let Σ_u^{3,y_0} be the analogue of the operator considered in [B5], §13.11. The minor difference with [B5], §13.11, is that here, only the Clifford variables $c(e_i)$ $(1 \leqq i \leqq 2l')$ are rescaled, while in [B5], §13.11, the Clifford variables $c(e_i)$ $(1 \leqq i \leqq 2l''')$ were rescaled (cf. [B4], §13j)). Now we have the obvious extension of [B5], Theorem 13.22:

Theorem 7.6. *Over $B_{y_0}^{TY}(0, \varepsilon/2u)$, the following identity holds*:

$$\Sigma_u^{3,y_0} = \Xi_u^{y_0}. \tag{7.28}$$

Again (4.3) plays an important role to get (7.28) (cf. [B5], p. 233). Using Theorem 7.5 and (7.28), and proceeding as in [BL], §13q), we get Theorem 7.2. □

8. A proof of Theorem 0.2

The proof of Theorem 0.2 is the same as the proof of [B5], Theorem 0.2. We just need to add g at each step in [B5], §14.

This concludes the proof of Theorem 0.2.

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Université Paris-Sud, Département de Mathématique, Bâtiment 425, 91405 Orsay Cedex, France
e-mail: Jean-Michel.Bismut@math.u-psud.fr

Centre de Mathématiques, UMR 7640 du CNRS, École Polytechnique, 91128 Palaiseau Cedex, France
e-mail: ma@math.polytechnique.fr

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