It is a pleasure and an honor to be able to write this note on the occasion of a conference celebrating 110 years since Chern's birth. And it is a special pleasure to do this for an event at Nankai University, which is Chern's alma mater and is where some fifty years ago I first visited to give a lecture series on exterior differential systems, a subject whose basics I learned from Chern. I will first make a few general observations and then, as Chern would have done, I will discuss a little mathematics.

Chern was a singular figure in 20th century geometry and indeed in the overall mathematics community. Through his research work, personal interactions and collaborations with students and colleagues and service to our community, his contributions to and impact on our subject are unexcelled.

Many of the areas in which he worked bear witness to his contributions by the names attached to them. Chern classes, which he created as a subject bridging differential geometry and topology, are ubiquitous in mathematics. Evidence of their fundamental nature is provided by the appearance of Chern classes in many diverse areas, including algebraic geometry where they are the central topological invariants, number theory, K-theory and of course topology to name a few. An important point in his original paper defining Chern classes was that one representation of them is by polynomials in the curvature matrices of Hermitian vector bundles. In so doing Chern basically initiated the subject of complex differential geometry: a Hermitian metric in a general holomorphic vector bundle, not just the tangent bundle, has a unique natural Chern connection, a phenomenon not present in the real case. Moreover, as he frequently emphasized the sign properties of the curvature have profound implications in algebraic geometry and complex function theory.

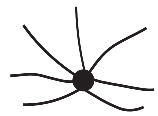
There are many more well-known areas I could mention, some of which bear his name such as Chern-Simons invariants (which have significant applications in physics), and Bott-Chern classes (initiated to study complex function theory). However I want to briefly describe an area in which Chern worked that is perhaps less well known. This is the subject of *webs*, which also has special meaning to me as it is the area in which he and I collaborated and wrote joint papers. Although we were not able to prove the main result we wanted, as is characteristic of Chern's work the paper stimulated considerable interest and the subsequent work by others led not only to a proof of the result we were after but also to results that I, and perhaps also Chern, did not even dream about.

Before turning to the mathematics, a bit of history. When Chern left China he went to Hamburg (Germany), which at that time had a flourishing mathematics community, one on at least a par in Germany with the other leading center at Göttingen. Among the prominent mathematicians in Hamburg was Wilhelm Blaschke, a geometer-analyst of great originality. One of the areas that he basically created was web geometry, and it was from Blaschke and the people around him such as Bol that Chern learned about webs and about which he wrote his thesis solving a problem posed by Blaschke. This work exhibited a number of characteristics that were to be prevalent in Chern's research. One is that the objective was to solve a specific problem, not to further develop a general theory. A second was that significant and subtle computations were required; however these computations had a conceptual framework, one that was furthered through the computations. Chern returned to the subject in the 1970s and this was when we did our joint work.

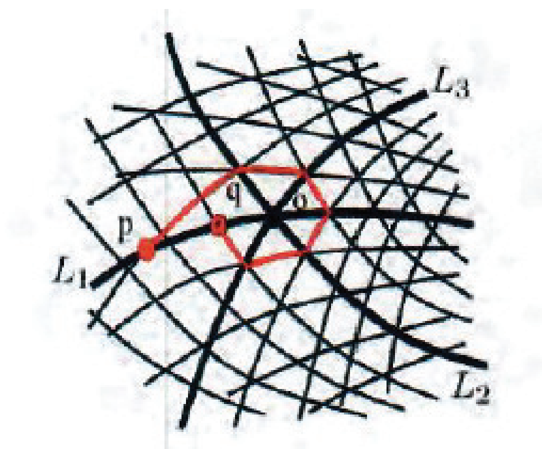
So what is a web and why are they interesting?[1] We will work in a small open set $U \subset \mathbb{C}^n$ and all data will be holomorphic. A *d-web* is given by a set $\mathcal{W} = \{W_1, \ldots, W_d\}$ of hypersurface foliations of U. It is assumed that these foliations are in general position. The real picture

[1]Web geometry is a branch of local differential geometry. As will be seen it has deep connections with algebraic geometry and, although we will not discuss it, also with certain special functions. For a reference about web geometry and a guide to both the literature and its history see "An invitation to web geometry" by Jorge Vitório Pereira and Luc Pirio, arXiv:1107.0595vi, 4 Feb 2011.

of a 4-web in $\mathbb{C}^2$ is through each point we have something like



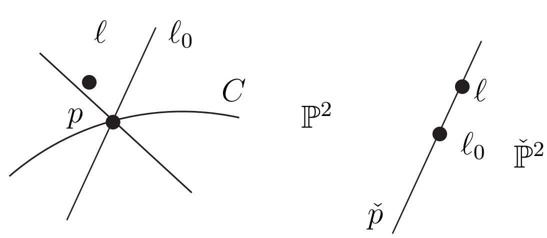
In general we shall restrict to the case $n = 2$ so we can draw pictures. The web is *linearizable* if there is a change of coordinates in U so that the W_i are (parts of) hyperplanes. For $d \leqq n$ any web is linearizable, but this is not the case for $d \geqq n+1$. For $n = 2, d = 3$ there is a web curvature given the map $\mathbf{q} \to \mathbf{p}$ in the picture



and that vanishes if, and only if, $\mathbf{p} = \mathbf{q}$.[2] This is the necessary and sufficient condition that the web be linearizable.

An arc C in $\mathbb{C}^2$ defines a web in the dual projective plane $\check{P}^2$ by the pictures

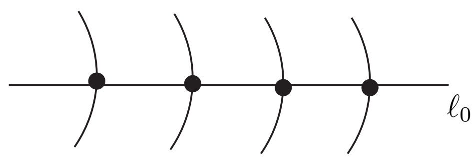
(1)



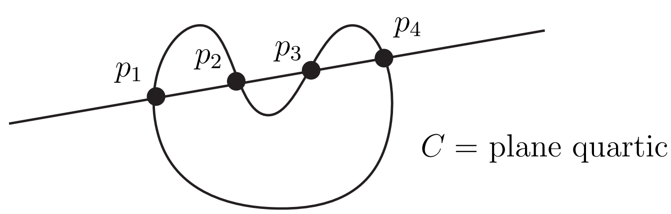
[2]In this picture taken from Pereira-Pirio, with $L_i = H_i$ starting at $\mathbf{q}$ one sequentially travels ϵ-distance along the L_i as pictured until you reach $\mathbf{p}$. For details see loc. cit.

- by projective duality the lines ℓ through the point $p = \ell_0 \cap C$ define a line $\check{p}$ in $\check{\mathbb{P}}^2$ determined by ℓ_0 and ℓ is a point on that line;
- any line in an open set $U \subset \check{\mathbb{P}}^2$ around $\check{p}$ will meet C, so the above picture will hold in U and determine a hypersurface through any point $\ell_0 \in U$.

To get a d-web we take d-arcs and use the picture



Algebraic geometry enters when the arcs are part of an algebraic curve



In general a degree d non-degenerate algebraic curve $C \subset \mathbb{P}^n$ generates a d-web in the dual projective space $\check{\mathbb{P}}^n$ of hyperplanes in $\mathbb{P}^n$.

Analytically a web is given by the level sets u_i = constant where $u_1, \dots, u_d \in \mathcal{O}(U)$ are holomorphic functions in U. We may make a change $u_i \to f_i(u_i) \cdot u_i$ where f_i is non-zero. In the picture (1) if $\omega = f(z)dz$ is a non-zero holomorphic 1-form on C, then the level sets of the function

$$u(\ell_0) = \int^{\ell_0 \cap C} \omega \tag{2}$$

defines a 1-web, in this case a line through the point $\ell_0 \in \check{\mathbb{P}}^2$.

An *abelian equation* is given by a linear relation

$$a_1(u_1)u_1 + \cdots + a_d(u_d)u_d = 0$$

in U. The name stems from Abel's theorem, which for a non-singular plane quartic gives

$$\sum_{i=1}^{d} \int_{p_0}^{p_i} \omega = 0 \tag{3}$$

where ω is a regular 1-form on C. One uses (2) to define the web and (3) to give an abelian equation. The dimension of the space of such 1-forms is $g(C) = 3$ where $g(C)$ is the genus of C.

The vector space of abelian relations is finite dimensional of dimension $r(\mathcal{W}) :=$ *rank* of $\mathcal{W}$. In his thesis Chern proved that for $n > 2$

$$r(\mathcal{W}) \leqq \pi(d, w) \tag{4}$$

where the *Castelnuovo number*

$$\pi(n,d) = \left\{ \begin{matrix} \text{maximum genus of a non-degenerate} \\ \text{algebraic curve of degree } d \text{ in } \mathbb{P}^n \end{matrix} \right\}.^{3}$$

The problem Chern took up in the 1970s, and on which I worked with him, has as a special case the converse to (4). More specifically, we wanted to prove that if $n > 2$ and $d \geqq 2n$, then any web of maximum rank for which equality holds in (4) is algebraizable and linearizable in the above way. As noted in the Pereira-Pirio book, even through we were not able to give a complete proof of this result the paper stimulated renewed interest in webs and the subsequent period saw a flurry of activity in web geometry, one that is continuing today. A complete proof of the result was finally given by Tréspreau in 2005, and together with the other works in the area led to a Bourbaki seminar in 2008 in which the renewed interest and significant results the whole area of webs was presented to the mathematical community.

[3]The result for $n = 2$ had been established by Bol. For $n > 2$ the result is significantly more complicated. Castelnuovo's number is given by an explicit formula involving binomial coefficients that for $n = 2$ is $\binom{d-1}{2}$, the usual formula for the genus of a smooth plane curve of degree d.