Maximal inequalities in noncommutative analysis

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Abstract. Maximal inequalities are of paramount importance in analysis. Here "analysis" is understood in a wide sense and includes harmonic analysis, probability theory and ergodic theory. Consider, for instance, the following three fundamental examples, each in one of the previous fields:

- **Hardy-Littlewood maximal function.** Given $f \in L_1(\mathbb{R})$ define

$$M(f)(t) = \sup_{t\in I} \frac{1}{|I|} \int_I |f(s)|ds, \quad t \in \mathbb{R},$$

where I denotes an interval $\mathbb{R}$ and $|I|$ the length of I.

- **Maximal martingale function.** Given $f = (f_n)_{n\geq 0}$ a martingale on a probability space $(\Omega, \mathcal{F}, P)$ define

$$M(f)(t) = \sup_{n\geq 0} |f_n|.$$

- **Maximal ergodic function.** Let T be a contraction on $L_p(\Omega)$ for every $1 \leq p \leq \infty$. Form the ergodic averages of T

$$A_n = \frac{1}{n+1} \sum_{k=0}^{n} T^k$$

and define

$$M(f)(t) = \sup_{n\geq 0} \big|A_n(f)\big|.$$

All three maximal functions satisfy the following inequality: For $1 < p \leq \infty$

$$\big\|M(f)\big\|_p \leq C_p \|f\|_p, \quad \forall f \in L_p(\mathbb{R}) \text{ or } L_p(\Omega),$$

where C_p is a constant depending only on p. This inequality fails for $p = 1$ but we have a weak type $(1, 1)$ substitute:

$$\sup_{\lambda>0} \lambda\big|\{M(f) > \lambda\}\big| \leq C_1 \|f\|_1,$$

where $|\{M(f) > \lambda\}|$ denotes the measure of the subset where $M(f)$ is bigger than λ. This classical result is due to Hardy-Littlewood, Doob or Dunford-Schwartz according to one of the three cases.

We will consider in this survey talk the analogues of these classical inequalities (and some others) in the noncommutative analysis. Then the usual L_p-spaces are replaced by noncommutative L_p-spaces associated to von Neumann algebras. The theory of noncommutative martingale/ergodic inequalities was remarkable developed in the last 20 years. Many classical results were successfully transferred to the noncommutative setting. This theory has fruitful interactions with operator spaces, quantum stochastic analysis and noncommutative harmonic analysis. We will discuss some of these noncommutative results and explain certain substantial difficulties in proving them.