Finite Group Actions in Four-Manifolds

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I. Finite Group Actions in SU(2)-Gauge Theory

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I. Finite Group Actions in SU(2)-Gauge Theory

I.1 Review of Donaldson Invariant

• $M$: a simply connected, smooth, closed 4-manifold with positive definite intersection form.

• $E \longrightarrow M$: SU(2)-bundle with $c_2(E) = -1$.

• A connection $\nabla \in \mathcal{C}(E)$: self-dual if $\star R^\nabla = R^\nabla$.

$$\Rightarrow R^\nabla : \text{harmonic}$$

$\nabla$ minimizes the Yang-Mills action

$$\mathcal{Y}_m(\nabla) = \frac{1}{2} \int_M \| R^\nabla \|^2 \, d\text{vol}.$$
The gauge group $g$ of bundle automorphisms of $E$ acts on the space $\mathcal{D}$ of self-dual connections.

$\mathcal{M}(E) = \mathcal{D}/g$ : the moduli space of self-dual connections.

By perturbation or generic metric and compactification.

Theorem I.1.1. $\overline{\mathcal{M}}(E)$ : a smooth, orientable, 5-manifold with $\lambda$ singular points each of which has a cone nbd on $\mathbb{CP}^2$, where $\lambda = \text{rank } H^2(M; \mathbb{Z})$.

Theorem I.1.2. If $H^2(M; \mathbb{Z}) = \langle a_1, \cdots, a_\lambda \rangle$, then

$$M \simeq \bigsqcup_{1}^{\lambda} \mathbb{CP}^2 : \text{cobordant.}$$
Theorem I.1.3.

(1) The positive definite intersection form (p.d.i.f) \[ \omega(M) \cong (1) \oplus \cdots \oplus (1). \]

(2) \exists Non smoothable topological 4-manifold.
   Ex. \( nE_8 \).

(3) \exists Exotic smooth structures on \( \mathbb{R}^4 \).

(4) Nondecomposability, \( M \neq M_1 \# M_2 \) (\( b_2^+(M_i) > 0, i = 1, 2 \)) if \( M \) has nontrivial Donaldson invariants.
I.2. Obstruction for $G$-moduli space

- Let $G$ be a finite group.
  Suppose that $G$ acts smoothly, semi-freely, isometrically on a simply connected, closed, smooth 4-manifold $M$ with a p.d.i.f.

Let $\phi : E \to M$ : a $G$-bundle with $c_2(E) = -1$,
$M^G = F := \{P_i\}_{i=1}^{n_1} \sqcup \{T^\lambda_i\}_{i=1}^{n_2}$, $G$ acts trivially on $E|_F$.

- $\overline{M}(E)$ is a $G$-space, but may not be a $G$-manifold. Transform the $G$-space $\overline{M}(E)$ into a smooth $G$-manifold with some singularities.

**Theorem I.2.1.** There is a Baire set in the $G$-invariant metrics such that $\widehat{M}^G$ is a smooth manifold in the $\widehat{M}^G$ of irreducible self-dual connections.
For each $\nabla \in \mathcal{M}^G$, there is $G$-invariant fundamental elliptic complex

$$o \rightarrow \Omega^0(g_E) \overset{d\nabla}{\longrightarrow} \Omega^1(g_E) \overset{d\nabla}{\longrightarrow} \Omega^2(g_E) \rightarrow o.$$  

Associate a Dirac operator

$$\Gamma(V_+ \otimes V_- \otimes g_C) \xrightarrow{D} \Gamma(V_- \otimes V_- \otimes g_C).$$

For each $g \in G$,

$$\text{ind}_g(D) = (-1)^m \frac{\text{ch}_g(j^*\sigma(D)) \cdot \text{td}(T^g \otimes \mathbb{C})}{\text{ch}_g(\Lambda_{-1}N^g \otimes \mathbb{C})} [TX^g],$$

$m = \dim M^g$,

$j : M^g \rightarrow M$,

$N^g = N(M, M^g)$ : normal bundle.
• Perturb the Fredholm $G$-map $\Psi : C/\mathfrak{g} \longrightarrow [C \times \Omega^2_- (\mathfrak{g}_E)]/\mathfrak{g}$ given by $\Psi([\nabla]) = [\nabla, R^\nabla]$ to transverse the zero section.

The Kuranishi map $\Psi$ is locally equivalent to the sum of a $G$-equivariant linear map and a nonlinear $G$-equivariant map in finite dimensional range.

• Now we assume that $G = \mathbb{Z}_2 = \langle h \rangle$ is the group of order 2.
Proposition I.2.2. Let \( A = \chi(F), \nabla \in \mathcal{M}(E)^G \) and \( h(\nabla) = g(\nabla) \) for some \( g \in g \).

(i) If \( (hg)^2 = I, \nabla \in \hat{\mathcal{M}}, \) then

\[
\begin{align*}
\dim H^1_{\nabla^+} - \dim H^2_{\nabla^+} &= \frac{1}{4}(10 + 3A) \\
\dim H^1_{\nabla^-} - \dim H^2_{\nabla^-} &= \frac{1}{4}(10 - 3A)
\end{align*}
\]

where \( H^*_{\nabla^\pm} \) is \((\pm 1)\)-eigen space of \((hg)\).

(ii) If \( (hg)^2 = -I, \quad \nabla \in \hat{\mathcal{M}}(E)^G, \) then

\[
\begin{align*}
\dim H^1_{\nabla^+} - \dim H^2_{\nabla^+} &= \frac{1}{4}(10 + A) \\
\dim H^1_{\nabla^-} - \dim H^2_{\nabla^-} &= \frac{1}{4}(10 - A)
\end{align*}
\]

where \( H^*_{\nabla^\pm} \) is \((\pm 1)\)-eigen space of \((hg)^2\),
(iii) If $\nabla$ is reducible, then

$$\begin{align*}
\dim H_{\nabla+}^1 - \dim H_{\nabla+}^2 &= \frac{1}{4}(14 + A) \\
\dim H_{\nabla-}^1 - \dim H_{\nabla-}^2 &= \frac{1}{4}(10 - A),
\end{align*}$$

where $H_{\nabla\pm}^*$ is $(\pm 1)$-eigen space of $g_1 h g g_2$ for some $g_i \in \Gamma_{\nabla}$.

- Apply a $G$-transversality technique of T. Petrie.
  $X = \mathcal{M}(E)^G,$
  $X_0 = (\text{end}(\mathcal{M}(E)) \cup \text{nbd of reducible con. in } \mathcal{M}(E)) \cap X,$

For $\nabla \in X$, there is a fiber bundle $V \longrightarrow X$ with fiber

$V_{\nabla} = \text{Hom}^S_G(H_{\nabla-}^1, H_{\nabla-}^2)$

: the space of surjective $G$-homomorphisms

: a Stiefel manifold with homotopy groups

$$\pi_i(V_{\nabla}) = \begin{cases} 
\mathbb{Z} & \text{if } (hg)^2 = -I \text{ and } i = 2, \\
0 & \text{if } (hg)^2 = I \text{ and } i \leq 3.
\end{cases}$$
Theorem I.2.3.

(i) To perturb $\Psi$ to be $G$-transversal throughout of $\mathcal{M}(E)^G$ there are obstruction classes $\Theta_3(\Psi) \in H^3(X, X_0; \mathbb{Z}(= \pi_2(V_\nabla)))$.

(ii) If the obstruction cohomology classes $\Theta_3(\Psi) = 0$, then we may have a smooth $G$-manifold $\mathcal{M}(E)$ of dimension 5 with $\lambda$ singular points each of which has a cone nbd of $\mathbb{C}P^2$, where $\lambda = \text{rank} H^2(M; \mathbb{Z})$. 
I.3. Involution and Donaldson invariant

- $X$: a closed, simply connected, smooth 4-manifold with an orientation-preserving smooth involution $\sigma$.

  $X^\sigma = F$: a 2-dim submanif.

  $E \rightarrow X$: SU(2) vector bundle, $-c_2(E)$: even.

  $p: X \rightarrow X/\sigma = X'$: projection, $F' = p(F)$.

- For a generic $\sigma$-invariant metric on $X$, let $\mathcal{M}(E)^\sigma$ be the moduli space of $\sigma$-invariant self-dual connections.
**Theorem 1.3.1.** There is 1-1 correspondence between $\mathcal{M}(E)^\sigma$ and $\mathcal{M}(E')$, where $\mathcal{M}(E')$ is the moduli space of self-dual connections on $E' \to X'$.

Let $\dim \mathcal{M}(E) = \dim \mathcal{M}(E') = 2d$.

For the coupled $\text{SU}(2)$-bundle $E^* \to \mathcal{M}(E) \times X$, the Donaldson $\mu : H_2(X) \xrightarrow{c_2(E^*)/} H^2(\mathcal{M}(E)) \xrightarrow{PD} H_{2d-2}(\mathcal{M}(E'))$. 

\[ \xymatrix{ H_2(X) \ar[r]^{c_2(E^*)/} & H^2(\mathcal{M}(E)) \ar[d]^{PD} \\ & H_{2d-2}(\mathcal{M}(E')) } \]
The Donaldson invariant

\[ D^\sigma : H_2(X)^\sigma \times \cdots \times h_2(X)^\sigma \rightarrow \mathbb{Z}, \]
\[ D^\sigma(\alpha_1, \cdots, \alpha_d) = \#[\overline{\mathcal{M}}^\sigma(E) \cap \mu(\alpha_1) \cap \cdots \cap \mu(\alpha_d)]. \]

**Theorem 1.3.2. (Wang)** Let \( \alpha_1, \cdots, \alpha_d \in H_2(X; \mathbb{Z})^\sigma \) and \( p_*(\alpha_i) = 2\beta_i \in H_2(X'; \mathbb{Z}), i = 1, \cdots, d. \) Then \( D_\sigma(\alpha_1, \cdots, \alpha_d) = D'(\beta_1, \cdots, \beta_d), \) where \( D' : H_2(X') \times \cdots \times H_2(X') \rightarrow \mathbb{Z} \) is the Donaldson invariant on the quotient \( E' \rightarrow X'. \)
II. Finite Group Actions in U(1)-Gauge Theory

II.1. Review of Seiberg-Witten Invariant

- $X$ : a closed, oriented, smooth 4-manifold, $b_2^+(x) > 1$.
  $L \rightarrow X$ : a $U(1)$–bundle with $c_1(L) = c_1(X)$ mod 2.
- $W^\pm$ : twisted spinor bundles associated with $L$.
- Clifford multi. $W^+ \otimes T^*X \rightarrow W^-$
- $\tau : W^+ \times W^+ \rightarrow \text{End}(W^+)$, given by $\tau(\phi, \phi) = (\phi \otimes \bar{\phi}^t) :$ traceless endomorphism of $W^+$.

- Levi-Civita connection. on $X$ and a connection.
  $A$ on $L$ induce a Dirac operator $D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$.
  Seiberg-Witten (SW) equations :

\[
\begin{cases}
D_A\phi = 0 \\
F_A^+ = -\tau(\phi, \phi)
\end{cases}
\]
• Gauge group $c^\infty(X,\mathbb{U}(1))$ of $L$ acts on $\mathcal{SW}(L) = \{(A, \phi)|(\boxtimes)\}$.

$\mathcal{M}(L) = \mathcal{SW}(L)/c^\infty(X,\mathbb{U}(1))$ : the moduli space associated with the spin$^c$ structure $L$ on $X$.

• For a generic metric on $X$, $\mathcal{M}(L)$ : a compact orientable manifold with dimension $d = \frac{1}{4}[c_1(L)^2 - (2\chi + 3\sigma)]$.

• Fix a point $x_0 \in X$, $\rho : c^\infty(X,\mathbb{U}(1)) \to \mathbb{U}(1)$ : evaluation induces a $\mathbb{U}(1)$-bundle

$E = (\mathcal{SW}(L) \times \mathbb{C})/c^\infty(X,\mathbb{U}(1)) \to \mathcal{M}(L)$.

$\mathcal{SW}(L) = < c_1(E)^s, \mathcal{M}(L) >$

: the Seiberg-Witten invariant of $L$, where $d = 2s$. 
Theorem II.1.1.

(1) \( \exists \) finitely many spin\(^c\) str. \( L \) on \( X \) for which \( \text{SW}(L) \neq 0 \).

(2) If \( X = X_1 \# X_2, b_2^+(X_i > 0), i = 1, 2 \), then \( \text{SW}(L) = 0 \) for spin\(^c\) str. \( L \) on \( X \).

(3) \( \text{SW}(L) \) : indep. on the metrics on \( X \), dep. only on \( c_1(L) \).

(4) \( f \) : a self-diffeo. of \( X \), \( \text{SW}(L) = \pm \text{SW}(f^*L) \).

(5) If \( X \) admits a metric of positive scalar curvature, then \( \text{SW} \equiv 0 \) on \( X \).

(6) If \( X \) : symplectic, then \( \text{SW}(K_X) = \pm 1 \).

(7) Thom conjecture is true.
II.2. Finite Group Action and spin$^c$ structure

Let $\pi_1(X)$ be finite, and $G$ be a finite group.

**Theorem II.2.1.** If $G$ acts smoothly, freely on $X$, $\text{SW}_X \not\equiv 0$. Then $X/G$ cannot decomposed as a smooth $X_1\#X_2$ with $b_2^+(X_i) > 0, i = 1, 2$.

**Theorem II.2.2.** Let $X$ be symplectic, $c_1(X)^2 > 0$, $b_2^+(X) > 3$. If $\sigma :$ free anti-sym. involution on $X$, then $\text{SW} \equiv 0$ on $X/\sigma$ and $X/\sigma$ can not be symplectic.

**Theorem II.2.3.** Let $\text{SW}_X \not\equiv 0$ on $X$, $b_2^+(X) > 1$. If $Y$ has negative definite intersection form and $n_1, \ldots, n_k :$ even integers such that $4b_1(Y) = 2n_1 + \cdots + 2n_k + n_1^2 + \cdots + n_k^2$, and $\pi_1(Y)$ has a nontrivial finite quotient. Then $X\#Y$ has a nontrivial SW-invariant, but does not admit a symplectic str.
Theorem II.2.4. Let $X$ : symplectic, and $\sigma$ : anti-sympl. involution on $X$,

$$X^\sigma = \bigsqcup_\lambda \Sigma_\lambda$$

: lagrangian surfaces, $\text{genus}(\Sigma_\lambda) > 1$ for some $\lambda$, $b_2^+(X/\sigma) > 1$. Then $X/\sigma$ has vanishing SW-invariants.

Example II.2.5. Let $X = (\Sigma_g \times \Sigma_g, w \oplus w)$, and $f : \Sigma_g \to \Sigma_g$ : an involution such that $f^*w = -w$.
Let $\sigma_f : \Sigma_g \times \Sigma_g \to \Sigma_g \times \Sigma_g$, be given by $\sigma_f(x, y) = (f^{-1}(y), f(x))$. Then $\sigma_f$ is an anti-symplectic involution, the fixed point set is $(\Sigma_g \times \Sigma_g)^{\sigma_f} \simeq \Sigma_g$.
By Hirzebruch signature thm.,

$$b_2^+(X/\sigma_f) = \frac{1}{2}(b_2^+(X) - 1) = g^2 > 1 \text{ if } g > 1.$$
Theorem II.2.6. Let $X: \text{Kähler surf.}$, $b_2^+(X) > 3$, $H_2(X; \mathbb{Z})$ has no 2-torsion.

$\sigma: X \rightarrow X : \text{anti-holom. involution},$

$X^\sigma = \Sigma \text{ has genus } > 0$, $[\Sigma] \in 2H_2(X; \mathbb{Z})$.

If $K_X^2 > 0$ or $K_X^2 = 0$ and $g(\Sigma) > 1$, then $SW \equiv 0$ on $X/\sigma$. 
Let $\mathbb{Z}_p$ act on $X$, $p$ : prime, $H_1(X, \mathbb{R}) = 0, b_2^+(X) > 1$.

**Theorem II.2.7. (Fang)** Supp. $\mathbb{Z}_p$ acts trivially on $H^{2+}(X, \mathbb{R})$. $L : \mathbb{Z}_p$-equivariant spin$^c$ str. on $X$, the equivariant Dirac operator $D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$ has the form

$$\text{ind } \mathbb{Z}_p(D_A) = \sum_{j=0}^{p-1} k_j t^j \in R(\mathbb{Z}_p) = \frac{\mathbb{Z}[t]}{(t^p - 1)}.$$

Then $SW(L) = 0 \mod p$

if $k_j \leq \frac{1}{2}(b_2^+(X) - 1), j = 0, 1, \cdots, p-1.$
III. Finite Group Actions in Gromov-Witten Theory.

III.1. Finite Group Action in Symplectic 4-Manifolds

- Let \((X, \omega)\) be a closed symplectic 4-manifold. Let a finite group \(G\) act pseudo-holomorphically, semifreely on \(X\) with a codim.2 fixed pt. set \(F\).

- Let \(p : X \rightarrow X/G = X'\) be the projection.

\[
\begin{align*}
H^2(X; \mathbb{Z})^G & \xrightarrow{p^*} H^2(X'; \mathbb{Z}) \\
PD & \downarrow \quad \downarrow |G| \cdot PD \\
H_2(X; \mathbb{Z})^G & \xrightarrow{p_*} H_2(X'; \mathbb{Z})
\end{align*}
\]

commutes, where \(|G|\) : the order of \(G\), and PD : Poincaré dual.
• Let $A' \in H_2(X'; \mathbb{Z})$, $A' = PD(\alpha') \in H_2(X'; \mathbb{Z})$, and

$L' \to X': U(1)$ vector bundle $c_1(L') = \alpha'$.

$L \equiv p^* L' \to X$, $A = PD(c_1(L))$.

Then $A \in H_2(X; \mathbb{Z})^G$ and $p_*(A) = |G|A'$.

• Let $J$ be $G$-invariant, $\omega$-comptible almost complex st. on $X$. $\omega', J'$: push downs of $\omega$, $J$, resp., on $X'$.

$p_* : H = \text{im}(PD \circ P^*) \subset H_2(X; \mathbb{Z})^G \to H' = \text{im}(|G|PD) \subset H_2(X'; \mathbb{Z})$

is an isomorphism.
For $A \in H, p_*(A) = |G|A', F = X^G, F' = p(F) \subset X'$,

\[
\overline{\mathcal{M}}_{0,k}(X, F : A, J)^G
= \left[ \{ u : (C, x_1, \ldots, x_k) \to (X, F; F) \} \middle| \begin{array}{l}
u : J \to \text{holo. stable map representing } A \text{ relative to } F, \\
C \text{ is a curve with arithmetic genus } 0 \text{ and } k \text{ marked points,} \\
u(C) : G \text{ -- invariant.} \} / \text{Aut}(u) \right]
\]

\[
\overline{\mathcal{M}}_{0,k}(x', F' ; A', J')
= \left[ \{ u' : (C, x_1, \ldots, x_k) \to (X', F'; F') \} | u' : J' \to \text{holo. stable map representing } A' \text{ relative to } F', C \text{ is a curve with} \\
\text{arithmetic genus } 0 \text{ and } k \text{ marked points,} \} / \text{Aut}(u') \right].
\]
Theorem III.1.1.

(i) There is a homeomorphism

\[ \psi : \overline{\mathcal{M}}_{0,k}(X, F; A, J)^G \to \overline{\mathcal{M}}_{0,k}(X', F'; A', J') \]

which is an orientation preserving diffeomorphism on each strata.

(ii) And we have

\[ \dim (\overline{\mathcal{M}}_{0,k}(X, F; A, J)^G) = \dim (\overline{\mathcal{M}}_{0,k}(X', F'; A', J')) = 2c_1(X')A' + 2k - 2 - 2A' \cdot F' = d'. \]
III.2. Relative Gromov-Witten Invariant

The evaluation map

\[ ev_k : \overline{M}_{0, k}(X, F; A, J)^G \rightarrow X^k \]

\[ ev_k([C, x_1, \ldots x_k; u]) = (u(x_1), \ldots , u(x_k)). \]

\[ \text{im}(ev_k) \subset X^k : \text{a } d'-\text{dim. pseudo-cycle whose boundary has at most dim. } d' - 2. \]

The relative \( G \)-invariant Gromov-Witten invariant

\[ \Phi_{\text{A, } k}^{F, G} : H_d(X^k) \rightarrow Q \]

is given by the integral

\[ \Phi_{\text{A, } k}^{F, G}(D) = \int_{\overline{M}_{0, k}(X, F; A, J)^G} ev_k^* PD(D) = (\text{im } ev_k) \cdot D \]

: intersection number in \( X^k \), where \( d' + d = 4k \).
Similarly, the evaluation map

\[ ev_k' : \overline{M}_{0,k}(X', F'; A', J') \rightarrow X'^k \]

is given by

\[ ev_k'([C, x_1, \cdots x_k; u']) = (u'(x_1), \cdots, u'(x_k)). \]

The relative Gromov-Witten invariant on the quotient

\[ \Phi_{A', k}^{F'} : H_d(X'^k) \rightarrow Q \]

is defined by the integral

\[ \Phi_{A', k}^{F'}(D') = \int_{\overline{M}_{0,k}(X', F'; A', J')} ev_k^* PD(D') = (\text{im} (ev_k')) \cdot D \]

: intersection number in \( X^k \), where \( d' + d = 4k \).

- If \( 2c_1(X')A' = 2 + 2k + 2F' \cdot A' \), then the invariant is the degree of the \( ev_k' \) which is the number of \( J' \)-holomorphic curves representing the homology \( A' \) meeting generic \( k \) distinct points in \( F' \) tangently.
Theorem III.2.1. Let $D \in H \subset H_d(X^k; \mathbb{Z})^G$ and $D' \in H' \subset H_d(X'^k; \mathbb{Z})$ such that $p_*(D) = |G|^k D'$, then

$$\Phi_{A,k}^{F,G}(D) = \Phi_{A',K}^{F',G}(D').$$

[i.e., $D \in H_d(X^k; \mathbb{Z})^G = \sum_{d_1+\cdots+d_k=d} H_{d_1}(X)^G \times \cdots \times H_{d_k}(X)^G$

$\exists D_{d_i} \in H_{d_i}(X)^G, i = 1, \cdots, k$ such that

$D = \sum_{d_1+\cdots+d_k=d} D_{d_1} \times \cdots \times D_{d_k}$.

$p_*(D_{d_i}) = |G|^k D_{d_i}'$, $i = 1, \cdots, k$.

$p_*(D) = p_*(\sum_{d_1+\cdots+d_k=d} D_{d_1} \times \cdots \times D_{d_k})$

$= \sum_{d_1+\cdots+d_k=d} p_*(D_{d_1}) \times \cdots \times p_*(D_{d_k})$

$= |G|^k p_*(D').$

$\Phi_{A,k}^{F,G}(D) = (\text{im}(ev_k)) \cdot D : \text{intersection number in } X^k.$]
### III.3. Example

- Recall the number $R(d)$ of rational curves of the degree $d$ in $\mathbb{C}P^2$.

$$\dim \overline{M}_{0,k}(\mathbb{C}P^2, d[S^2]) = 2(3d - 1 + k).$$

If $k = 3d - 1$, then $\dim \overline{M}_{0,k}(\mathbb{C}P^2, d[S^2]) = 4(3d - 1)$,

$$ev_k : \overline{M}_{0,k}(\mathbb{C}P^2, d[S^2]) \to (\mathbb{C}P^2)^k.$$

$$R(D) = \int_{\overline{M}_{0,k}(\mathbb{C}P^2,d[S^2])} \prod_{i=1}^{k} ev_k^*(c_1(\mathcal{O}(1)^2_i)) = \deg(ev_k)$$

$$= \left\{ \begin{array}{ll} \sum_{d_1+d_2=d} R(d_1)R(d_2)d_1^2d_2 \left[ k_2\left(\frac{3d-4}{3k_1-2}\right) - k_1\left(\frac{3d-4}{3k_1-1}\right) \right], & d \geq 2 \\ 1, & d = 1 \end{array} \right.$$__

computed by the WDVV-equations for the potential.
• For \( \mathbb{CP}^1 \subset \mathbb{CP}^2 \),

\[
\dim \overline{\mathcal{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1) = 2(2d - 1 + k).
\]

If \( k = 2d - 1 \), then \( \dim = 4(2d - 1) \).

The number of rational curves in \( \mathbb{CP}^2 \) of degree \( d \) relative to \( \mathbb{CP}^1 \) passing through generic \((2d - 1)\) points in \( \mathbb{CP}^1 \) is

\[
R'(d) = \int_{\overline{\mathcal{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1)} \prod_{i=1}^{k} ev_k'^*(c_1(Q(1)i)^2) = \text{degree of } [ev_k' : \overline{\mathcal{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1) \to (\mathbb{CP}^2)^k].
\]

• Let \( \sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1 \) be the involution given by \( \sigma(z_1, z_2) = (z_2, z_1) \).

Then \( (\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma = \Delta \sim \mathbb{CP}^1 \), and \( \mathbb{CP}^1 \times \mathbb{CP}^1 / \sigma \sim \mathbb{CP}^2 \).
Commutative diagram

\[
\begin{array}{ccc}
H^2(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma & \xrightarrow{p^*} & H^2(\mathbb{CP}^2) \\
\text{PD} & \downarrow & \text{2 \cdot PD} \\
H_2(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma & \xrightarrow{p^*} & H_2(\mathbb{CP}^2).
\end{array}
\]

Let \( A, B \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1), C \in H_2(\mathbb{CP}^2) \): generators.

\[
p_*(\triangle) = p_*(A, B) = 2C,
\]

\[
dim \overline{\mathcal{M}}_{0,k}(\mathbb{CP}^1 \times \mathbb{CP}^1, \triangle; d(\triangle))^\sigma
\]

\[
= 2C_1(\mathbb{CP}^1 \times \mathbb{CP}^1) \cdot d(\triangle) + 2k - 2 - 2 \cdot \triangle \cdot d(\triangle)
\]

\[
= 2(2d - 1 + k).
\]

For \( D \in H[(\mathbb{CP}^1 \times \mathbb{CP}^1)^k]^\sigma, p_*(D) = 2^k D' \in H(\mathbb{CP}^2)^k \),
the relative Gromov-Witten invariant

\[
\psi^C_{d(\triangle),k}(D) = \psi^C_{dC,k}(D')
\]

\[
= \psi^C_{dC,k}(pt, \ldots, pt) \text{ if } k = 2d - 1
\]

\[
= R'(d)
\]

\[
= \deg[ev_k' : \overline{\mathcal{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1, dC) \to (\mathbb{CP}^2)^k].
\]