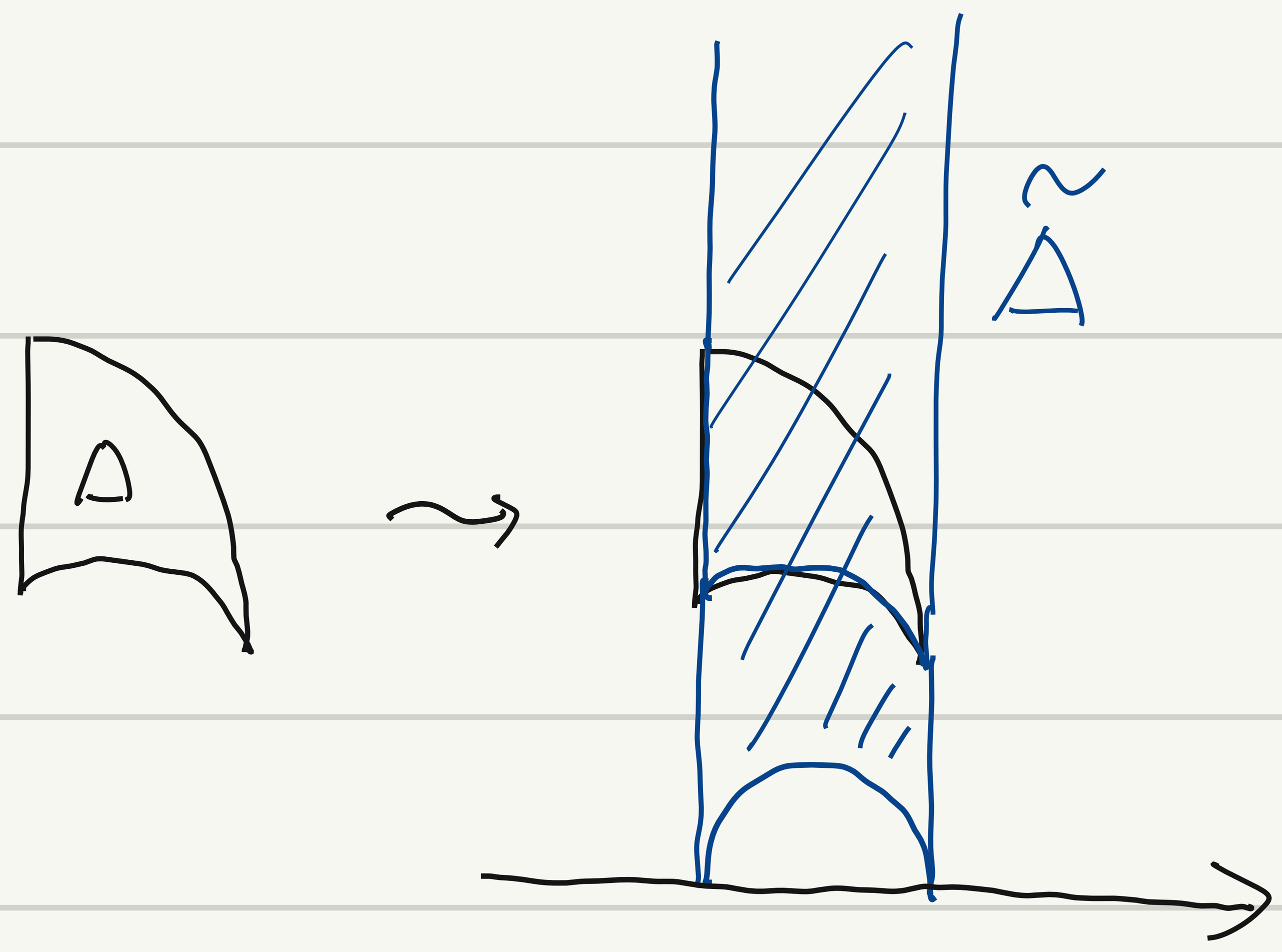
Thin Triangle

Prop: $\forall\, \Delta$ triangle in $\mathbb{H}$, $\exists$ an ideal triangle $\tilde{\Delta}$ s.t. $\Delta \subset \tilde{\Delta}$

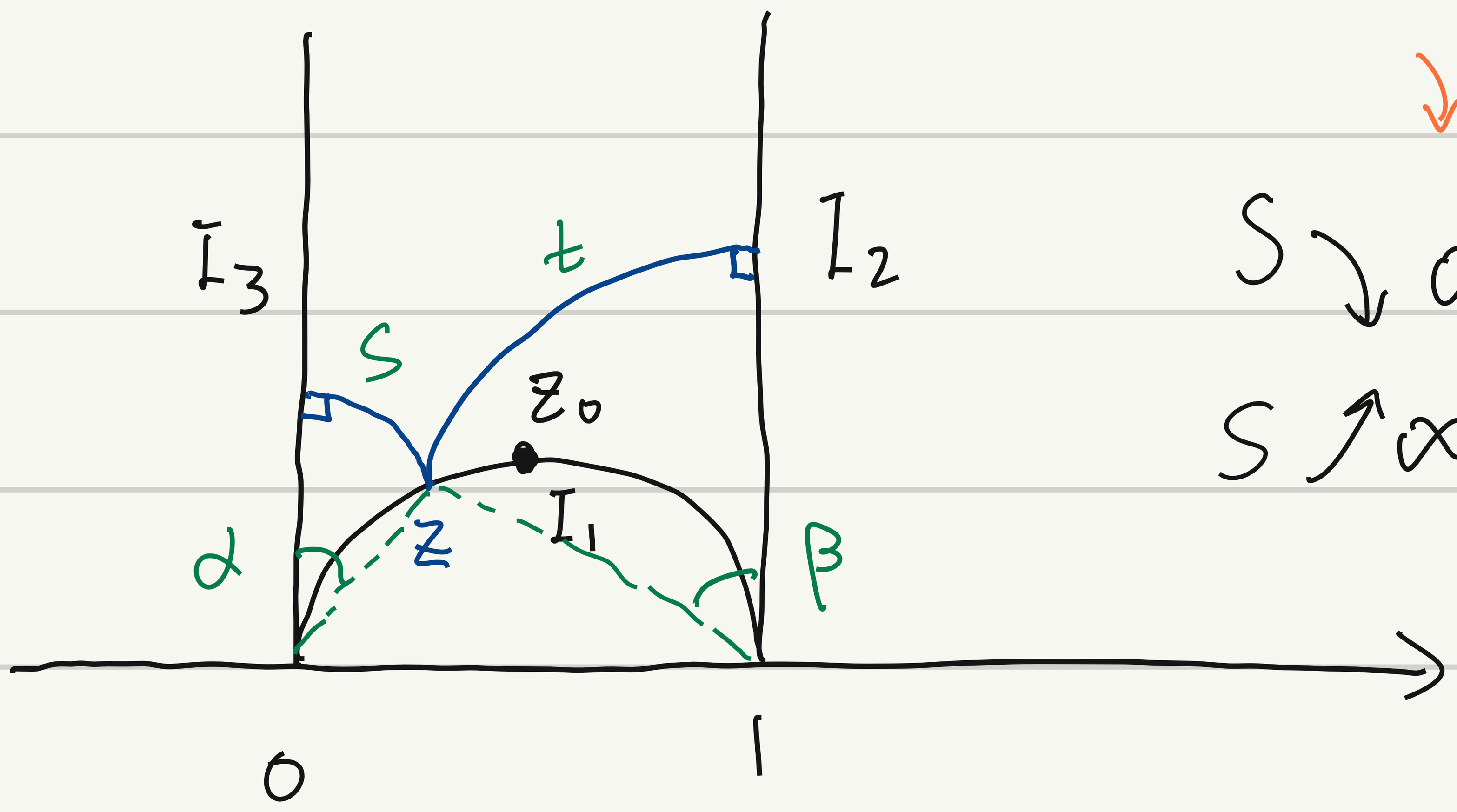
Proof:



Prop: Consider the ideal triangle of vertices $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$

$\forall\, z \in I_1$, $\quad d_{\mathbb{H}}(z, I_2 \cup I_3) \leq \log(1+\sqrt{2})$

Proof:



↓ decreasing ↗ increasing

$S \searrow 0$, $t \nearrow \infty$, as $z \to 0$ along I_1

$S \nearrow \infty$, $t \searrow 0$, as $z \to 1$

$d_{\mathbb{H}}(z, I_2 \cup I_3) = \min\{S(z), t(z)\}$

We should maximize $\min\{\alpha(z), \beta(z)\}$. $\left(\alpha(z) + \beta(z) = \frac{\pi}{2} \quad \forall z \in I_1\right)$

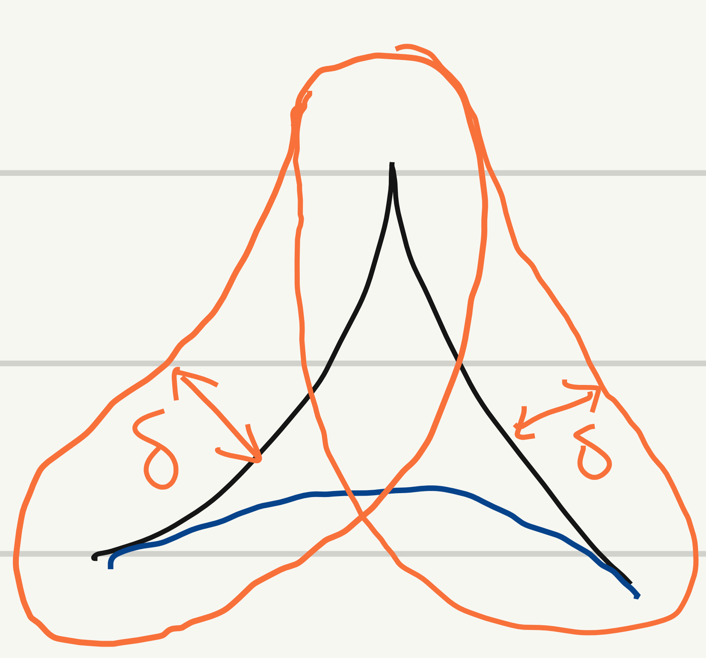
Notice that $\max\limits_{z \in I_1} \min\{\alpha(z), \beta(z)\} = \frac{\pi}{4}$ realized by $z_0 = \frac{1}{2} + \frac{i}{2}$

Hence $\max\limits_{z \in I_1} d_{\mathbb{H}}(z, I_2 \cup I_3) = d_{\mathbb{H}}(z_0, I_2) = d_{\mathbb{H}}(z_0, I_3)$

$$= \log \frac{\cos\frac{\pi}{4} + 1}{\sin\frac{\pi}{4}} = \log \frac{\frac{\sqrt{2}}{2} + 1}{\frac{\sqrt{2}}{2}} = \log(1+\sqrt{2})$$

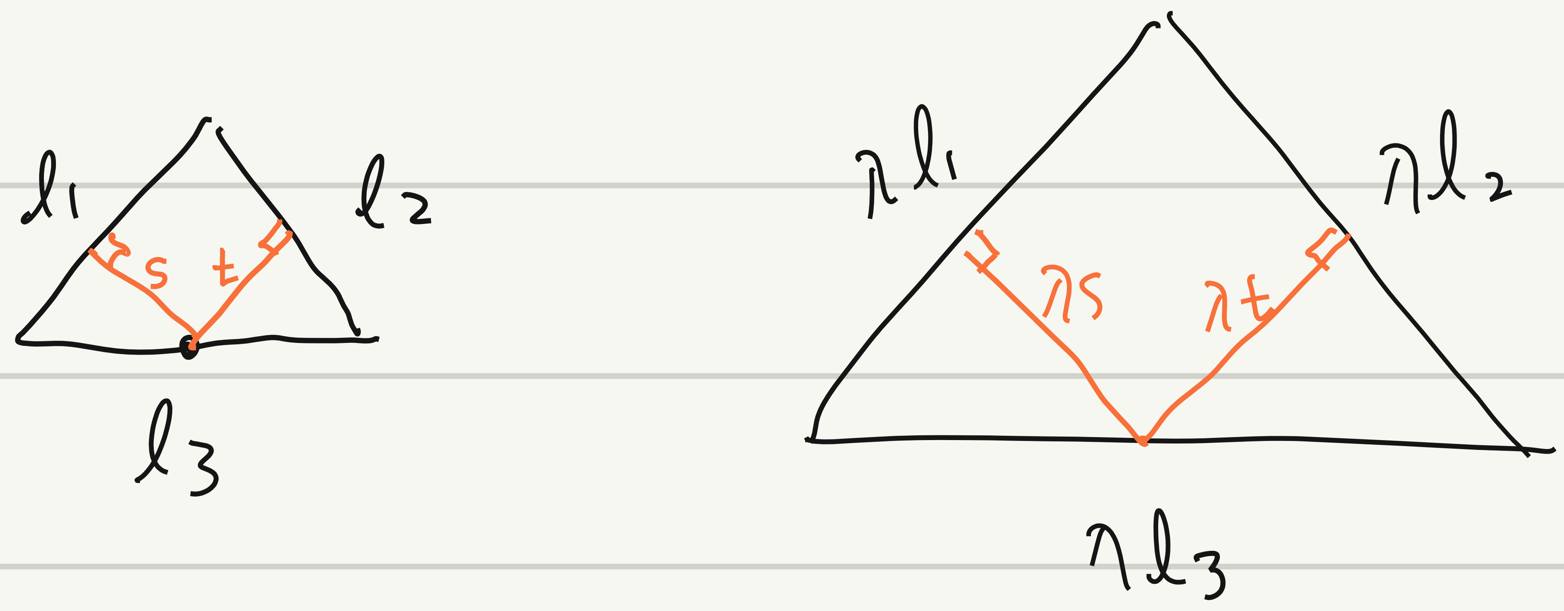
Cor: $\exists\, \delta\ (> \log(1+\sqrt{2}))$, s.t. $\forall\, \Delta$ triangle, $I_j \subset U_\delta(I_k \cup I_\ell)$, $\{j,k,\ell\} = \{1,2,3\}$

Ex:



$\delta > \log(1+\sqrt{2})$

Rmk: This is not true for Δ in $\mathbb{E}$.



λs, $\lambda t \longrightarrow \infty$, as $\lambda \to \infty$